Gambler’s Ruin Problem with Relative Wealth Perception

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Abstract

We consider a modified version of Gambler’s Ruin Problem in which the gambler decides to quit based on the relative change of his or her wealth. For this purpose we consider two possible changes of wealth, namely the upward rally and the downward fall. We define upward rally as the difference of the current wealth and its historical minimum, while the downward fall is the difference of the historical maximum of the gambler’s wealth and its current value. The gambler stops playing once either the upward rally or the downward fall reach some pre-specified levels for the first time. This paper determines probabilities of stopping on the upward rally in contrast to stopping on the downward fall both in the discrete and in the continuous time models.

1 Introduction

In the traditional setup of the Gambler’s Ruin Problem, the gambler quits once his or her wealth reaches some upper or lower level for the first time. The evolution of the gambler’s wealth is assumed to be a biased random walk in the discrete time model, and a Brownian motion with non-positive drift in the continuous time model. It is quite straightforward to compute explicitly the probability of stopping the game at the upper level in contrast to stopping the game at the lower level. Reaching the upper bound can be viewed as winning in the betting game, while reaching the lower bound as losing in the game. Computing these probabilities is an easy consequence of the Optional Sampling Theorem and this result is reviewed in the Appendix.

However, people often make decisions based on relative change in contrast to absolute change of their wealth. As a consequence, some gamblers (and investors in general) may have a tendency to stop after their wealth makes a significant positive or negative movement. In this paper we consider this situation, i.e., the case when the gambler decides to stop either when his or her current wealth is above a certain level in comparison to the historical minimum of his wealth (upward rally), or when his or her current wealth is below a certain level in comparison to the historical maximum of his or her wealth (downward fall). The gambler would stop as soon as either the upward rally or the downward fall reach some pre-specified values.

In the setting described above we compute the probabilities that the gambler stops after his or her wealth makes a significant upward rally or a significant downward fall. The probabilities of quitting the game on upward rally (or downward fall) are computed both in the discrete and in the continuous time framework. The probabilities are computed by means of the distribution function of the random variables $Y^+_{T_1(a)}$ and $Y^-_{T_2(b)}$, where $Y^+_{T_1(a)}$ represents the value of the upward
rally when the downward fall reaches the level $a$ for the first time, and $Y_{T_2(b)}^-$ represents the value of the downward fall when the upward rally reaches the level $b$ for the first time.

In the discrete time framework, it is shown that the distribution of each of $Y_{T_1(a)}^+$ and $Y_{T_2(b)}^-$ is geometric with a parameter that is related to the gambler’s ruin probability in the traditional setting, but with an additional mass at 0. The mass at 0 is computed in terms of the expected values of the time it takes the downward fall or the upward rally to reach their respective thresholds. This is achieved using the method described by Siegmund (see [2]) in the computation of the expected value of the one-sided CUSUM stopping time.

In the continuous time framework, the computation of the probabilities is achieved using the distributional properties of $y_{T_1(a)}^+$ and $y_{T_2(b)}^-$, the continuous time counterparts of the above mentioned random variables. Using results of Taylor [3] and Lehoczky [1] for distribution of stopped drifted Brownian motion at the first time of the downfall of level $a$, we are able to show that the probability density function of $y_{T_1(a)}^+$ and $y_{T_2(b)}^-$ is exponential, but with a discrete mass at 0. The mass at 0 can be computed in a similar fashion as in the discrete case.

This paper is structured as follows. The concept of relative wealth is presented in Section 2.1 and the probabilities of quitting the game on upward rally (or downward fall) are computed along with the probability distribution functions of the random variables $Y_{T_1(a)}^+$ and $Y_{T_2(b)}^-$ in the discrete time framework. Probabilities for an unbiased random walk follow as a special case. In Section 2.2, the explicit formulas for the probabilities of quitting the game on upward rally (or downward fall) as well as the probability distribution functions of the random variables $y_{T_1(a)}^+$ and $y_{T_2(b)}^-$ are determined in the continuous time framework. Probabilities for the case of standard Brownian motion are also given as a consequence of the general result. Concluding remarks appear in Section 3 and the Appendix reviews the probabilities of winning (or losing) in the traditional gambler’s ruin problem both in the discrete and in the continuous time framework.

2 The Gambler’s Ruin Problem with Relative Wealth Perception

2.1 The discrete time framework

Assume that the evolution of the gambler’s wealth $S_n$ follows biased random walk, i.e., at time $n$

$$S_n = \sum_{i=1}^{n} Z_i,$$

where

$$Z_i = \begin{cases} 1, & \text{with probability } p, \\ -1, & \text{with probability } q, \end{cases}$$

with $p + q = 1$ and $p < q$. The quantity

$$S_n - \min_{k \in [0,n]} S_k$$

measures the size of the upward rally comparing the present value of the wealth to its historical minimum, while the quantity

$$\max_{k \in [0,n]} S_k - S_n$$

measures the size of the downward fall.
measures the size of the downward fall comparing the present value of the wealth to its historical maximum.

The aim of this section is to determine the probability that the gambler would quit the game on the upward rally in contrast to quitting the game on the downward fall. To this effect, we introduce the stopping times:

\[ T_1(a) = \inf\{n \in \mathcal{N} : \max_{k \in [0,n] \cap \mathcal{N}} S_k - S_n = a, \ a \in \mathcal{N}\}, \]

and

\[ T_2(b) = \inf\{n \in \mathcal{N} : S_n - \min_{k \in [0,n] \cap \mathcal{N}} S_k = b, \ b \in \mathcal{N}\}. \]

The gambler stops at \( T(a,b) = T_1(a) \wedge T_2(b) \). The stopping times \( T_1(a) \) and \( T_2(b) \) indicate the first time of reaching the critical level of the downward fall \( T_1(a) \), or the first time of reaching the critical level of the upward rally \( T_2(b) \). In this section, we compute probabilities of the events \( \{T(a,b) = T_1(a)\} \), which represents stopping the game on the downward fall, and \( \{T(a,b) = T_2(b)\} \), which represents stopping the game on the upward rally.

In order to simplify notation we introduce the following processes:

\[ M^+_n := \min_{k \in [0,n] \cap \mathcal{N}} S_k, \]
\[ M^-_n := \min_{k \in [0,n] \cap \mathcal{N}} (-S_k) = - \sup_{k \in [0,n] \cap \mathcal{N}} S_k, \]
\[ Y^+_n := S_n - M^+_n, \]
\[ Y^-_n := -S_n - M^-_n. \]

Therefore we can re-express \( T_1(a) \) and \( T_2(b) \) as:

\[ T_1(a) = \min\{n \in \mathcal{N} : Y^-_n = a, \ a \in \mathcal{N}\}, \]
\[ T_2(b) = \min\{n \in \mathcal{N} : Y^+_n = b, \ b \in \mathcal{N}\}. \]

**Theorem 2.1** Let \( S_n = \sum_{i=1}^n Z_i \) be the evolution of the wealth of the gambler and let \( T(a,b) \), \( T_1(a) \) and \( T_2(b) \) be stopping times defined as above. We distinguish the following three cases:

1. \( b \geq a + 1 > 1 \)

   The probabilities of stopping the game on downward fall or upward rally are given by

\[ (2.1) \quad P\left( T(a,b) = T_1(a) \right) = m_A + (1 - m_A) \cdot (1 - R_A^{b-a}), \]
\[ (2.2) \quad P\left( T(a,b) = T_2(b) \right) = (1 - m_A) \cdot R_A^{b-a}, \]

respectively, where

\[ (2.3) \quad m_A = \frac{\left( \frac{q}{p} \right)^{a+1} - (a+1) \left( \frac{q}{p} \right) + a}{\left[ 1 - \left( \frac{q}{p} \right)^{-a} \right] \cdot \left( \frac{q}{p} \right)^{a+1} - 1}, \]
and

\[ R_A = \frac{1 - \left(\frac{q}{p}\right)^a}{1 - \left(\frac{q}{p}\right)^{a+1}}. \]  

2. \( a \geq b + 1 > 1 \)

The probabilities of stopping the game on downward fall or upward rally are given by

\[ P(T(a, b) = T_1(a)) = (1 - m_B) \cdot R_B^{a-b}, \]  

\[ P(T(a, b) = T_2(b)) = m_B + (1 - m_B) \cdot (1 - R_B^{a-b}), \]  

respectively, where

\[ m_B = \frac{\left(\frac{q}{p}\right)^{-(b+1)}}{1 - \left(\frac{q}{p}\right)^{-b}} \cdot \left(\frac{q}{p}\right)^{b+1} \cdot \left[\frac{1}{\left(\frac{q}{p}\right)^{b+1} - 1}\right], \]  

and

\[ R_B = \left(\frac{q}{p}\right)^{1 - \left(\frac{q}{p}\right)^b} \cdot \frac{1}{\left(\frac{q}{p}\right)^{b+1}}. \]  

3. \( a = b \)

The probabilities of stopping the game on downward fall or upward rally are given by

\[ P(T(a, a) = T_1(a)) = \frac{\left(\frac{q}{p}\right)^{a+1} - (a + 1)\left(\frac{q}{p}\right) + a}{\left[1 - \left(\frac{q}{p}\right)^{-a}\right] \cdot \left(\frac{q}{p}\right)^{a+1} - 1}, \]  

\[ P(T(a, a) = T_2(a)) = \frac{\left(\frac{q}{p}\right)^{-a+1} - (a + 1)\left(\frac{q}{p}\right)^{-1} + a}{\left[1 - \left(\frac{q}{p}\right)^{-a}\right] \cdot \left(\frac{q}{p}\right)^{a+1} - 1}, \]  

respectively.

The proof of the above theorem uses the following proposition:

**Proposition 2.2** The probability distribution functions of the random variables \( Y^+_{T_1(a)} \) and \( Y^-_{T_2(b)} \) are given by the following:

1. \( p_0^A = P(Y^+_{T_1(a)} = 0) = m_A + (1 - m_A) \cdot (1 - R_A), \)

\[ p_k^A = P(Y^+_{T_1(a)} = k) = (1 - m_A) \cdot (1 - R_A) \cdot R_A^k, \quad \forall \ k \in \mathbb{N}^*, \]  

where \( m_A \) and \( R_A \) are given by equations (2.3) and (2.4) respectively.
\[ p_0^B = P(Y_{T_2(b)}^- = 0) = m_B + (1 - m_B) \cdot (1 - R_B), \]
\[ p_k^B = P(Y_{T_2(b)}^- = k) = (1 - m_B) \cdot (1 - R_B) \cdot R_k^B, \quad \forall \ k \in \mathcal{N}^*, \]

where \( m_B \) and \( R_B \) are given by equations (2.7) and (2.8) respectively.

In order to prove Proposition 2.2 and Theorem 2.1, we will need the following two lemmas.

**Lemma 2.3**  For \( a, b \in \mathcal{N} \), we have:

\[ E[T_1(a)] = \frac{1}{p} \cdot \left( \frac{q}{p} \right)^{-(a+1)} \cdot \frac{(q/p)^{-1} - (a+1)(q/p)^{-1} + a}{((q/p)-1) \cdot [1-(q/p)]}, \]

\[ E[T_2(b)] = \frac{1}{q} \cdot \left( \frac{q}{p} \right)^{b+1} \cdot \frac{(q/p)^{-b+1} - (b+1)(q/p)^{-b} + b}{((q/p)-1) \cdot [1-(q/p)]}. \]

**Proof.** The proof is similar to the procedure that appears in Siegmund (1985) (see [2]) for the purpose of computing the expectation of the CUSUM stopping time. With \( S_n = \sum_{i=1}^{n} Z_i \), define the sequence of stopping times \( \{N_k\} \) in the following way:

\[ N_1 = \inf\{n \geq 1; S_n \notin (-1, b)\}. \]

If \( S_{N_1} = b \), then \( T_2 = N_1 \), otherwise

\[ S_{N_1} = \min_{k \in [0, N_1] \cap \mathcal{N}} S_k, \]

and

\[ N_2 = \inf\{n \geq 1; S_{N_1+n} - S_{N_1} \notin (-1, b)\}. \]

If \( S_{N_1+N_2} = b \), then \( T_2 = N_1 + N_2 \), else

\[ S_{N_1+N_2} = \min_{k \in [0, N_1+N_2] \cap \mathcal{N}} S_k. \]

In general we have:

\[ N_k = \inf\{n \geq 1; S_{N_1+\ldots+N_{k-1}+n} - S_{N_1+\ldots+N_{k-1}} \notin (-1, b)\}, \]

and \( T_2(b) = \sum_{i=1}^{M} N_i \), where

\[ M = \inf\{k; S_{N_1+\ldots+N_k} - S_{N_1+\ldots+N_{k-1}} = b\}. \]

Since the \( Z_i' \)s and the \( N_i' \)s are independent, from Wald’s identity it follows that

\[ E[S_{N_1}] = E[Z_1]E[N_1] = (p - q) \cdot E[N_1], \]

\[ E[T_2(b)] = E[N_1]E[M] = \frac{E[N_1]}{P(S_{N_1} = b)}, \]
since $M \sim \text{Geometric}(P(S_{N_1} = b))$. From Theorem 3.1 mentioned in the Appendix, we can write

\begin{align}
P(S_{N_1} = b) &= P(U(b, 1) = U_1(b)) = \frac{1 - \left(\frac{q}{p}\right)^b}{1 - \left(\frac{q}{p}\right)^{b+1}}, \tag{2.17} \\
P(S_{N_1} \leq -1) &= P(U(b, 1) = U_2(1)) = \left(\frac{q}{p}\right) \frac{1 - \left(\frac{q}{p}\right)^b}{1 - \left(\frac{q}{p}\right)^{b+1}}. \tag{2.18}
\end{align}

Finally, we have

\begin{align}
E[N_1] &= b \cdot P(S_{N_1} = b) + 1 \cdot P(S_{N_1} = -1). \tag{2.19}
\end{align}

Using equations (2.19), (2.18), (2.17), (2.16), and (2.15), we get (2.14).

Equation (2.13) follows similarly by noticing that

\begin{align*}
Y^-_n &= \sum_{i=1}^{n} R_i - \inf_{k \in [0,n] \cap \mathbb{N}} \sum_{i=1}^{k} R_i, \\
Y^+_n &= \sum_{i=1}^{n} R_i - \inf_{k \in [0,n] \cap \mathbb{N}} \sum_{i=1}^{k} R_i,
\end{align*}

where

\begin{align*}
R_i &= \begin{cases} 
1, & \text{with probability } q, \\
-1, & \text{with probability } p.
\end{cases}
\end{align*}

Equation (2.15) becomes:

\begin{align}
E[S_{N_1}] &= E[Z_1] E[N_1] = (q - p) \cdot E[N_1]. \tag{2.20}
\end{align}

The result follows by using equations (2.19), (2.18), (2.17), (2.16), where we substitute $p$ in place of $q$ and $q$ in place of $p$, and (2.20). This concludes the proof of the lemma. \hfill \diamond

**Lemma 2.4** We have

\begin{align*}
Y^+_k + Y^-_k &= \max_{i \in [0,k] \cap \mathbb{N}} \{Y^+_i, Y^-_i\}.
\end{align*}

**Proof.** Observe that

\begin{align}
Y^+_k + Y^-_k &= -M^+_k - M^-_k. \tag{2.21}
\end{align}

We notice that the process $Y^+_k + Y^-_k$ can only increase when either $S_k = M^+_k$ or $-S_k = M^-_k$, both of which cannot happen since that would imply that (2.21) is 0. Therefore, $Y^+_k + Y^-_k$ is constant in time unless either $Y^+_k = 0$ or $Y^-_k = 0$, at which instant

\begin{align*}
\max\{Y^+_k, Y^-_k\} &= \max_{i \in [0,k] \cap \mathbb{N}} \left\{\max\{Y^+_i, Y^-_i\}\right\}.
\end{align*}

This completes the proof of the lemma. \hfill \diamond

An important consequence of this lemma is that

\begin{align}
Y^+_{T_1(a)} &= (\max_{n \leq T_1(a)} Y^+_n - a) \vee 0, \tag{2.22} \\
Y^-_{T_2(b)} &= (\max_{n \leq T_2(b)} Y^-_n - b) \vee 0. \tag{2.23}
\end{align}
We can now proceed to the proof of Proposition 2.2 and then to the proof of Theorem 2.1.

Proof of Proposition 2.2. Let us compute the probability distribution function of the random variable \( Y_{T_1}^+ \), since the computation of the probability mass function of the random variable \( Y_{T_2}^- \) is done in a similar way. From equation (2.22), it follows that

\[
P \left( Y_{T_1}^+ = 0 \right) = P \left( \max_{n \leq T_1} Y_n^+ < a \right) + P \left( \max_{n \leq T_1} Y_n^+ \geq a \right) \cdot P \left( Y_{T_1}^+ = 0 \mid \max_{n \leq T_1} Y_n^+ \geq a \right),
\]

while

\[
P \left( Y_{T_1}^+ = k \right) = P \left( \max_{n \leq T_1} Y_n^+ \geq a \right) \cdot P \left( Y_{T_1}^+ = k \mid \max_{n \leq T_1} Y_n^+ \geq a \right).
\]

We prove this proposition in three basic steps:

In the first step we compute the distribution of the random variable

\[
\max_{n \leq T_1} S_n.
\]

In the second step we show that

\[
P \left( Y_{T_1}^+ = k \mid \max_{n \leq T_1} Y_n^+ \geq a \right) = P \left( \max_{n \leq T_1} S_n = k \right), \quad k \in \mathcal{N}.
\]

In the last step we compute \( P \left( \max_{n \leq T_1} Y_n^+ < a \right). \)

Beginning with the distribution of

\[
\max_{n \leq T_1} S_n,
\]

we notice that \( \max_{n \leq T_1} S_n = k \) is the same event as \( k \) times going up by 1 before going down by \( a \), and then going down by \( a \) before going up by 1. Thus we have

\[
P \left( \max_{n \leq T_1} S_n = k \right) = P \left( U_2(1) < U_1(a) \right)^k \cdot P \left( U_1(a) < U_2(1) \right),
\]

where the last equality follows from the definition of \( U_1(a) \) and \( U_2(b) \) as it appears in the Appendix. Therefore, using the result of Theorem 3.1, we get that

\[
\max_{n \leq T_1} S_n \sim \text{Geometric}(\pi),
\]

where \( \pi = \frac{\left( \frac{a}{p} \right)^a - \left( \frac{a}{p} \right)^{a+1}}{1 - \left( \frac{a}{p} \right)^{a+1}} \).

Let us proceed to the second step where we demonstrate

\[
\mathcal{L} \left( Y_{T_1}^+ \mid \max_{n \leq T_1} Y_n^+ \geq a \right) = \mathcal{L} \left( \max_{n \leq T_1} S_n \right).
\]

To see this, let

\[
R_1 = \sup \{ n \leq T_1 ; Y_n^+ = 0 \}.
\]
Fix $k \in \mathcal{N}$. Then

$$P\left(Y^+_T = k \mid \max_{n \leq T(a)} Y^+_n \geq a\right) = \frac{P\left(S_{T(a)} - \inf_{n \leq T(a)} S_n = k\right)}{P\left(\max_{n \leq T(a)} Y^+_n \geq a\right)}$$

$$= \frac{P\left(S_{T(a)} - S_{R_1} + S_{R_1} - \inf_{n \leq T(a)} S_n = k \mid R_1 < T(a)\right) \cdot P\left(R_1 < T(a)\right)}{P\left(\max_{n \leq T(a)} \left(S_n - S_{R_1} + S_{R_1} - \inf_{k \leq n} S_k\right) \geq a \mid R_1 < T(a)\right) \cdot P\left(R_1 < T(a)\right)}$$

$$= \frac{P\left(S_{T(a)} - S_{R_1} + S_{R_1} - \inf_{n \leq T(a)} S_n = k \mid R_1 < T(a)\right)}{P\left(\max_{n \leq T(a)} \left(S_n - S_{R_1} + S_{R_1} - \inf_{k \leq R_1} S_k\right) \geq a \mid R_1 < T(a)\right)}$$

$$= \frac{\frac{\left(1-\pi\right)^{k+a} \pi}{1-\frac{\left(1-\pi\right)^a}{1-a}}} = P\left(\max_{n \leq T(a)} S_n = k\right),$$

where $\pi = \frac{\left(\frac{a}{a}\right)^a - \left(\frac{b}{b}\right)^a+1}{1-\left(\frac{b}{b}\right)^a}$. Therefore we get

$$P\left(Y^+_T = k \mid \max_{t \leq T(a)} Y^+_t \geq a\right) \sim \text{Geometric}(\pi), \ k \in \mathcal{N}.$$ (2.30)

What remains to be computed is $P(\max_{n \leq T(a)} Y^+_n < a)$. From equation (2.22), it follows that

$$P\left(\max_{n \leq T(a)} Y^+_n < a\right) = P\left(T_1(a) < T_2(a)\right).$$ (2.31)

To compute $P\left(T_1(a) < T_2(a)\right)$, we first notice that

$$T_1(a) = T(a, b) + (T_1(a) - T(a, b))\mathbf{1}_{\{T(a,b)=T_2(b)\}},$$ (2.32)

$$T_2(b) = T(a, b) + (T_2(b) - T(a, b))\mathbf{1}_{\{T(a,b)=T_1(a)\}}.$$ (2.33)

Taking expectations we get

$$E[T_1(a)] = E[T(a,b)] + E\left[\left(T_1(a) - T(a,b)\right)\mathbf{1}_{\{T(a,b)=T_2(b)\}}\right],$$ (2.34)

$$E[T_2(b)] = E[T(a,b)] + E\left[\left(T_2(b) - T(a,b)\right)\mathbf{1}_{\{T(a,b)=T_1(a)\}}\right].$$ (2.35)

With $a = b$ and equation (2.22), it follows that

$$E[T_1(a)] = E[T(a,a)] + E[T_1(a)] \cdot P\left(T_2(a) < T_1(a)\right),$$ (2.36)

$$E[T_2(a)] = E[T(a,a)] + E[T_2(a)] \cdot P\left(T_1(a) < T_2(a)\right).$$ (2.37)

Using

$$P\left(T_1(a) < T_2(a)\right) + P\left(T_2(a) < T_1(a)\right) = 1$$
and equations (2.36) and (2.37), we conclude that

\begin{equation}
(2.38) \quad P\left( T_1(a) < T_2(a) \right) = \frac{E[T_2(a)]}{E[T_2(a)] + E[T_1(a)]}.
\end{equation}

The result now follows by substituting (2.30) and (2.38) into (2.24) and (2.25), using Lemma 2.3 and the fact that

\[ P(\max_{n \leq T_1(a)} Y_n^+ < a) + P(\max_{n \leq T_1(a)} Y_n^+ \geq a) = 1. \]

This concludes the proof of the proposition.

\( \Box \)

We can now proceed to the proof of Theorem 2.1.

**Proof of Theorem 2.1.** We will prove the result in the case \( b \geq a + 1 \) since the result is proven similarly in the case when \( a \geq b + 1 \).

From Lemma 2.4 and equation (2.22), it follows that on the event \( \{ T_1(a) < T_2(b) \} \), we have

\begin{equation}
(2.39) \quad Y^+_{T_1(a)} = \begin{cases} 
0 & \text{if } \max_{n \leq T_1(a)} Y_n^+ < a, \\
\max_{n \leq T_1(a)} Y_n^+ - a & \text{if } a \leq \max_{n \leq T_1(a)} Y_n^+ < b.
\end{cases}
\end{equation}

From equation (2.39) it becomes obvious that on the event \( \{ T_1(a) < T_2(b) \} \), \( Y^+_{T_1(a)} \) cannot exceed the level \( b - a \), or cannot be exactly equal to this level. Therefore

\begin{equation}
(2.40) \quad P\left( T_1(a) < T_2(b) \right) = \sum_{k=0}^{b-a-1} P\left( Y^+_{T_1(a)} = k \right).
\end{equation}

Using Proposition 2.2 the result follows. This completes the proof of the Theorem 2.1. \( \Box \)

It is interesting to see the probabilities of stopping on downward fall or upward rally for an unbiased random walk.

**Corollary 2.5** Let \( S_n = \sum_{i=1}^{n} Z_i \) be the evolution of the wealth of the gambler in a game of equal odds (\( p = q = \frac{1}{2} \)), and let \( T(a, b) \), \( T_1(a) \) and \( T_2(b) \) be stopping times defined as above. We distinguish the following three cases:

1. \( b \geq a + 1 > 1 \)

   The probabilities of stopping the game on downward fall or upward rally are given by

   \begin{align}
   (2.41) \quad P\left( T(a, b) = T_1(a) \right) &= 1 - \frac{1}{2} \cdot \left( \frac{a}{a+1} \right)^{b-a}, \\
(2.42) \quad P\left( T(a, b) = T_2(b) \right) &= \frac{1}{2} \cdot \left( \frac{a}{a+1} \right)^{b-a}.
   \end{align}

2. \( a \geq b + 1 > 1 \)

   The probabilities of stopping the game on downward fall or upward rally are given by

   \begin{align}
   (2.43) \quad P\left( T(a, b) = T_1(a) \right) &= \frac{1}{2} \cdot \left( \frac{b}{b+1} \right)^{a-b}, \\
(2.44) \quad P\left( T(a, b) = T_2(b) \right) &= 1 - \frac{1}{2} \cdot \left( \frac{b}{b+1} \right)^{a-b}.
   \end{align}
3. $a = b$

The probabilities of stopping the game on downward fall or upward rally are given by

$$P\left( T(a,a) = T_1(a) \right) = P\left( T(a,a) = T_2(a) \right) = \frac{1}{2}. \tag{2.45}$$

**Proof.** All of the above results are a simple consequence of taking the limit as $p \to \frac{1}{2}$ in Theorem 2.1.

**Corollary 2.6** Let $S_n = \sum_{i=1}^{n} Z_i$ be the evolution of the wealth of the gambler in a game of equal odds ($p = q = \frac{1}{2}$). The probability distribution functions of the random variables $Y^+_{T_1(a)}$ and $Y^-_{T_2(b)}$ are given by the following:

1. \hspace{1cm}

$$p^A_0 = P(Y^+_{T_1(a)} = 0) = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{a+1},$$

$$p^A_k = P(Y^+_{T_1(a)} = k) = \frac{1}{2} \cdot \frac{1}{a+1} \cdot \left(\frac{a}{a+1}\right)^k, \quad \forall k \in \mathbb{N}^*.$$

2. \hspace{1cm}

$$p^B_0 = P(Y^-_{T_2(b)} = 0) = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{b+1},$$

$$p^B_k = P(Y^-_{T_2(b)} = k) = \frac{1}{2} \cdot \frac{1}{b+1} \cdot \left(\frac{b}{b+1}\right)^k, \quad \forall k \in \mathbb{N}^*.$$

**Proof.** This corollary is a simple consequence of Proposition 2.2 by taking the limit as $p \to \frac{1}{2}$.

### 2.2 The continuous time framework

In the continuous time framework, the wealth of the gambler at each time point $t$ is assumed to follow

$$X_t = W_t - \mu t,$$

where $\mu > 0$ and $W_t$ is a Brownian motion.

The quantity

$$X_t - \inf_{s \in [0,t]} X_s$$

measures the size of the upward rally comparing the present value of the wealth to its historical minimum, while the quantity

$$\sup_{s \in [0,t]} X_s - X_t$$

measures the size of the downward fall comparing the present value of the wealth to its historical maximum.
The aim of this section is to determine the probability that the gambler would quit the game on the upward rally in contrast to quitting the game on the downward fall. We introduce the stopping times:

\[ T_c^1 (a) = \inf \{ t \geq 0 : \sup_{s \in [0,t]} X_s - X_t = a, \ a \in \mathbb{R}_+ \}, \]

and

\[ T_c^2 (b) = \inf \{ t \geq 0 : X_t - \inf_{s \in [0,t]} X_s = b, \ b \in \mathbb{R}_+ \}. \]

The gambler stops at \( T_c (a, b) = T_c^1 (a) \land T_c^2 (b) \). The stopping times \( T_c^1 (a) \) and \( T_c^2 (b) \) indicate the first time of reaching the critical level of the downward fall \( T_c^1 (a) \), or the first time of reaching critical level of the upward rally \( T_c^2 (b) \). In this section we compute the probabilities of the events \( \{ T_c (a, b) = T_c^1 (a) \} \), which represents stopping the game on downward fall, and \( \{ T_c (a, b) = T_c^2 (b) \} \), which represents stopping the game on upward rally.

In order to simplify notation we introduce the following processes:

\[ m^+_t := \inf_{s \in [0,t]} X_s, \]
\[ m^-_t := \inf_{s \in [0,t]} (-X_s) = - \sup_{s \in [0,t]} X_s, \]
\[ y^+_t := X_t - m^+_t, \]
\[ y^-_t := -X_t - m^-_t. \]

Using the above notation, the stopping times \( T_c^1 (a) \) and \( T_c^2 (b) \) become

\[ T_c^1 (a) = \inf \{ t \geq 0 : y^-_t = a, \ a \in \mathbb{R}_+ \} \]
\[ T_c^2 (b) = \inf \{ t \geq 0 : y^+_t = b, \ b \in \mathbb{R}_+ \} \]

**Theorem 2.7** Let \( X_t = W_t - \mu t \) be the evolution of the wealth of the gambler and let \( T_c^1 \) and \( T_c^2 \) be stopping times defined as above and \( \mu > 0 \). We distinguish the following two cases:

1. \( b \geq a > 0 \)
   
   **The probabilities of stopping at downward fall or upward rally are given by**

   \[ P \left( T_c^1 (a) = T_c^1 (a) \right) = m_A^c + (1 - m_A^c) \cdot \left[ 1 - \exp \left( -\frac{2\mu}{1-e^{-2\mu}} \cdot (b - a) \right) \right], \quad (2.48) \]
   \[ P \left( T_c^2 (b) = T_c^2 (b) \right) = (1 - m_A^c) \cdot \exp \left( -\frac{2\mu}{1-e^{-2\mu}} \cdot (b - a) \right), \quad (2.49) \]
   
   where

   \[ m_A^c = \frac{e^{2\mu a} - 2\mu a - 1}{e^{2\mu a} + e^{-2\mu a} - 2}. \quad (2.50) \]

2. \( a \geq b > 0 \)
The probabilities of stopping at downward fall or upward rally are given by

\[ P(T^c(a, b) = T^1_c(a)) = (1 - m_B^c) \cdot \exp \left( -\frac{2\mu}{e^{2\mu} - 1} \cdot (a - b) \right), \]

\[ P(T^c(a, b) = T^2_c(b)) = m_B^c + (1 - m_B^c) \cdot \left[ 1 - \exp \left( -\frac{2\mu}{e^{2\mu} - 1} \cdot (a - b) \right) \right], \]

where

\[ m_B^c = \frac{e^{-2\mu b} + 2\mu b - 1}{e^{2\mu b} + e^{-2\mu b} - 2}. \]

The proof of the theorem uses the next proposition:

**Proposition 2.8**  The probability distribution functions of the random variables \( y_{T^1_c(a)}^+ \) and \( y_{T^2_c(b)}^- \) are given by:

1. \[ P(y_{T^1_c(a)}^+ = 0) = m_A^c, \]
   \[ P(y_{T^1_c(a)}^+ \in dr) = (1 - m_A^c) \cdot \left[ 1 - \exp \left( -\frac{2\mu}{e^{2\mu} - 1} \cdot (a - b) \right) \right] dr, r > 0, \]

   where \( m_A^c \) is given by equation (2.50).

2. \[ P(y_{T^2_c(b)}^- = 0) = m_B^c, \]
   \[ P(y_{T^2_c(b)}^- \in dr) = (1 - m_B^c) \cdot \left[ 1 - \exp \left( -\frac{2\mu}{e^{2\mu} - 1} \cdot (a - b) \right) \right] dr, r > 0, \]

   where \( m_B^c \) is given by equation (2.53).

In order to prove Proposition 2.8 and Theorem 2.7, we will need the following two lemmas.

**Lemma 2.9**  For \( a, b \in \mathbb{R}_+ \), we have:

\[ E[T^c_1(a)] = \frac{e^{-2\mu a} + 2\mu a - 1}{2\mu^2}, \]

\[ E[T^c_2(b)] = \frac{e^{2\mu b} - 2\mu b - 1}{2\mu^2}. \]

**Proof.** Let \( g_2(x) = e^{2\mu x} - 2\mu x - 1 \). By applying Itô’s rule to the processes \( g_2(y_t^+) \) we get

\[ dg_2(y_t^+) = g''_2(y_t^+)dW_t - \mu g'_2(y_t^+)dt - g'_2(y_t^+)dm_t^+ + \frac{1}{2} g''_2(y_t^+)dt. \]

We notice that the third term in the right hand side of the above equality disappears because \( dm_t^+ \neq 0 \) only when \( y_t^+ = 0 \) and \( g'_2(0) = 0 \). We also notice that the function \( g_2 \) satisfies the second order differential equation

\[ -\mu g'_2(x) + \frac{1}{2} g''_2(x) = 2\mu^2. \]
By integrating from 0 to $T^c_2(b)$, we have

$$g_2(y^+(T^c_2(b))) - g_2(0) = \int_0^{T^c_2(b)} g_2(y^+_t) dW_t + \int_0^{T^c_2(b)} \left(-\mu g'_2(y^+_t) + \frac{1}{2}g''_2(y^+_t)\right) dt.$$ 

Using equation (2.61), $y^+(T^c_2(b)) = b$, $g_2(0) = 0$ and taking expectations we get

\begin{equation}
(2.62)
g_2(b) = 2\mu^2 E[T^c_2(b)].
\end{equation}

Consequently,

\begin{equation}
(2.63)
E[T^c_2(b)] = \frac{g_2(b)}{2\mu^2}.
\end{equation}

Similarly, by applying Itô’s rule to $g_1(y^-)$, where $g_1(x) = e^{-2\mu x} + 2\mu x - 1$, we have

$$g_1(y^-_{T^c_1(a)}) - g_1(0) = -\int_0^{T^c_1(a)} g'_1(y^-_t) dW_t + \int_0^{T^c_1(a)} \left(\mu g'_1(y^-_t) + \frac{1}{2}g''_1(y^-_t)\right) dt,$$

from which it follows that

\begin{equation}
(2.64)
E[T^c_1(a)] = \frac{g_1(a)}{2\mu^2}.
\end{equation}

This concludes the proof of the lemma.

\boxproof

**Lemma 2.10** We have

$$y^+_t + y^-_t = \max_{s \leq t} \{y^+_s, y^-_s\}.$$ 

**Proof.** Observe that

\begin{equation}
(2.65)
y^+_t + y^-_t = -m^+_t - m^-_t.
\end{equation}

We notice that the process $y^+_t + y^-_t$ can only increase when either $X_t = m^+_t$ or $-X_t = m^-_t$, both of which cannot happen at the same time since that would imply that $y^+_t + y^-_t$ is 0. Therefore, $y^+_t + y^-_t$ is a constant as a function of time unless either $y^+_t = 0$ or $y^-_t = 0$, at which instant

$$\max\{y^+_t, y^-_t\} = \sup_{s \in [0,t]} \{\max\{y^+_s, y^-_s\}\}.$$ 

This completes the proof of the lemma.

\boxproof

As a consequence of this lemma we have

\begin{equation}
(2.66)
y^+_{T^c_1(a)} = (\max_{t \leq T^c_1(a)} y^+_t - a) \lor 0,
\end{equation}

\begin{equation}
(2.67)
y^-_{T^c_2(b)} = (\max_{t \leq T^c_2(b)} y^-_t - b) \lor 0.
\end{equation}

Finally, in order to proceed to the proof of Proposition 2.8 and Theorem 2.7, we will use the results of Taylor in [3] and Lehoczky in [1]. Taylor computes the bivariate Laplace transform of
$X_{T_1^c}^c(a)$ and $T_1^c(a)$, where $T_1^c$ is defined as above. Lehoczky pointed out that the random variable $X_{T_1^c}^c(a) + a = \sup_{t \leq T_1^c(a)} X_t$ has the exponential distribution:

$$X_{T_1^c}(a) + a \sim \text{Exp} \left( \frac{2\mu}{1-e^{-2\mu a}} \right).$$

Note that the exponential parameter becomes equal to $\frac{1}{a}$ in the case when $\mu = 0$. Now we can proceed to the proof of Proposition 2.8 and then to the proof of Theorem 2.7.

**Proof of Proposition 2.8.** We will only compute the probability density function of the random variable $y_{T_1^c(a)}^+$ since the computation of the probability density function of the random variable $y_{T_2^c(b)}^+$ is done in a similar way. From equation (2.66), it follows that

$$P \left( y_{T_1^c(a)}^+ = 0 \right) = P \left( \max_{t \leq T_1^c(a)} y_t^+ < a \right),$$

while

$$P \left( y_{T_1^c(a)}^+ \in dr \right) = P \left( \max_{t \leq T_1^c(a)} y_t^+ \geq a \right) \cdot P \left( y_{T_1^c(a)}^+ \in dr \mid \max_{t \leq T_1^c(a)} y_t^+ \geq a \right) \cdot P \left( y_{T_1^c(a)}^+ > 0 \right) \cdot P \left( \max_{t \leq T_1^c(a)} y_t^+ \in dr \mid y_{T_1^c(a)}^+ > 0 \right), r > 0.$$

In the next discussion we first demonstrate

$$\mathcal{L} \left( y_{T_1^c(a)}^+ \mid y_{T_1^c(a)}^+ > 0 \right) = \mathcal{L} \left( X_{T_1^c}(a) + a \right).$$

To see this, let

$$R_1^c = \sup\{t \leq T_1^c(a); y_t^+ = 0\}.$$

Fix $r > 0$. Then

$$P \left( y_{T_1^c(a)}^+ \in dr \mid y_{T_1^c(a)}^+ > 0 \right) = \frac{P \left( X_{T_1^c(a)} - \inf_{s \leq T_1^c(a)} X_s \in dr \right)}{P \left( \max_{t \leq T_1^c(a)} y_t^+ \geq a \right)} \cdot \frac{P \left( X_{T_1^c(a)} - R_t^c \in dr \right)}{P \left( \max_{t \leq T_1^c(a)} X_t - X_{R_1^c} \in dr \mid R_t^c < T_1^c(a) \right)} \cdot P \left( R_t^c < T_1^c(a) \right)$$

$$= \frac{P \left( X_{T_1^c(a)} - R_t^c \in dr \right)}{P \left( \max_{t \leq T_1^c(a)} X_t - X_{R_t^c} \in dr \mid R_t^c < T_1^c(a) \right)} \cdot \frac{P \left( X_{T_1^c(a)} - \inf_{s \leq R_t^c} X_s \geq a \mid R_t^c < T_1^c(a) \right)}{P \left( \max_{t \leq T_1^c(a)} X_t - X_{R_t^c} \in dr \mid R_t^c < T_1^c(a) \right)} \cdot \frac{P \left( R_t^c < T_1^c(a) \right)}{P \left( \max_{t \leq T_1^c(a)} X_t \in dr \right)}$$

$$= \frac{P \left( X_{T_1^c(a)} - \inf_{s \leq R_t^c} X_s \geq a \mid R_t^c < T_1^c(a) \right)}{P \left( \max_{t \leq T_1^c(a)} X_t - X_{R_t^c} \in dr \mid R_t^c < T_1^c(a) \right)} \cdot \frac{P \left( \max_{t \leq T_1^c(a)} X_t \geq a \mid R_t^c < T_1^c(a) \right)}{P \left( \max_{t \leq T_1^c(a)} X_t \geq a \right)}$$

$$= \frac{\lambda e^{-\lambda r} e^{-\lambda a} dr}{e^{-\lambda a}} = \lambda e^{-\lambda r} dr = P \left( X_{T_1^c(a)} + a \in dr \right),$$

where $\lambda = \frac{2\mu}{1-e^{-2\mu a}}$. Therefore we get

$$P \left( y_{T_1^c(a)}^+ \in dr \mid y_{T_1^c(a)}^+ > 0 \right) \sim \text{Exp} \left( \frac{2\mu}{1-e^{-2\mu a}} \right), r > 0.$$
¿From equation (2.66), it follows that

\[(2.74)\]  
\[P\left(y^+_{T^c_1(a)} = 0\right) = P\left(T^c_1(a) < T^c_2(a)\right).\]

With \(T^c_1, T^c_2\) in place of \(T_1, T_2\) respectively in equations (2.36) and (2.37), we get

\[(2.75)\]  
\[E\left[T^c_1(a)\right] = E\left[T^c(a, a)\right] + E\left[T^c_2(a)\right] \cdot P\left(T^c_2(a) < T^c_1(a)\right),\]

\[(2.76)\]  
\[E\left[T^c_2(a)\right] = E\left[T^c(a, a)\right] + E\left[T^c_1(a)\right] \cdot P\left(T^c_1(a) < T^c_2(a)\right).\]

Using

\[P\left(y^+_{T^c_1(a)} < T^c_2(b)\right) = 1\]

and equations (2.75) and (2.76), we conclude that

\[(2.77)\]  
\[P\left(T^c_1(a) < T^c_2(b)\right) = \frac{E\left[T^c_2(a)\right]}{E\left[T^c_2(a)\right] + E\left[T^c_1(a)\right]}.\]

The result now follows by substituting (2.73), (2.74), (2.77) into equation (2.70) using Lemma 2.9. This completes the proof of Proposition 2.8.

**Proof of Theorem 2.7.** We will prove the theorem in the case that \(b \geq a\) since the proof is similar in the case \(a \geq b\). Suppose that \(b \geq a\).

From Lemma 2.10 and equation (2.66), it follows that on the event \(\{T^c_1(a) < T^c_2(b)\}\) we have

\[(2.78)\]  
\[y^+_{T^c_1(a)} = \begin{cases} 0 & \text{if } \max_{s \leq T^c_1(a)} y^+_s < a, \\ \max_{s \leq T^c_1(a)} y^+_s - a & \text{if } a \leq \max_{s \leq T^c_1(a)} y^+_s < b. \end{cases}\]

Therefore,

\[(2.79)\]  
\[P\left(T^c_1(a) < T^c_2(b)\right) = P\left(y^+_{T^c_1(a)} = 0\right) + \int_{0^+}^{b-a} P\left(y^+_{T^c_1(a)} \in dr\right),\]

and the result is obtained from Proposition 2.8. This completes the proof of the Theorem 2.7.

**Corollary 2.11** Let \(X_t = W_t\) be the evolution of the wealth of the gambler and let \(T^c, T^c_1\) and \(T^c_2\) be stopping times defined as above in a game of equal chances. We distinguish the following two cases:

1. \(b \geq a > 0\)

The probabilities of stopping at downward fall or upward rally are given by

\[(2.80)\]  
\[P\left(T^c(a, b) = T^c_1(a)\right) = \frac{1}{2} + \frac{1}{2} \cdot \left[1 - e^{-\frac{1}{a}(b-a)}\right],\]

\[(2.81)\]  
\[P\left(T^c(a, b) = T^c_2(b)\right) = \frac{1}{2} e^{-\frac{1}{a}(b-a)}.\]
2. $a \geq b > 0$

The probabilities of stopping at downward fall or upward rally are given by

\[
P(T_c(a, b) = T_1^c(a)) = \frac{1}{2} \cdot e^{-\frac{1}{b}(a-b)},
\]

\[
P(T_c(a, b) = T_2^c(b)) = \frac{1}{2} + \frac{1}{2} \cdot \left[1 - e^{-\frac{1}{b}(a-b)}\right].
\]

**Proof.** It is a simple consequence of Theorem 2.7 by taking the limit as $\mu \to 0$. \hfill \Box

**Corollary 2.12** Let $X_t = W_t$ be the evolution of the wealth of the gambler. The probability distribution function of the random variables $y_{T_1}^+$ and $y_{T_2}^-$ are given by

1.

\[
P(y_{T_1}^+(a) = 0) = \frac{1}{2},
\]

\[
P(y_{T_1}^+(a) \in dr) = \frac{1}{2} \cdot \left[\frac{1}{a} e^{-\frac{1}{a} r}\right] dr, \ r > 0.
\]

2.

\[
P(y_{T_2}^-(b) = 0) = \frac{1}{2},
\]

\[
P(y_{T_2}^-(b) \in dr) = \frac{1}{2} \cdot \left[\frac{1}{b} e^{-\frac{1}{b} r}\right] dr, \ r > 0.
\]

**Proof.** The above corollary is a consequence of Proposition 2.8 by letting $\mu \to 0$. \hfill \Box

3 Conclusion

In this paper we are able to compute explicitly the probabilities of winning or losing in a game of chance based on quitting the game after a significant upward rally or downward fall both in the continuous and in the discrete time framework. In doing so, we have also managed to compute the distributions of the random variables $Y_{T_1}^+$, $Y_{T_2}^-$ in discrete time with their continuous counterparts $y_{T_1}^+$ and $y_{T_2}^-$ respectively.

**Appendix: Review of the Gambler’s Ruin Problem in the Traditional Setup**

This section reviews the well known result of the gambler’s ruin problem. We distinguish between the discrete time and the continuous time framework.
The discrete time framework

Let $Z_i$, $i \in \mathcal{N}$ be a sequence of independent identically distributed random variables with the following distribution

$$Z_i = \begin{cases} 1, & \text{with probability } p, \\ -1, & \text{with probability } q, \end{cases}$$

where $p + q = 1$ and $p < q$. Each $Z_i$ represents a win or loss of the gambler on the $i$-th bet. The wealth (or cumulative winnings) of the gambler after $n$ bets is given by

$$S_n = \sum_{i=1}^n Z_i.$$

The gambler stops as soon as his or her wealth reaches some upper level $b$ or some lower level $-a$, where $a, b \in \mathcal{N}$. This event occurs at the stopping time

$$U(a, b) = \inf\{n \in \mathcal{N} : S_n = -a \text{ or } S_n = b\}.$$ 

Let us introduce the stopping times

$$U_1(a) = \inf\{n \in \mathcal{N} : S_n = -a\},$$

and

$$U_2(b) = \inf\{n \in \mathcal{N} : S_n = b\}.$$ 

In other words, $U_1(a)$ is the time when gambler’s wealth reaches the level $-a$, and $U_2(b)$ is the time at which his or her wealth reaches the level $b$. We are interested in computing the probabilities of the events $\{U(a, b) = U_1(a)\}$, i.e., exiting the game on a loss, and $\{U(a, b) = U_2(b)\}$, i.e., exiting the game on a win. We have the following result which determines these probabilities:

**Theorem 3.1** Let $S_n = \sum_{i=1}^n Z_i$ be the evolution of the wealth of the gambler and let $U(a, b)$, $U_1(a)$ and $U_2(b)$ be stopping times defined as above, with $a, b \in \mathcal{N}$. Then

\begin{align*}
(3.1) \quad P\left(U(a, b) = U_1(a)\right) &= P\left(U_1(a) < U_2(b)\right) = \frac{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^{a+b}}{1 - \left(\frac{q}{p}\right)^{a+b}}, \\
(3.2) \quad P\left(U(a, b) = U_2(b)\right) &= P\left(U_2(b) < U_1(a)\right) = \frac{1 - \left(\frac{q}{p}\right)^a}{1 - \left(\frac{q}{p}\right)^{a+b}}.
\end{align*}

**Proof.** The result is a simple consequence of the Optional Sampling Theorem applied to the discrete time martingale $M_n = \left(\frac{q}{p}\right)^{S_n}$. In particular,

$$1 = E\left[M_{U(a,b)}\right] = P\left(U(a, b) = U_1(a)\right) \cdot \left(\frac{q}{p}\right)^{-a} + P\left(U(a, b) = U_2(b)\right) \cdot \left(\frac{q}{p}\right)^{b}.$$ 

The fact that $P\left(U(a, b) = U_1(a)\right) + P\left(U(a, b) = U_2(b)\right) = 1$ concludes the proof. \qed
Corollary 3.2 For the case of equal odds \((p = q = \frac{1}{2})\), we can pass to the limit in the previously computed probabilities to conclude

\[
P(U(a, b) = U_1(a)) = P(U_1(a) < U_2(b)) = \frac{b}{a + b},
\]
and

\[
P(U(a, b) = U_2(b)) = P(U_2(b) < U_1(a)) = \frac{a}{a + b}.
\]

The continuous time framework

In the continuous time framework, the wealth \(X_t\) of the gambler follows a drifted Brownian motion

\[
X_t = W_t - \mu t,
\]
for \(\mu > 0\), where \(W_t\) is a standard Brownian motion. The analogous stopping times introduced above now become

\[
U^c(a, b) = \inf\{t \geq 0 : X_t = -a \text{ or } X_t = b\},
\]

\[
U_1^c(a) = \inf\{t \geq 0 : X_t = -a\},
\]
and

\[
U_2^c(b) = \inf\{t \geq 0 : X_t = b\},
\]
with \(a, b \in \mathbb{R}_+\). The following theorem determines probabilities of events \(\{U^c(a, b) = U_1^c(a)\}\) and \(\{U^c(a, b) = U_2^c(b)\}\).

Theorem 3.3 Let \(X_t = W_t - \mu t\) be the evolution of the wealth of the gambler and let \(U^c(a, b)\), \(U_1^c(a)\) and \(U_2^c(b)\) be the stopping times defined above, with \(a, b \in \mathbb{R}_+, \mu > 0\). Then

\[
P(U^c(a, b) = U_1^c(a)) = P(U_1^c(a) < U_2^c(b)) = \frac{e^{2\mu b} - 1}{e^{2\mu b} - e^{-2\mu a}}.
\]

and

\[
P(U^c(a, b) = U_2^c(b)) = P(U_2^c(b) < U_1^c(a)) = \frac{1 - e^{-2\mu a}}{e^{2\mu b} - e^{-2\mu a}}.
\]

Proof. Consider the martingale \(M_t = e^{2\mu X_t}\). Then, according to the Optional Sampling Theorem,

\[1 = E[M_{U^c(a,b)}] = P(U^c(a,b) = U_1^c(a)) \cdot e^{-2\mu a} + P(U^c(a,b) = U_2^c(b)) \cdot e^{2\mu b}.
\]

Since

\[P(U^c(a,b) = U_1^c(a)) + P(U^c(a,b) = U_2^c(b)) = 1,
\]
simple algebra concludes the proof. \(\diamond\)

Corollary 3.4 When \(\mu = 0\), we can take the limit as \(\mu \to 0\) in the previously computed probabilities to conclude

\[
P(U^c(a, b) = U_1^c(a)) = P(U_1^c(a) < U_2^c(b)) = \frac{b}{a + b},
\]

\[
P(U^c(a, b) = U_2^c(b)) = P(U_2^c(b) < U_1^c(a)) = \frac{a}{a + b}.
\]
References

