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Optimal Decision for the Squash Player

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Abstract. The player of squash has a choice between two options when his/her opponent changes the score from 8:7 to 8:8. The player to receive has to decide whether to play until 9th or until 10th point. We observe in this paper that the score of the squash game can be modelled as a Markov chain. Thus using standard techniques of Markov processes, we can derive the optimal strategy for the squash player.

Key words: Markov Chains, Optimal Control, Squash Game.

Mathematics Subject Classification: 60J20, 91A35.

1 Introduction

World Squash Singles Rules 2001 define squash game in the following way: "The game of Singles Squash is played between two players, each using a racquet, with a ball and in a court. Only the server scores points. The server, on winning a rally, scores a point; the receiver, on winning a rally, becomes the server. The player who scores nine points wins the game, except that on the score reaching eight-all for the first time, the receiver shall choose, before the next service, to continue that game either to nine points (known as "Set one") or to ten points (known as "Set two")."

When the score reaches 8:8 for the first time, the receiver has to choose between "Set one" and "Set two". The objective of this article is to determine the optimal strategy for this decision. This situation can be modelled as a Markov Chain. For the choice of "Set one", we may consider the state space to be: {Win, Loss, 8S:8, 8:8S}. We have to keep information about who serves, so 8S:8 means that the first player serves, and 8:8S means that the second player (the opponent) serves. Win means that the first player wins and Loss means that he/she loses the game. For the choice of "Set two", we have the following state space: {Win, Loss, 9S:9, 9S:8, 8S:8, 8S:9, 8:9S, 8:8S, 9:8S, 9:9S}. In both cases, states {Win, Loss} are recurrent, all other states are transient.

The decision is made at the point when the score is 8:8S, i.e., the second player has the serve, but the first player has an option to choose between "Set one" and "Set two" when this score is reached for the first time. From each possible score there is a certain probability that the first player would either win or lose the rally, thus changing the state of the Markov chain. For illustration, from the 8:8S state, the game can move to the 8S:8 state, which means that the first player won the rally and thus the serve without changing the score. If the first player loses the rally, the second player would make a point and keep the serve, changing the score to 8:9S. In the "Set one" option, the game would be over since it is played only until one of the players reaches the 9th point, resulting in the loss of the whole game for the first player. In the "Set two" option, the game goes on even from 8:9S state until one of the players reaches the 10th point.

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Eventually, the Markov chain would reach one of the two recurrent states, which represent the win or the loss of the first player. Thus our problem can be viewed as computing the probability of being absorbed in the win situation given that we start at the state 8:8S. When the probability of winning is bigger in "Set one", the player to receive should choose to play to the 9th point, otherwise he/she should choose to play to the 10th point.

2 General Results from Markov Chain Theory

General theory of Markov chains explains how to compute the probability that the chain ends up in a particular recurrent state under the condition that the chain started in state i. Good explanation of this problem can be found in any introductory book on Markov chains, such as Lawler [1].

Suppose that the Markov chain, which can be represented by transition matrix P, has some transient states and let Q be the submatrix of P which includes only the rows and columns for the transient states. Assume that there are at least two different recurrent classes and that the recurrent classes consist of single points $r_1, \dots, r_m$ with $p(r_j, r_j) = 1$.

If we order the states so that the recurrent states $r_1, \dots, r_m$ precede the transient states $t_1, \dots, t_s$, then we can write

$$P = \begin{pmatrix} I & 0 \\ S & Q \end{pmatrix}.$$

The matrix Q is a substochastic matrix, i.e., a matrix with nonnegative entries whose row sums are less than or equal to 1. Since the states represented by Q are transient, $Q^n \to 0$. This implies that all of the eigenvalues of Q have absolute values strictly less than 1. Therefore $I - Q$ is invertible matrix and we can define

$$M = (I - Q)^{-1}.$$

For $i = 1, \dots, s$, $j = 1, \dots, m$, let $\alpha(t_i, r_j)$ be the probability that the chain starting at t_i eventually ends up in recurrent state r_j. We have $\alpha(r_j, r_j) = 1$ and $\alpha(r_l, r_j) = 0$ for $l \neq j$. For any transient state t_i,

$$\begin{aligned} \alpha(t_i, r_j) &= \mathbb{P}(X_n = r_j \text{ eventually } | X_0 = t_i) \\ &= \mathbb{P}(X_1 = r_j | X_0 = t_i) + \sum_{x \in S} \mathbb{P}(X_1 = x | X_0 = t_i) \mathbb{P}(X_n = r_j \text{ eventually } | X_1 = x) \\ &= p(t_i, r_j) + \sum_{x \in S} p(t_i, x) \alpha(x, r_j). \end{aligned}$$

If A is the $s \times m$ matrix with entries $\alpha(t_i, r_j)$, then the above can be written in matrix form

$$A = S + Q \cdot A,$$

or

$$A = (I - Q)^{-1} \cdot S = M \cdot S.$$

Thus the probability of being absorbed at state r_j under the condition that the chain starts at state t_i is given by the $[i, j]$ entry of the matrix $A = M \cdot S$.

The matrix M also provides us information about expected number of visits to a transient state k before the time of absorbtion. Let Y_k denotes the total number of visits to k:

$$Y_k = \sum_{n=0}^{\infty} I(X_n = k).$$

Since k is transient, $Y_k < \infty$ with probability 1. Suppose $X_0 = i$, where i is another transient state. Then

$$\begin{aligned}
\mathbb{E}(Y_k|X_0 = i) &= \mathbb{E}\left(\sum_{n=0}^{\infty} I(X_n = k)|X_0 = i\right) \\
&= \sum_{n=0}^{\infty} \mathbb{P}(X_n = k|X_0 = i) \\
&= \sum_{n=0}^{\infty} p_n(i,k).
\end{aligned}$$

In other words, $\mathbb{E}(Y_k|X_0 = i)$ is the $[i,k]$ entry of the matrix $I + P + P^2 + \dots$, which is the same as the $[i,k]$ entry of the matrix $I + Q + Q^2 + \dots$. Using the fact that

$$(I + Q + Q^2 + \dots)(I - Q) = I,$$

we conclude that

$$I + Q + Q^2 + \dots = (I - Q)^{-1} = M.$$

This is equivalent to saying that $\mathbb{E}(Y_k|X_0 = i)$ is the $[i,k]$ entry of the matrix M. If we want to compute the expected number of steps until the chain enters a recurrent class, assuming $X_0 = i$, we need only sum $[i,k]$ entries of the matrix M over all transient states k.

3 Application to the Squash Game

Suppose that the first player has a probability $0 \le p \le 1$ of winning a single rally (regardless if he/she serves of if he/she receives). Transition matrix for the "Set one" case becomes:

$$P1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ p & 0 & 0 & 1-p \\ 0 & 1-p & p & 0 \end{pmatrix}$$

with states {Win, Loss, 8S:8, 8:8S}, "Set two" transition matrix is

$$P2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-p \\
p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-p & 0 \\
0 & 0 & 0 & p & 0 & 0 & 0 & 1-p & 0 & 0 \\
0 & 0 & p & 0 & 0 & 0 & 1-p & 0 & 0 & 0 \\
0 & 1-p & 0 & 0 & 0 & p & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & p & 0 & 1-p & 0 & 0 & 0 \\
0 & 0 & 0 & p & 0 & 0 & 0 & 0 & 0 & 1-p \\
0 & 1-p & p & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

with states {Win, Loss, 9S:9, 9S:8, 8S:8, 8S:9, 8:9S, 8:8S, 9:8S, 9:9S}.

3.1 Set One

For the case of "Set one" we have

$$S1 = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}, \quad Q1 = \begin{pmatrix} 0 & 1-p \\ p & 0 \end{pmatrix}.$$

Matrix $M1$ is given by

$$M1 = (I - Q1)^{-1} = \begin{pmatrix} \frac{1}{1-p+p^2} & \frac{1-p}{1-p+p^2} \\ \frac{p}{1-p+p^2} & \frac{1}{1-p+p^2} \end{pmatrix}.$$

Also,

$$M1 \cdot S1 = \begin{pmatrix} \frac{p}{1-p+p^2} & \frac{(1-p)^2}{1-p+p^2} \\ \frac{p^2}{1-p+p^2} & \frac{1-p}{1-p+p^2} \end{pmatrix}.$$

The probability $W1(p)$ of winning the game is given by the [2,1] entry of the matrix $M1 \cdot S1$:

$$W1(p) = (M1 \cdot S1)[2,1] = \frac{p^2}{1-p+p^2}.$$

Moreover, the expected number of balls played is given by the following expression:

$$B1(p) = M1[2,1] + M1[2,2] = \frac{1+p}{1-p+p^2}.$$

3.2 Set Two

For the "Set two" we have

$$S2 = \begin{pmatrix} p & 0 \\ p & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1-p \\ 0 & 0 \\ 0 & 0 \\ 0 & 1-p \end{pmatrix}, \quad Q2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-p \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-p & 0 \\ 0 & p & 0 & 0 & 0 & 1-p & 0 & 0 \\ p & 0 & 0 & 0 & 1-p & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 1-p & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 & 0 & 1-p \\ p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Matrix $M2$ is given by

$$M2 = (I - Q2)^{-1},$$

which gives a quite complicated expression. The exact form of this matrix can be computed by a suitable program which handles matrix algebra, such as *Mathematica 4.1.* The probability $W2(p)$ of winning the game is given by the [6,1] entry of the matrix $M2 \cdot S2$, which happens to be:

$$W2(p) = (M2 \cdot S2)[6,1] = \frac{p^3 \left(2 - p - p^2 + p^3\right)}{\left(1 - p + p^2\right)^3}.$$

The expected number of balls played is given by the sum of the 6th row elements of the matrix $M2$, which gives

$$B2(p) = \sum_{k=1}^{8} M2[6,k] = \frac{2 - 2\,p + 6\,p^2 - 5\,p^3 + p^4 + p^5}{\left(1 - p + p^2\right)^3}.$$

3.3 Optimal Decision

The optimal decision for the squash player is to choose "Set one" whenever $W1(p) > W2(p)$ and "Set two" when $W1(p) < W2(p)$. If we solve for $W1(p) = W2(p)$, or in other words for

$$\frac{p^2}{1-p+p^2} = \frac{p^3 \left(2 - p - p^2 + p^3\right)}{\left(1 - p + p^2\right)^3},$$

we get that this equation is satisfied for

$$p = \frac{3 - \sqrt{5}}{2}.$$

This equation has no other solution for $0 < p < 1$. It is optimal for the squash player to choose "Set one" if $p < \frac{3-\sqrt{5}}{2}$ and "Set two" otherwise.

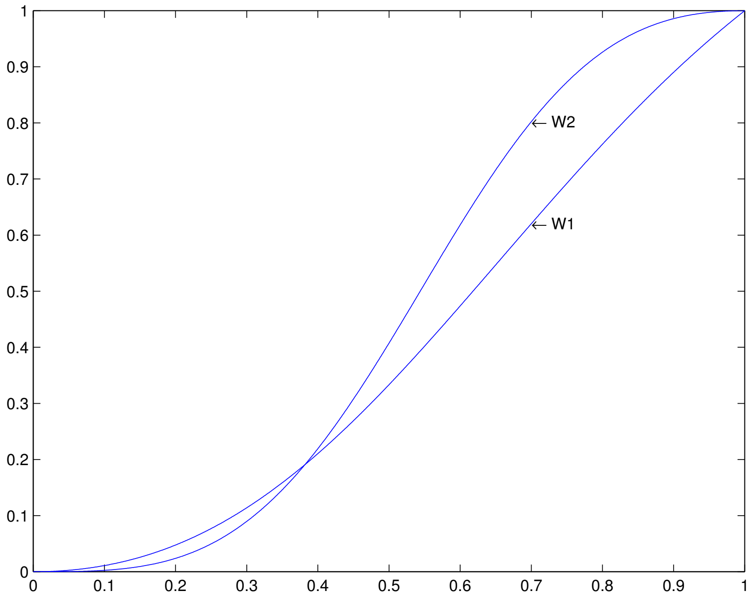

Figure 1: Probability of winning

We illustrate the probability of winning for "Set one" and "Set two" in Figure 1. Expected number of balls played in both cases is given in Figure 2.

References

[1] Lawler, G., *Introduction to Stochastic Processes,* Chapman & Hall, 1995.

[2] *World Squash Singles Rules 2001,* `http://www.worldsquash.org/rules.htm`

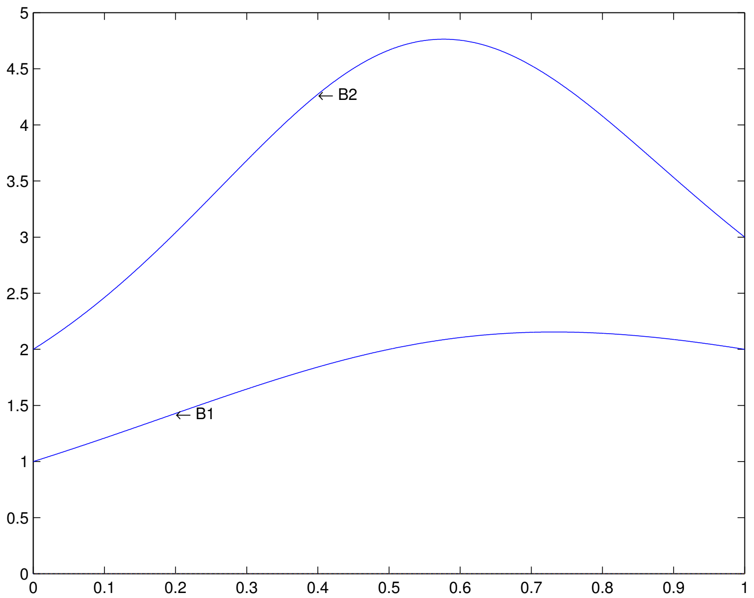

Figure 2: Expected number of balls played