Multi-Dimensional Reflective BSDE

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Multi-Dim Reflective BSDE
Applications

Lipschitz Growth

\begin{equation}
\begin{aligned}
Y(t) &= \xi + \int_t^T g(s, Y(s), Z(s)) ds - \int_t^T Z(s) dB_s + K(T) - K(t); \\
Y(t) &\geq L(t), \quad 0 \leq t \leq T, \quad \int_0^T (Y(t) - L(t))' dK(t) = 0.
\end{aligned}
\end{equation}

(1)
Lipschitz Growth

Entry by entry,

\[
\begin{aligned}
Y_1(t) &= \xi_1 + \int_t^T g_1(s, Y(s), Z(s)) ds - \int_t^T Z_1(s) dB_s + K_1(T) - K_1(t);\\
Y_1(t) &\geq L_1(t), 0 \leq t \leq T, \int_0^T (Y_1(t) - L_1(t)) dK_1(t) = 0, \\
&\ldots \\
Y_m(t) &= \xi_m + \int_t^T g_m(s, Y(s), Z(s)) ds - \int_t^T Z_m(s) dB_s + K_m(T) - K_m(t);\\
Y_m(t) &\geq L_m(t), 0 \leq t \leq T, \int_0^T (Y_m(t) - L_m(t)) dK_m(t) = 0.
\end{aligned}
\]

(2)
Seek solution \((Y, Z, K)\) in the spaces

\[
Y = (Y_1, \cdots, Y_m)' \in M^2(m; 0, T) = \{m\text{-dimensional predictable process } \phi \text{ s.t. } \mathbb{E}[\sup_{0,T} \phi_t^2] \leq \infty\};
\]

\[
Z = (Z_1, \cdots, Z_m)' \in L^2(m \times d; 0, T) = \{m \times d\text{-dimensional predictable process } \phi \text{ s.t. } \mathbb{E}[\int_0^T \phi_t^2 dt] \leq \infty\};
\]

\[
K = (K_1, \cdots, K_m)' = \text{continuous, increasing process in } M^2(m; 0, T).
\]

(3)
Assumption A 3.1

1. The random field

\[ g = (g_1, \cdots, g_m)' : [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \rightarrow \mathbb{R}^m \]  

is predictable in \( t \), and is uniformly Lipschitz in \( y \) and \( z \), i.e. there exists a constant \( b > 0 \), such that

\[ |g(t, y, z) - g(t, \bar{y}, \bar{z})| \leq b(\|y - \bar{y}\| + \|z - \bar{z}\|), \forall t \in [0, T]. \]

Further more,

\[ \mathbb{E}[\int_0^T g(t, 0, 0)^2 dt] < \infty. \]  

2. The random variable \( \xi \) is \( \mathcal{F}_T \)-measurable and square-integrable. The lower reflective boundary \( L \) is progressively measurable, and satisfy \( \mathbb{E}[\sup_{[0,T]} L^+(t)^2] < \infty. \) \( L \leq \xi, \mathbb{P}\text{-a.s.} \)
Lipschitz Growth

Results:

- existence and uniqueness of solution, via Picard iteration
- 1-dim Comparison Theorem (El Karoui et al, 1997)
- continuous dependency property
Linear Growth, Markovian System

\( l(elle)\)-dim forward equation

\[
\begin{cases}
X^{t,x}(s) = x, \ 0 \leq s \leq t; \\
\frac{dX^{t,x}(s)}{ds} = f(s, X^{t,x}(s))ds + \sigma(s, X^{t,x}(s))dB_s, \ t < s \leq T.
\end{cases}
\] (7)

\( m\)-dim backward equation

\[
\begin{cases}
Y^{t,x}(s) = \xi(X^{t,x}(T)) + \int_s^T g(r, X^{t,x}(r), Y^{t,x}(r), Z^{t,x}(r))dr \\
\quad - \int_s^T Z^{t,x}(r)dB_r + K^{t,x}(T) - K^{t,x}(s); \\
Y^{t,x}(s) \geq L(s, X^{t,x}(s)), \ t \leq s \leq T, \ \int_t^T (Y^{t,x}(s) - L(s, X^{t,x}(s)))' dK^{t,x}(s)
\end{cases}
\] (8)
Assumption A 4.1

1. Drift $f : [0, T] \times \mathbb{R}^l \to \mathbb{R}^l$, and volatility $\sigma : [0, T] \times \mathbb{R}^l \to \mathbb{R}^{l \times d}$, are deterministic, measurable mappings, locally Lipschitz in $x$ uniformly for all $t \in [0, T]$. And for all $(t, x) \in [0, T] \times \mathbb{R}^l$, 
   
   \[ |f(t, x)|^2 + |\sigma(t, x)|^2 \leq C(1 + |x|^2), \]
   for some constant $C$.

2. $g : [0, T] \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to \mathbb{R}^m$ is deterministic, measurable, and for all $(t, x, y, z) \in [0, T] \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$,
   
   \[ |g(t, x, y, z)| \leq b(1 + |x|^p + |y| + |z|), \]
   for some positive constant $b$;

3. for every fixed $(t, x) \in [0, T] \times \mathbb{R}$, $g(t, x, \cdot, \cdot)$ is continuous.

4. $\xi : \mathbb{R}^l \to \mathbb{R}^m$ deterministic, measurable. $L : [0, T] \times \mathbb{R}^l \to \mathbb{R}^m$ deterministic, measurable, continuous. $\mathbb{E}[\xi(X(T))^2] < \infty$;

\[ \mathbb{E}[\sup_{[0,T]} L^+(t, X(t))^2] < \infty. \quad L \leq \xi, \mathbb{P}\text{-a.s.} \]
Linear Growth, Markovian System

Results

- existence of solution, via Lipschitz approximation
- 1-dim Comparison Theorem
- continuous dependency property
Applications

What for?
Applications

Connections with

- Multi-dim variational inequalities (Feynman-Kac formula)
- Non-zero-sum stoch. differential games (esp. Dynkin games)
- Financial market sensitive to large traders’ transactions


References


THAT’S ALL
THANK YOU