Measuring Severity of the Market Crashes by Using Contracts on Maximum Relative Drawdown

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Abstract

Maximum Relative Drawdown measures the largest percentage drop of the price process on a given time interval. More recently, Maximum Relative Drawdown has become more popular as an alternative measure of risk. In contrast to the Value at Risk measure, it captures the path property of the price process. In this article, we propose a partial differential equation approach to determine the theoretical distribution of the Maximum Relative Drawdown. We also discuss the possibility of constructing an option contract that would insure the event that the Maximum Relative Drawdown exceeds a certain fixed percentage. We call these contracts *Crash Options.* We compute the theoretical prices and hedging strategies for the Crash Option. This gives us a new method how to measure the severity of the market crash by comparing the actual size of the market drop to the theoretical values of the crash option contract. Although the crash option is not currently traded contract, we can use our methods to find its market value by constructing the hedging strategy which replicates this option.

1 Introduction

The maximum relative drawdown D_T^δ of a stock price S_t is defined as the largest percentage drop of the asset price from its maximum on a given time interval $[T-\delta, T]$. We can write the maximum relative drawdown D_T^δ as

$$D_T^\delta = \sup_{T-\delta \le t \le T} \left(\frac{M_t - S_t}{M_t} \right), \tag{1}$$

where $M_t = \sup_{T-\delta \le s \le t} S_s$. Closely related is the concept of maximum (absolute) drawdown MDD_T^δ, defined as

$$MDD_T^\delta = \sup_{T-\delta \le t \le T} (M_t - S_t), \tag{2}$$

the largest absolute value drop on a given time interval $[T-\delta, T]$. In particular, notice that the maximum drawdown of the Brownian motion is the maximum relative drawdown of the geometric Brownian motion.

The drawdown has been extensively studied in recent literature. Portfolio optimization using constraints on the drawdown has been considered in Chekhlov, Uryasev and Zabarkin (2005). Harmantzis and Miao (2005) considered the impact of heavy tail returns on maximum drawdown risk measure. Analytical results linking the maximum drawdown to the mean return appeared in the paper of Magdon-Ismail and Atiya (2004). In a related paper, Magdon-Ismail et. al. (2004) determined the distribution of the maximum drawdown of Brownian motion.

Our paper extends the results obtained by Magdon-Ismail et. al. (2004), giving an alternative characterization of the distribution of the maximum drawdown by using the methods of partial differential equations.

The advantage of our method is that it could be used for more general dynamics of the underlying process. We illustrate this concept by giving an analytical characterization of the distribution of the maximum (absolute) drawdown of geometric Brownian motion.

The concept of the relative drawdown is depicted in Figures 1 and 2. Figure 1 is the S&P500 index for period 01/1970 – 12/2005. Figure 2 is the corresponding maximum relative drawdown for a 3 month window and 1 year window respectively. Notice that the plot of the maximum relative drawdown peaks during the periods of market crises and is low in stable periods. Thus it can serve as an excellent indicator of the market stability.

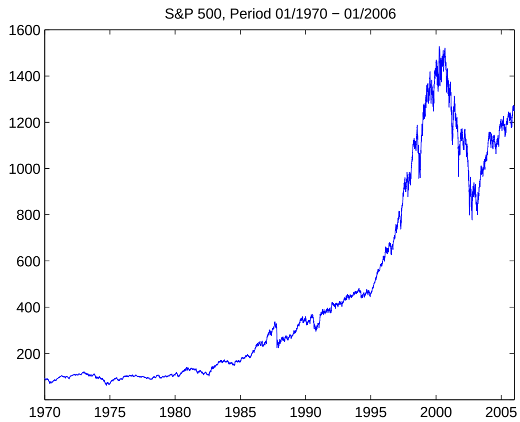

Figure 1: Index S&P 500 from 01/1970 to 12/2005.

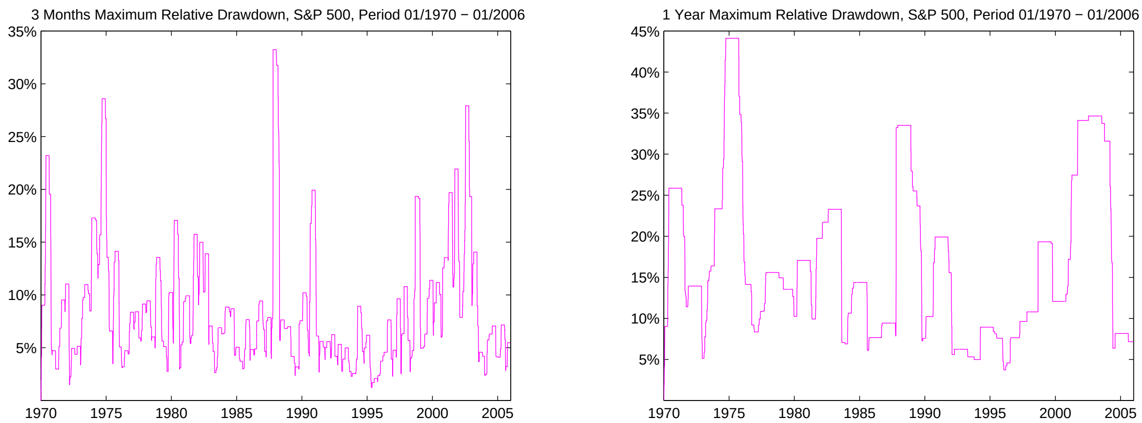

Figure 2: Maximum relative drawdown for δ=3 months (left) and δ=1 year (right) of S&P500 index, period from 01/1970 to 12/2005.

The graphs of the maximum relative drawdown also suggest the following definition of a market crash. The first time when the maximum relative drawdown $\left(1-\frac{S_t}{M_t}\right)$ exceeds the level x could be regarded as a market crash, denoted by T_x:

(3) $$T_x = \inf\left\{u \geq 0 \quad \left(1-\frac{S_u}{M_u}\right) \geq x\right\}$$

This time is directly observable by the market and thus can serve as the trigger point for contracts which insure the event of a market crash.

Previous literature on market crashes is mostly limited to empirical research as opposed to creating active trading strategies which could hedge out such events. An excellent review of the existing techniques for analysis and potential prediction of such events is given in Sornette (2004). Our research provides additional tools for managing the adverse market moves.

We introduce two contracts whose payoffs are linked to the time of the market crash T_x. We call them crash options. The first contract we consider is a crash option with a payoff of \$ 1 at the time when the maximum relative drawdown exceeds a certain percentage x. The second contract is triggered at the same instant, but the payoff resets the holder's account to the maximum of the asset price.

The introduction of crash option contracts is giving us a new method for measuring the severity of the market crashes. We can compare the realized values of the maximum relative drawdown to the prices of the corresponding crash options. Maximum relative drawdown is an increasing function of volatility, so it is natural to expect higher maximum relative drawdown in periods of high volatility. By computing the prices of crash options, we can directly take this volatility effect into consideration, thus getting the crash size estimate in terms of the replication costs only. In this perspective, the worst market crashes are the cheapest to replicate, meaning that the odds of this event were small to start with.

Contingent claims linked to the crash also introduce new tools for managing these adverse market movements. Existing contracts, such as deep out of the money puts, are weakly path dependent and thus have only limited predictive ability of the potential future drawdown. When the market is in a bubble, it is reasonable to expect that the prices of drawdown contracts would be significantly higher than when the market is stable, or when it exhibits mean reversion behavior. The prices of contracts linked to the maximum drawdown can serve as an indicator of the risk of future market crises.

The crash option described in this article is a novel concept, although some existing financial contracts have embedded features resembling the insurance of the market crash. For instance, equity default swaps are triggered by significant drops in the asset value. As for the pricing of EDS, see Albanese and Chen (2005). The list of other possible contracts which depend on the maximum (absolute) drawdown is given in Vecer (2006).

The paper is structured as follows. First we give the partial differential equation for the distribution of the maximum relative drawdown. We use a Brownian motion model in this paper for its simplicity, although it is possible to extend our techniques to more general settings (such as jumps, etc.). The next part of the paper introduces the crash options and studies their prices and the corresponding hedging strategies. The last section covers a new method of measuring the severity of the market crashes by comparing the realized values of the maximum relative drawdown to the theoretical prices of the values of the crash option for the given time period. We illustrate these concepts on S&P500 data.

2 Distribution of the Maximum Relative Drawdown

Our concern is to find the theoretical distribution of the random variable D_T^δ for a given T. Without the loss of generality we may assume that $\delta = T$, and find the distribution of the maximum relative drawdown on a time interval of length T. Let us denote

$$D_T = D_T^T.$$

Let us consider a geometric Brownian motion for the underlying dynamics of the asset price S_t

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{4}$$

The approach we are using is based on the heat equation with specific boundary and terminal conditions. The equation allows us to calculate the probability $\mathbb{P}(D_T \geq x)$ for a given value $x \in (0,1)$.

Theorem 2.1 *Let D_T be the maximum relative drawdown of a geometric Brownian motion. Then*

$$\mathbb{P}(D_T \leq x) = 1 - u(0,1),$$

where function $u(t,z)$ is a solution of the partial differential equation

$$u_t(t,z) + \mu z u_z(t,z) + \tfrac{1}{2}\sigma^2 z^2 u_{zz}(t,z) = 0, \tag{5}$$

defined in region $(0,T) \times (1-x, 1)$ with the boundary conditions

$$u(T,z) = 0 \text{ for } 1-x < z \leq 1, \tag{6}$$
$$u_z(t,1) = 0 \text{ for } t \in [0,T], \tag{7}$$
$$u(t,1-x) = 1 \text{ for } t \in [0,T]. \tag{8}$$

Proof of the theorem is based on the martingale techniques and is given in the appendix.

As an illustration, Figure 3 shows the distribution function of D_T which was obtained from solutions of the partial differential equation for the function u.

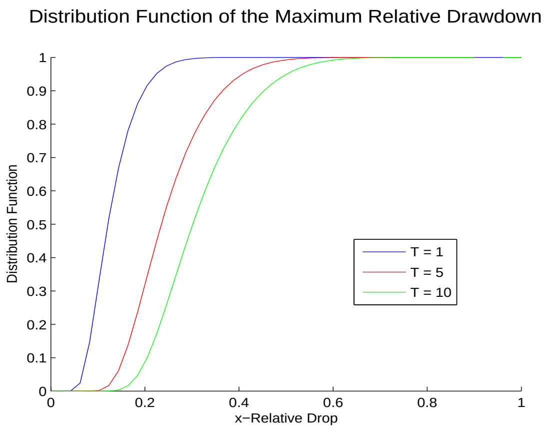

Figure 3: The distribution functions $P(D_T \leq x)$ for various horizons T, based on the numerical solutions of the partial differential equation u. The parameters are $\mu = 3\%$ and $\sigma = 12\%$. The plot shows that the distribution of D_T is positively skewed and $P(D_T \leq x)$ decreases as T goes up.

The same martingale techniques lead to the following characterization of the distribution of the maximum (absolute) drawdown of geometric Brownian motion. However, in this case the resulting partial differential equation is two dimensional in space and the reduction of the dimension is not possible.

Remark 2.2 *Let MDD_T be the maximum absolute drawdown of a geometric Brownian motion. Then*

$$\mathbb{P}(MDD_T \leq x) = 1 - v(0, S_0, S_0),$$

where function v is the solution of the following partial differential equation

$$v_t(t,s,m) + \mu s v_s(t,s,m) + \tfrac{1}{2}\sigma^2 s^2 v_{ss}(t,s,m) = 0, \tag{9}$$

satisfied on $(0,T) \times \{(s,m);\ s>0\ \&\ m-x<s\le m\}$ *with the boundary conditions*

$$
\begin{aligned}
v(T,s,m) &= 0 \text{ for } m-x<s\le m, \\
v_m(t,s,s) &= 0 \text{ for } t\in[0,T], \\
v(t,s,s+x) &= 1 \text{ for } t\in[0,T].
\end{aligned}
\tag{10}
$$

3 Options on the Maximum Relative Drawdown

A portfolio manager concerned with a control of the maximum drawdown might want to insure the event the maximum relative drawdown exceeds a certain threshold, either by entering the corresponding option contract, or by creating a hedge which would replicate the payoff. Let us consider two types of closely related contracts. The first pays off \$1 at the time when the maximum relative drawdown exceeds a certain percentage (crash option with digital payoff), the other one resets the options holder's account to the historical maximum at the time when the maximum relative drawdown exceeds a certain percentage (crash option resetting to the maximum value).

3.1 Crash Option with Digital Payoff

Let us consider a contract which pays off \$1 at the time when the relative drop of S_t from its maximum exceeds a value x. If the the relative drawdown stays below x until maturity T, the contract expires worthless. If the option is knocked in (i.e. $T_x \le T$), \$1 is paid to the holder at time T_x.

We define the value of this digital option by standard option pricing formula:

$$V_t = \mathbb{E}[e^{-r(T_x-t)} I_{\{D_T>x\}}|\mathcal{F}_t] = \mathbb{E}[e^{-r(T_x-t)} I_{\{T_x\le T\}}|\mathcal{F}_t], \tag{11}$$

where we assume the following risk neutral dynamics for the stock price:

$$dS_t = rS_t dt + \sigma S_t dW_t. \tag{12}$$

Similar to the approach used in the previous section, V_t can be expressed as a function of time and (S_t, M_t) on the set $\{t<T_x\}$: $V_t = v(t,S_t,M_t)$. Definition (11) implies that $e^{-rt}V_t$ is a $\mathcal{F}_t$-martingale. Using similar reasoning as in the previous chapter leads to the following partial differential equation:

$$
\begin{gathered}
v_t(t,s,m) + rsv_s(t,s,m) + \tfrac{1}{2}\sigma^2 s^2 v_{ss}(t,s,m) = rv(t,s,m), \\
\text{on } \ (0,T)\times\left\{(s,m);\ s>0\ \&\ s<m<\tfrac{s}{1-x}\right\}
\end{gathered}
\tag{13}
$$

$$
\begin{aligned}
v(T,s,m) &= 0 \text{ for } s\le m<\tfrac{s}{1-x}, \\
v_m(t,s,s) &= 0 \text{ for } t\in[0,T], \\
v\left(t,s,\tfrac{s}{1-x}\right) &= 1 \text{ for } t\in[0,T].
\end{aligned}
\tag{14}
$$

Again, we can define the function $u\left(t,\frac{s}{m}\right) = v(t,s,m)$ to obtain the following result

Theorem 3.1 *The value of the digital option on the maximum relative drawdown is given by*

$$V_t = u\left(t, \tfrac{S_t}{M_t}\right), \tag{15}$$

where u is the solution of the following partial differential equation:

$$u_t(t,z) + rzu_z(t,z) + \tfrac{1}{2}\sigma^2 z^2 u_{zz}(t,z) = ru(t,z), \tag{16}$$

satisfied on $(0,T)\times(1-x,1)$ *with the boundary conditions*

$$
\begin{array}{rcl}
u(T,z) & = & 0 \text{ for } 1-x<z\leq 1, \\
u_z(t,1) & = & 0 \text{ for } t\in[0,T], \\
u(t,1-x) & = & 1 \text{ for } t\in[0,T].
\end{array} \tag{17}
$$

The hedge $\Delta(t)$ *is given by*

$$\Delta(t)=v_s(t,S_t,M_t)=\tfrac{1}{M_t}u_z\left(t,\tfrac{S_t}{M_t}\right). \tag{18}$$

Figure 4 gives the price and the hedging strategy for the digital crash option as a function of time and the drop level. Figure 5 gives graphs of the price as a function of time to maturity and the drop level. Table 1 lists prices of the contract for selected drop levels and maturities.

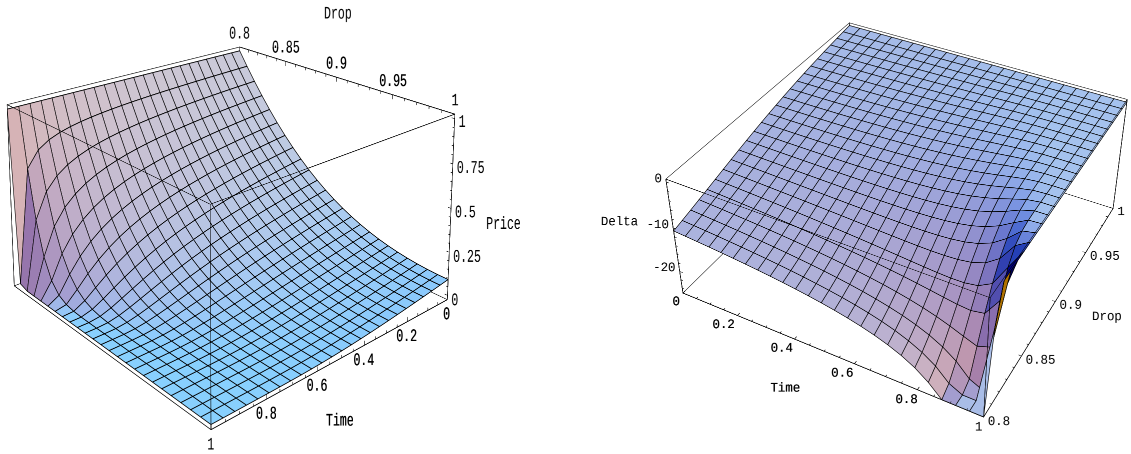

Figure 4: Left: The value of the digital crash option price $u(t,z)$ for $T=1$, $r=3\%$, $\sigma=12\%$ and $x=20\%$. Right: The hedge multiplied by the value of the maximum $mv_s(t,s,m)=u_z\left(t,\frac{s}{m}\right)$ as a function of time and the ratio $z=\frac{s}{m}$.

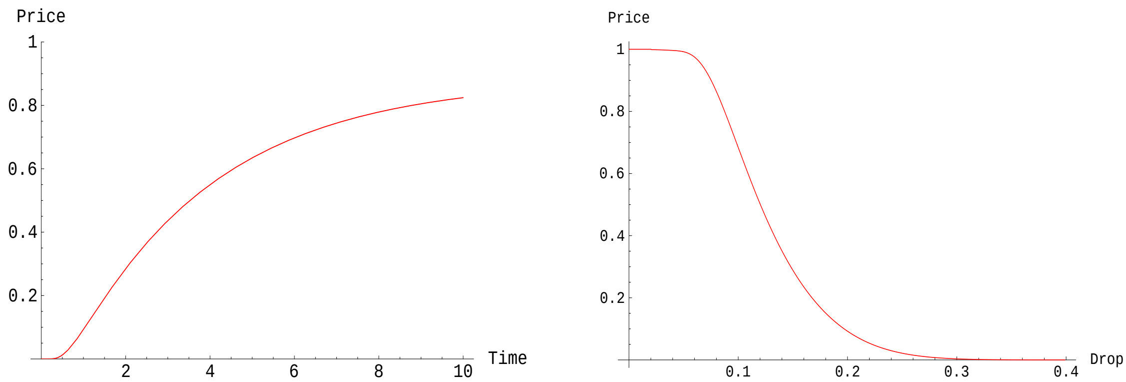

Figure 5: Left: The initial price of the digital crash option as a function of time to maturity T for $r=3\%$, $\sigma=12\%$ and $x=20\%$. Right: The initial price of the digital option as a function of the relative drop x.

Drop Percentage	Maturity						
	1M	**3M**	**6M**	**1Y**	**5Y**	**25Y**	∞
5%	.2641	.7399	.9423	.9921	.9942	.9942	.9942
10%	.0042	.1388	.3823	.6838	.9737	.9746	.9746
15%	.0000	.0108	.0891	.2887	.8720	.9377	.9377
20%	.0000	.0003	.0123	.0924	.6344	.8799	.8806
25%	.0000	.0000	.0009	.0216	.3958	.7901	.8022

Table 1: The price of the Digital Crash Option for selected drop levels and selected maturities using the parameters $r = 3\%$, $\sigma = 12\%$.

3.2 Crash Option Resetting to Maximum Value

Another possible contract to consider is a crash option which resets the holder's account to the historical maximum at the time when the maximum relative drawdown reaches the level x. Thus the payoff is given by xM_{T_x} at time T_x. For this contract, its value is given by

$$v(t,s,m) = x\mathbb{E}[e^{-r(T_x-t)}I(T_x < T)M_{T_x}|S_t = s, M_t = m].$$

We have the following partial differential equation

$$v_t(t,s,m) + rxv_s(t,s,m) + \tfrac{1}{2}\sigma^2 s^2 v_{ss}(t,s,m) = rv(t,s,m) \tag{19}$$

for the value function v, satisfied in the region $\{(t,s,m); 0 \le t < T, s \le m \le \frac{1}{1-x}s\}$, with boundary conditions

$$\begin{aligned} v(t,(1-x)m,m) &= xm, \quad 0 \le t \le T,\ m > 0, \\ v_m(t,m,m) &= 0, \quad 0 \le t \le T,\ m > 0, \\ v(T,s,m) &= 0, \quad s \le m < \tfrac{1}{1-x}s. \end{aligned}$$

Since the percentage value crash option satisfies the linear scaling property

$$v(t,\lambda s,\lambda m) = \lambda v(t,s,m),$$

we may reduce the dimensionality of the problem by introducing function u

$$u(t,z) = v(t,z,1), \quad 0 \le t \le T,\ 1-x \le z \le 1.$$

Then

$$v(t,s,m) = mu(t,\tfrac{s}{m}).$$

The following result easily follows:

Theorem 3.2 *The value of the crash option with the payoff xM_{T_x} at time T_x is given by*

$$V_t = M_t u(t,\tfrac{S_t}{M_t}), \tag{20}$$

where u satisfies the following partial differential equation

$$u_t(t,z) + rzu_z(t,z) + \tfrac{1}{2}\sigma^2 z^2 u_{zz}(t,z) = ru(t,z), \quad 0 \le t \le T,\ 1-x \le z \le 1, \tag{21}$$

with boundary conditions

$$\begin{aligned} u(T,z) &= 0, \quad 1-x < z \le 1, \\ u(t,1) &= u_z(t,1), \quad 0 \le t < T, \\ u(t,1-x) &= x, \quad 0 \le t \le T. \end{aligned}$$

The hedge is given by $\Delta(t)$

$$\Delta(t) = v_s(t,S_t,M_t) = u_z(t,\tfrac{S_t}{M_t}). \tag{22}$$

Figure 6 gives the price and the hedging strategy for the crash option which resets its holder's account to the maximum as a function of time and the drop level. Figure 7 gives graphs of the price as a function of time to maturity and the drop level. Table 2 lists prices of the contract for selected drop levels and maturities.

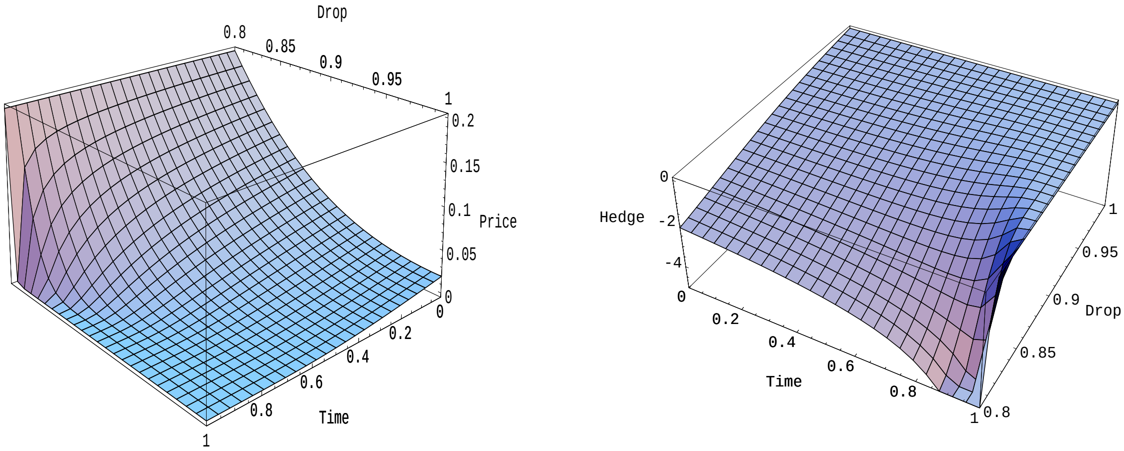

Figure 6: Left: The value of the crash option price $u(t,z)$ divided by the value of the maximum of the asset price M_t for $T=1$, $r=3\%$, $\sigma=12\%$ and $x=20\%$. Right: The hedge of the option given by $v_s(t,s,m) = u_z\left(t, \frac{s}{m}\right)$ as a function of time and the ratio $z=\frac{s}{m}$.

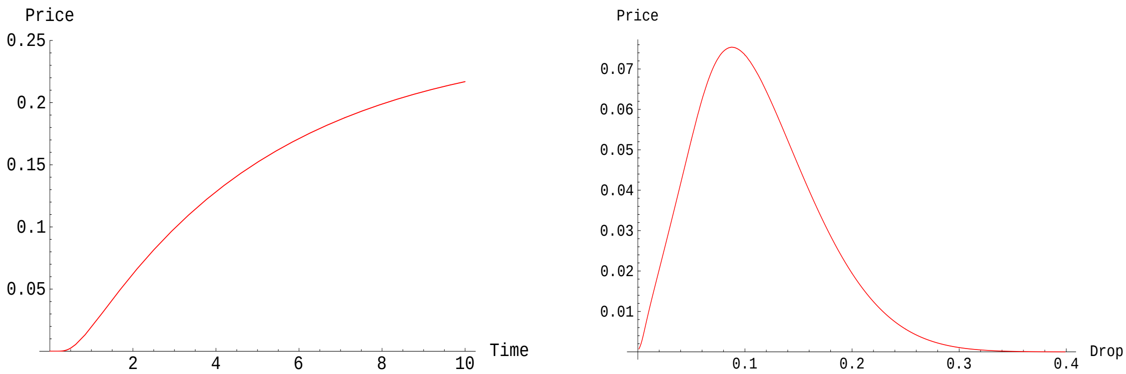

Figure 7: Left: The initial price of the crash option as a function of time to maturity T for $S_0=1$, $r=3\%$, $\sigma=12\%$ and $x=20\%$. Right: The initial price of the crash option as a function of the relative drop x.

4 Severity of the Market Crashes

By comparing the realized values of the maximum relative drawdown to the corresponding crash option values, we can measure the severity of the market crash. Let us consider the crash option with digital payoff. We can compute the barriers which correspond to crash options with initial values \$ 0.005 and \$ 0.995. This would give us 99% confidence interval for the maximum relative drawdown in terms of the replication costs of the crash option. The odds that the maximum relative drawdown exceeds the barrier corresponding to the crash option with initial value \$ 0.005 is 1 in 200. The odds that the maximum relative drawdown is below the barrier corresponding to the crash option with initial value \$ 0.995 is also 1 in 200. Thus the odds that the maximum relative drawdown stays within these two barriers is 99%.

Drop Percentage	Maturity						
	1M	**3M**	**6M**	**1Y**	**5Y**	**25Y**	∞
5%	1.34%	3.83%	4.94%	5.25%	5.26%	5.26%	5.26%
10%	0.04%	1.42%	3.99%	7.35%	11.07%	11.11%	11.11%
15%	0.00%	0.16%	1.38%	4.60%	15.65%	17.65%	17.65%
20%	0.00%	0.01%	0.25%	1.95%	15.21%	24.87%	25.00%
25%	0.00%	0.00%	0.02%	0.56%	11.69%	31.02%	33.33%

Table 2: The price of the Percentage Crash Option for selected drop levels and selected maturities using the parameters $r = 3\%$, $\sigma = 12\%$. The perpetual option has the price $\frac{x}{1-x}$. The percentage drop of any level happens in finite time, the payoff of the option being xM_{T_x}, which is a value exceeding xS_0.

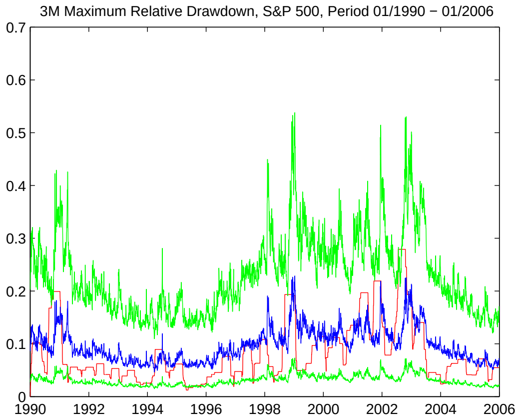

Figure 8: MDD (percentage, in red) taken over 3M interval, S&P500, 01/1990 to 12/2005, the expected MRDD (ex ante – using the values of VIX index, in blue), and the confidence intervals (in green). The upper green barrier corresponds to the crash option with initial value $ 0.005, the lower green barrier corresponds to the crash option with initial value $0.995.

We illustrate this concept on S&P500 index in years 1990 – 2005. The expected maximum relative drawdown is an increasing function of volatility. We use the values of VIX index to calculate the values of the expected maximum drawdown and we determine the barrier levels for crash options which cost $ 0.005 and $ 0.995 respectively. By using the volatility quoted by the VIX index, we obtain forward (ex ante) estimate of these values. Figure 8 shows the realized and the expected maximum relative drawdown, together with the barrier levels which give us 99% confidence interval.

Notice that there were 5 significant market drops during this period, namely in years 1991, 1998, two in 2001, and 2002. Four of them were outside of the 1 in 200 confidence interval, but the fifth crash (mid 2001) is safely within the confidence bounds. The reason is that this particular crash was preceded by a falling market with higher volatility, and thus a larger drop did not come as a surprise to the market. The four unexpected crashes were relatively cheap to replicate in terms of the crash option, the odds being less than 1 in 200.

Since the market parameters frequently change, we also conducted a study of the actual replications of these contracts using the realized values. We compute the expected maximum relative drawdown using the realized values of S&P500, and the corresponding barriers which could be obtained by adopting the hedging strategies described in the previous section. Figure 9 shows the values which could have been obtained by replication of these contracts using the S&P500 data. This situation is similar to the case when we used the VIX index as the input.

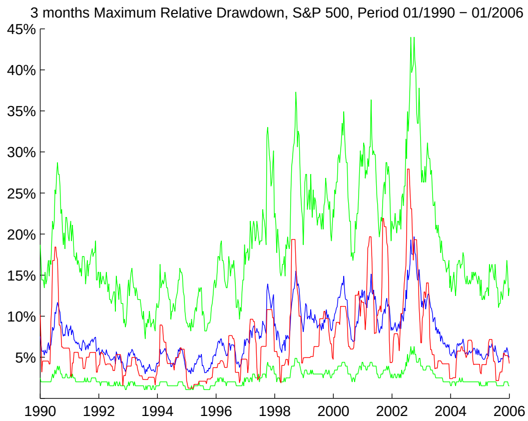

Figure 9: MDD (percentage, in red) taken over 3M interval, S&P500, 01/1990 to 12/2005, the expected MRDD (ex post – using the realized implied volatility, in blue) and the confidence intervals (green). The upper green barrier corresponds to the crash option with initial value \$ 0.005, the lower green barrier corresponds to the crash option with initial value \$0.995.

5 Appendix

Proof of Theorem 2.1: Fix a number $x \in (0,1)$ and express the distribution function of D_T as

$$\mathbb{P}(D_T \leq x) = 1 - \mathbb{P}(D_T > x) = 1 - \mathbb{E}I_{\{D_T > x\}}, \tag{23}$$

where the symbol I represents the indicator function.

The first time when the relative drop $\left(1 - \frac{S_t}{M_t}\right)$ reaches the level x is denoted by T_x:

$$T_x = \inf\left\{u \geq 0 \quad \left(1 - \frac{S_u}{M_u}\right) \geq x\right\}. \tag{24}$$

If the difference does not exceed x, we set T_x to be equal to infinity. The events $\{D_T > x\}$ and $\{T_x \leq T\}$ are identical. Therefore $\mathbb{P}(D_T \leq x) = 1 - \mathbb{E}I_{\{T_x \leq T\}}$.

Following Shreve (2004), section 7.4.2; let us define the process V_t

$$V_t = \mathbb{E}[I_{\{T_x \leq T\}} | \mathcal{F}_t]. \tag{25}$$

The following notation will be used: $T_{x,t} = \inf\left\{u \geq t \quad \left(1 - \frac{S_u}{M_u}\right) \geq x\right\}$. On the set $\{t < T_x\}$, it holds $I_{\{T_x \leq T\}} = I_{\{T_{x,t} \leq T\}}$. Hence, we can write:

$$V_t = \mathbb{E}[I_{\{T_{x,t} \leq T\}} | \mathcal{F}_t] \quad \text{on} \quad \{t < T_x\}. \tag{26}$$

Indicator $I_{\{T_{x,t} \leq T\}}$ depends on $(S_s,\ t \leq s \leq T)$ and $(M_s,\ t \leq s \leq T)$. By using the Markov property of (S_t, M_t), it is possible to express V_t as a function of time and (S_t, M_t):

$$V_t = v(t, S_t, M_t) \quad \text{on the set} \quad \{t < T_x\} \tag{27}$$

We can apply the Itô Formula to the process $v(t, S_t, M_t)$:

$$\begin{aligned} dv(t, S_t, M_t) &= v_t\ dt + v_s\ dS_t + v_m\ dM_t + \tfrac{1}{2} v_{ss}\ d\langle S_t \rangle \\ &= \left(v_t + \mu S_t v_s + \tfrac{1}{2}\sigma^2 S_t^2 v_{ss}\right) dt + v_m dM_t + \sigma S_t v_s dW_t. \end{aligned} \tag{28}$$

The definition of V_t implies that this process is a $\mathcal{F}_t$-martingale and $v(t, S_t, M_t)$ is a $\mathcal{F}_t$-martingale for $t < T_x$. According to this fact, the function $v(t, s, m)$ must satisfy the following partial differential equation:

$$v_t(t,s,m) + \mu s v_s(t,s,m) + \tfrac{1}{2}\sigma^2 s^2 v_{ss}(t,s,m) = 0, \tag{29}$$
$$\text{on } \ (0,T) \times \left\{(s,m);\ s > 0 \ \&\ s < m < \tfrac{s}{1-x}\right\},$$

$$\begin{array}{rcl} v(T,s,m) & = & 0 \text{ for } s \le m < \frac{s}{1-x}, \\ v_m(t,s,s) & = & 0 \text{ for } t \in [0,T], \\ v\left(t, s, \frac{s}{1-x}\right) & = & 1 \text{ for } t \in [0,T]. \end{array} \tag{30}$$

The terminal condition reflects the fact that if $\left(1 - \frac{S_t}{M_t}\right) < x$ for $t \in [0,T]$, then $I_{\{T_x \le T\}} = 0$ and $V_T = 0$. The first boundary condition ensures that $v_m dM_u = 0$. The other boundary condition corresponds to the following equality: $v(T_x, S_{T_x}, M_{T_x}) = V_{T_x} = 1$.

If v satisfies the above PDE, then $dv(t, S_t, M_t) = \sigma v_s dW_t$, and the process $v(t, S_t, M_t)$ is a $\mathcal{F}_t$-martingale.

Process V_t depends on values S_t and M_t only through the ratio $\frac{S_t}{M_t}$ (on the set $\{t < T_x\}$). Consequently, function v satisfies the equality below:

$$v(t,s,m) = v(t, \lambda s, \lambda m). \tag{31}$$

Such a property allows us to define a function $u(t,z)$ on $[0,T] \times [1-x, 1]$:

$$u\left(t, \frac{s}{m}\right) = v(t,s,m). \tag{32}$$

It holds that $v_t = u_t$, $v_s = \frac{1}{m}u_z$, $v_{ss} = \frac{1}{m^2}u_{zz}$, $v_m = -\frac{s}{m^2}u_z$.

Hence, the partial differential equation can be expressed in terms of u :

$$u_t(t,z) + \mu z u_z(t,z) + \tfrac{1}{2}\sigma^2 z^2 u_{zz}(t,z) = 0, \tag{33}$$
$$\text{on } \ (0,T) \times (1-x, 1),$$

$$\begin{array}{rcl} u(T,z) & = & 0 \text{ for } 1-x < z \le 1, \\ u_z(t,1) & = & 0 \text{ for } t \in [0,T], \\ u(t, 1-x) & = & 1 \text{ for } t \in [0,T]. \end{array} \tag{34}$$

Variable z stands for the ratio $\frac{s}{m}$.

Let u be the solution of the above equation. Then $v(0, S_0, S_0) = u(0,1)$. For the given $x \in (0,1)$, the following holds:

$$\mathbb{P}(D_T \le x) = 1 - \mathbb{E}I_{(T_x \le T)} = 1 - V_0 = 1 - v(0, S_0, S_0) = 1 - u(0,1). \tag{35}$$

◇

References

[1] Albanese, C., O. Chen (2005): "Pricing equity default swaps", *Risk*, Vol. 18, No. 6, 8387.

[2] Chekhlov, A., S. Uryasev, M. Zabarankin (2005): "Drawdown Measure in Portfolio Optimization," *International Journal of Theoretical and Applied Finance,* Vol. 8, No. 1, 13–58.

[3] Harmantzis, F., L., Miao (2005): "On the impact of heavy tails returns to popular risk measures: Evidence from Global Indices", working paper.

[4] Magdon-Ismail, M., A. Atiya (2004): “Maximum drawdown,” *Risk*, Vol. 17, No. 10, 99–102.

[5] Magdon-Ismail,M., A. Atiya, A. Pratap, Y. Abu-Mostafa (2004): “On the Maximum Drawdown of a Brownian Motion”, *Journal of Applied Probability*, Vol. 41, No. 1, 147-161.

[6] Shreve, S. (2004): *Stochastic Calculus for Finance II (Continuous-Time Models)*, *Springer*, New York.

[7] Sornette, D. (2004): *Why Stock Markets Crash*, Princeton Press.

[8] Vecer, J. (2006): “Maximum Drawdown and Directional Trading”, *Risk*, Vol. 19, No. 12.