How to Manage the Maximum Relative Drawdown

Jan Vecer, Petr Novotny, Libor Pospisil,
Columbia University, Department of Statistics, New York, NY 10027, USA

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Abstract

Maximum Relative Drawdown measures the largest percentage drop of the price process on a given time interval. Recently, Maximum Relative Drawdown has become more popular as an alternative measure of risk. In contrast to the Value at Risk measure, it captures the path property of the price process. In this article, we propose a partial differential equation approach to determine the theoretical distribution of the Maximum Relative Drawdown. We also discuss the possibility of constructing an option contract that would insure the event that the Maximum Relative Drawdown exceeds a certain fixed percentage. We call these contracts *Crash Options.* We compute the theoretical prices and hedging strategies for the Crash Option.

1 Introduction

The maximum relative drawdown D_T^δ of a stock price S_t is defined as the largest percentage drop of the asset price from its maximum on a given time interval $[T-\delta, T]$. We can write the maximum relative drawdown D_T^δ as

$$D_T^\delta = \sup_{T-\delta \le t \le T} \left(\frac{M_t - S_t}{M_t} \right), \tag{1}$$

where $M_t = \sup_{T-\delta \le s \le t} S_s$. Closely related is the concept of maximum (absolute) drawdown MDD_T^δ, defined as

$$MDD_T^\delta = \sup_{T-\delta \le t \le T} (M_t - S_t), \tag{2}$$

the largest absolute value drop on a given time interval $[T-\delta, T]$. In particular, notice that the maximum drawdown of the Brownian motion is the maximum relative drawdown of the geometric Brownian motion.

The drawdown has been extensively studied in recent literature. Portfolio optimization using constraints on the drawdown has been considered in Chekhlov, Uryasev and Zabarkin (2005). Harmantzis and Miao (2005) considered the impact of heavy tail returns on maximum drawdown risk measure. Analytical results linking the maximum drawdown to the mean return appeared in the paper of Magdon-Ismail and Atiya (2004). In a related paper, Magdon-Ismail et. al. (2004) determined the distribution of the maximum drawdown of Brownian motion.

Our paper extends the results obtained by Magdon-Ismail et. al. (2004), giving an alternative characterization of the distribution of the maximum drawdown by using the methods of partial differential equations. The advantage of our method is that it could be used for more general dynamics of the underlying process. We illustrate this concept by giving an analytical characterization of the distribution of the maximum (absolute) drawdown of geometric Brownian motion.

The concept of the relative drawdown is depicted in Figures 1 and 2. Figure 1 is the S&P500 index for period 01/1970 – 12/2005. Figure 2 is the corresponding maximum relative drawdown for a 3 month window and 1 year window respectively. Notice that the plot of the maximum relative drawdown peaks during the periods of market crises and is low in stable periods. Thus it can serve as an excellent indicator of the market stability.

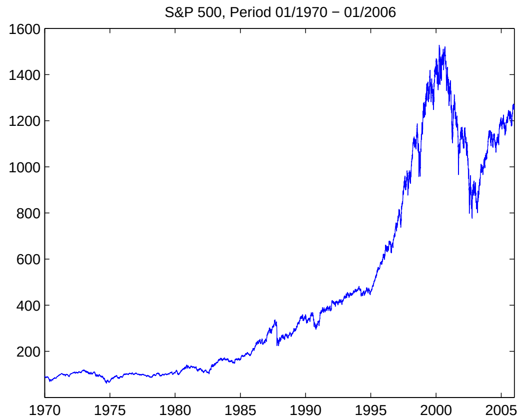

Figure 1: Index S&P 500 from 01/1970 to 12/2005.

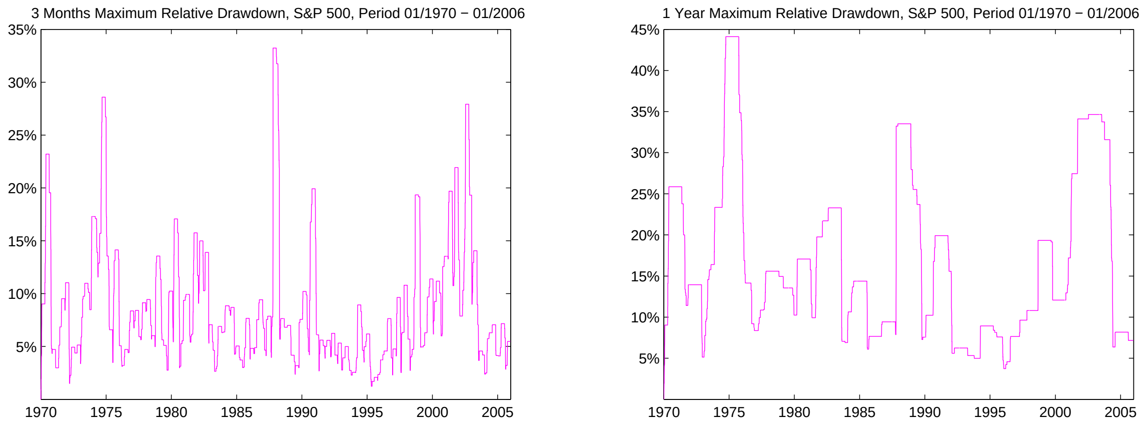

Figure 2: Maximum relative drawdown for δ=3 months (left) and δ=1 year (right) of S&P500 index, period from 01/1970 to 12/2005.

The graphs of the maximum relative drawdown also suggest the following definition of a market crash. The first time when the maximum relative drawdown $\left(1 - \frac{S_t}{M_t}\right)$ exceeds the level x could be regarded as a market crash, denoted by T_x:

$$T_x = \inf\left\{u \geq 0 \quad \left(1 - \frac{S_u}{M_u}\right) \geq x\right\} \tag{3}$$

This time is directly observable by the market and thus can serve as the trigger point for contracts which insure the event of a market crash.

Previous literature on market crashes is mostly limited to empirical research as opposed to creating active trading strategies which could hedge out such events. An excellent review of the existing techniques for analysis and potential prediction of such events is given in Sornette (2004). Our research provides additional tools for managing the adverse market moves.

We introduce two contracts whose payoffs are linked to the time of the market crash T_x. We call them crash options. The first contract we consider is a crash option with a payoff of \$ 1 at the time when the maximum relative drawdown exceeds a certain percentage x. The second contract is triggered at the same instant, but the payoff resets the holder's account to the maximum of the asset price.

The introduction of contingent claims linked to the crash would introduce new tools for managing these adverse market movements. Existing contracts, such as deep out of the money puts, are weakly path dependent and thus have only limited predictive ability of the potential future drawdown. When the market is in a bubble, it is reasonable to expect that the prices of drawdown contracts would be significantly higher then when the market is stable, or when it exhibits mean reversion behavior. The prices of contracts linked to the maximum drawdown can serve as an indicator of the risk of future market crises.

The crash option described in this article is a novel concept, although some existing financial contracts have embedded features resembling the insurance of the market crash. For instance, equity default swaps are triggered by significant drops in the asset value. As for the pricing of EDS, see Albanese and Chen (2005). The list of other possible contracts which depend on the maximum (absolute) drawdown is given in Vecer (2006).

The paper is structured as follows. First we give the partial differential equation for the distribution of the maximum relative drawdown. We use a Brownian motion model in this paper for its simplicity, although it is possible to extend our techniques to more general settings (such as jumps, etc.). The second part of the paper introduces the crash options and studies their prices and the corresponding hedging strategies. We use parameters $r = 0.03$ and $\sigma = 0.12$. We used volatility implied from one year options on S&P500 from the middle of 2005. It might seem to be a low estimate, however the realized payoffs of crash options tend to be lower than their theoretical prices, perhaps due to the presence of certain mean reversion in the market. Other factors, such as stochastic volatility or jumps are dominated by the presence (or absence) of the mean reversion.

2 Distribution of the Maximum Relative Drawdown

Our concern is to find the theoretical distribution of the random variable D_T^δ for a given T. Without the loss of generality we may assume that $\delta = T$, and find the distribution of the maximum relative drawdown on a time interval of length T. Let us denote

$$D_T = D_T^T.$$

Let us consider a geometric Brownian motion for the underlying dynamics of the asset price S_t

$$dS_t = rS_t dt + \sigma S_t dW_t. \tag{4}$$

The approach we are using is based on the heat equation with specific boundary and terminal conditions. The equation allows us to calculate the probability $\mathbb{P}(D_T \geq x)$ for a given value $x \in (0,1)$.

Theorem 2.1 *Let D_T be the maximum relative drawdown of a geometric Brownian motion. Then*

$$\mathbb{P}(D_T \leq x) = 1 - u(0,1),$$

where function $u(t,z)$ is a solution of the partial differential equation

$$u_t(t,z) + rzu_z(t,z) + \tfrac{1}{2}\sigma^2 z^2 u_{zz}(t,z) = 0 \tag{5}$$

defined in region $(0,T)\times(1-x,1)$ *with the boundary conditions*

$$u(T,z) = 0 \text{ for } 1-x<z\le 1 \quad (6)$$
$$u_z(t,1) = 0 \text{ for } t\in[0,T] \quad (7)$$
$$u(t,1-x) = 1 \text{ for } t\in[0,T]. \quad (8)$$

Proof of the theorem is based on the martingale techniques and is given in the appendix.

As an illustration, Figure 3 shows the distribution function of D_T which was obtained from solutions of the partial differential equation for the function u.

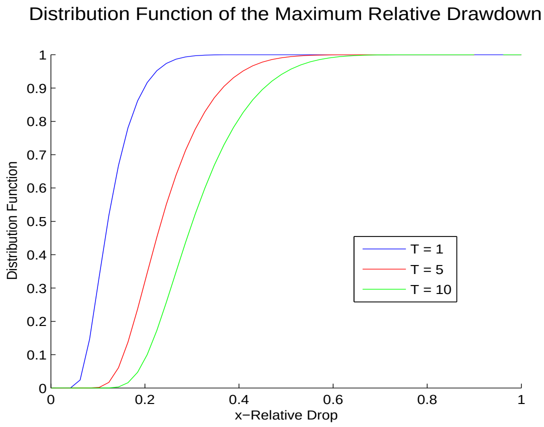

Figure 3: The distribution functions $P(D_T\le x)$ for various horizons T, based on the numerical solutions of the partial differential equation u. The parameters are $r=3\%$ and $\sigma=12\%$. The plot shows that the distribution of D_T is positively skewed and $P(D_T\le x)$ decreases as T goes up.

The same martingale techniques lead to the following characterization of the distribution of the maximum (absolute) drawdown of geometric Brownian motion. However, in this case the resulting partial differential equation is two dimensional in space and the reduction of the dimension is not possible.

Remark 2.2 *Let* MDD_T *be the maximum relative drawdown of a geometric Brownian motion. Then*

$$\mathbb{P}(MDD_T\le x)=1-v(0,S_0,S_0),$$

where function v *is the solution of the following partial differential equation*

$$v_t(t,s,m)+rsv_s(t,s,m)+\tfrac{1}{2}\sigma^2s^2v_{ss}(t,s,m)=0 \quad (9)$$

satisfied on $(0,T)\times\{(s,m);\ s>0\ \&\ m-x<s\le m\}$

$$\begin{aligned} v(T,s,m) &= 0 \text{ for } m-x<s\le m\\ v_m(t,s,s) &= 0 \text{ for } t\in[0,T]\\ v(t,s,s+x) &= 1 \text{ for } t\in[0,T]. \end{aligned} \quad (10)$$

3 Options on the Maximum Relative Drawdown

A portfolio manager concerned with a control of the maximum drawdown might want to insure the event the maximum relative drawdown exceeds a certain threshold, either by entering the corresponding option contract, or by creating a hedge which would replicate the payoff. Let us consider two types of closely related contracts. The first pays off \$1 at the time when the maximum relative drawdown exceeds a certain percentage (crash option with digital payoff), the other one resets the options holder's account to the historical maximum at the time when the maximum relative drawdown exceeds a certain percentage (crash option resetting to the maximum value).

3.1 Crash Option with Digital Payoff

Let us consider a contract which pays off \$1 at the time when the relative drop of S_t from its maximum exceeds a value x. If the the relative drawdown stays below x until maturity T, the contract expires worthless. If the option is knocked in (i.e. $T_x \leq T$), \$1 is paid to the holder at time T_x.

We define the value of this digital option by the standard pricing formula:

$$V_t = \mathbb{E}[e^{-r(T_x-t)} I_{\{D_T>x\}} | \mathcal{F}_t] = \mathbb{E}[e^{-r(T_x-t)} I_{\{T_x \leq T\}} | \mathcal{F}_t] \tag{11}$$

Similar to the approach used in the previous section, V_t can be expressed as a function of time and (S_t, M_t) on the set $\{t < T_x\}$: $V_t = v(t, S_t, M_t)$. Definition (11) implies that $e^{-rt}V_t$ is a $\mathcal{F}_t$-martingale. Using similar reasoning as in the previous chapter leads to the following partial differential equation:

$$\begin{gathered} v_t(t,s,m) + rsv_s(t,s,m) + \tfrac{1}{2}\sigma^2 s^2 v_{ss}(t,s,m) = rv(t,s,m) \\ \text{on} \quad (0,T) \times \left\{(s,m);\ s>0\ \&\ s<m<\tfrac{s}{1-x}\right\} \end{gathered} \tag{12}$$

$$\begin{aligned} v(T,s,m) &= 0 \text{ for } s \leq m < \tfrac{s}{1-x} \\ v_m(t,s,s) &= 0 \text{ for } t \in [0,T] \\ v\left(t, s, \tfrac{s}{1-x}\right) &= 1 \text{ for } t \in [0,T] \end{aligned} \tag{13}$$

Again, we can define the function $u\left(t, \frac{s}{m}\right) = v(t,s,m)$ to obtain the following result

Theorem 3.1 *The value of the digital option on the maximum relative drawdown is given by*

$$V_t = u\left(t, \frac{S_t}{M_t}\right), \tag{14}$$

where u is the solution of the following partial differential equation:

$$u_t(t,z) + rzu_z(t,z) + \tfrac{1}{2}\sigma^2 z^2 u_{zz}(t,z) = ru(t,z) \tag{15}$$

satisfied on $(0,T) \times (1-x, 1)$ with boundary conditions

$$\begin{aligned} u(T,z) &= 0 \text{ for } 1-x < z \leq 1 \\ u_z(t,1) &= 0 \text{ for } t \in [0,T] \\ u(t,1-x) &= 1 \text{ for } t \in [0,T] \end{aligned} \tag{16}$$

The hedge $\Delta(t)$ is given by

$$\Delta(t) = v_s(t, S_t, M_t) = \frac{1}{M_t} u_z\left(t, \frac{S_t}{M_t}\right). \tag{17}$$

Figure 4 gives the price and the hedging strategy for the digital crash option as a function of time and the drop level. Figure 5 gives graphs of the price as a function of time to maturity and the drop level. Table 1 lists prices of the contract for selected drop levels and maturities.

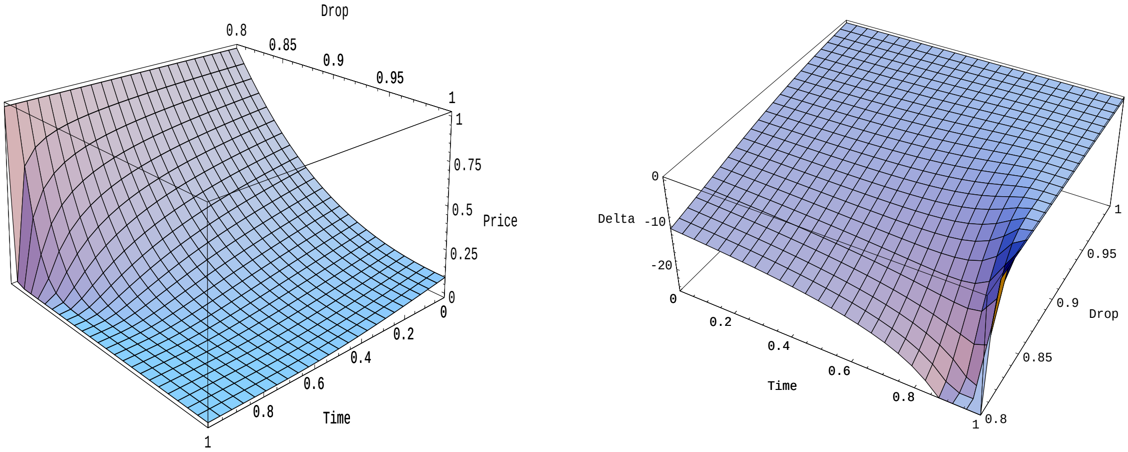

Figure 4: Left: The value of the digital crash option price $u(t,z)$ for $T = 1$, $r = 3\%$, $\sigma = 12\%$ and $x = 20\%$. Right: The hedge multiplied by the value of the maximum $mv_s(t,s,m) = u_z \;\; t, \frac{s}{m}$ as a function of time and the ratio $z = \frac{s}{m}$.

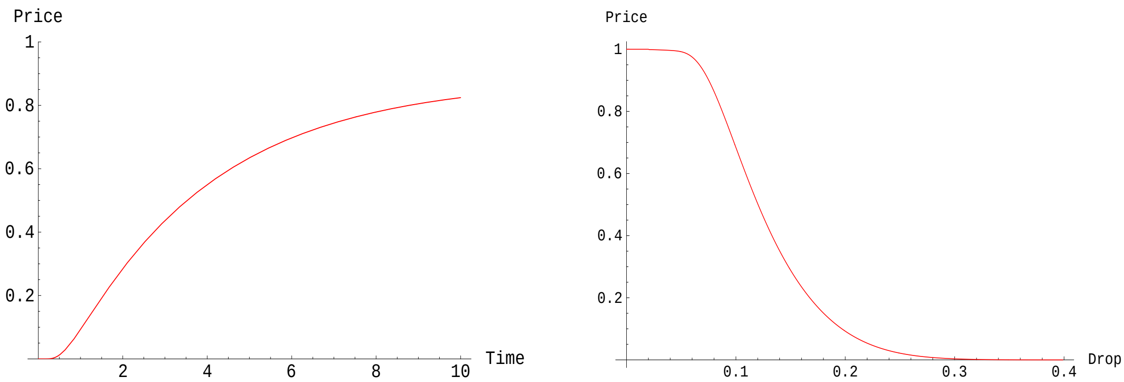

Figure 5: Left: The initial price of the digital crash option as a function of time to maturity T for $r = 3\%$, $\sigma = 12\%$ and $x = 20\%$. Right: The initial price of the digital option as a function of the relative drop x.

3.2 Crash Option Resetting to Maximum Value

Another possible contract to consider is a crash option which resets the holder's account to the historical maximum at the time when the maximum relative drawdown reaches the level x. Thus the payoff is given by xM_{T_x} at time T_x. For this contract, its value is given by

$$v(t,s,m) = x\mathbb{E}[e^{-r(T_x - t)} I(T_x < T) M_{T_x} | S_t = s, M_t = m],$$

We have the following partial differential equation

$$v_t(t,s,m) + rxv_s(t,s,m) + \tfrac{1}{2}\sigma^2 s^2 v_{ss}(t,s,m) = rv(t,s,m) \tag{18}$$

for the value function v, satisfied in the region $\{(t,s,m); 0 \le t < T, s \le m \le \frac{1}{1-x}s\}$, with boundary conditions

$$\begin{aligned} v(t,(1-x)m,m) &= xm, \quad 0 \le t \le T,\ m > 0, \\ v_m(t,m,m) &= 0, \quad 0 \le t \le T,\ m > 0, \\ v(T,s,m) &= 0, \quad s \le m < \tfrac{1}{1-x}s. \end{aligned}$$

Since the percentage value crash option satisfies the linear scaling property

$$v(t,\lambda s,\lambda m) = \lambda v(t,s,m),$$

Drop Percentage	Maturity						
	1M	**3M**	**6M**	**1Y**	**5Y**	**25Y**	∞
5%	.2641	.7399	.9423	.9921	.9942	.9942	.9942
10%	.0042	.1388	.3823	.6838	.9737	.9746	.9746
15%	.0000	.0108	.0891	.2887	.8720	.9377	.9377
20%	.0000	.0003	.0123	.0924	.6344	.8799	.8806
25%	.0000	.0000	.0009	.0216	.3958	.7901	.8022

Table 1: The price of the Digital Crash Option for selected drop levels and selected maturities using the parameters $r = 3\%$, $\sigma = 12\%$.

we may reduce the dimensionality of the problem by introducing function u

$$u(t,z) = v(t,z,1), \quad 0 \le t \le T, \ 1-x \le z \le 1.$$

Then

$$v(t,s,m) = mu(t, \tfrac{s}{m}).$$

The following result easily follows:

Theorem 3.2 *The value of the crash option with the payoff xM_{T_x} at time T_x is given by*

$$V_t = M_t u(t, \tfrac{S_t}{M_t}), \tag{19}$$

where u satisfies the following partial differential equation

$$u_t(t,z) + rzu_z(t,z) + \tfrac{1}{2}\sigma^2 z^2 u_{zz}(t,z) = ru(t,z), \quad 0 \le t \le T, \ 1-x \le z \le 1, \tag{20}$$

with boundary conditions

$$\begin{aligned} u(T,z) &= 0, \quad 1-x < z \le 1, \\ u(t,1) &= u_z(t,1), \quad 0 \le t < T, \\ u(t,1-x) &= x, \quad 0 \le t \le T. \end{aligned}$$

The hedge is given by $\Delta(t)$

$$\Delta(t) = v_s(t,S_t,M_t) = u_z(t, \tfrac{S_t}{M_t}). \tag{21}$$

Figure 6 gives the price and the hedging strategy for the crash option which resets its holder's account to the maximum as a function of time and the drop level. Figure 7 gives graphs of the price as a function of time to maturity and the drop level. Table 2 lists prices of the contract for selected drop levels and maturities.

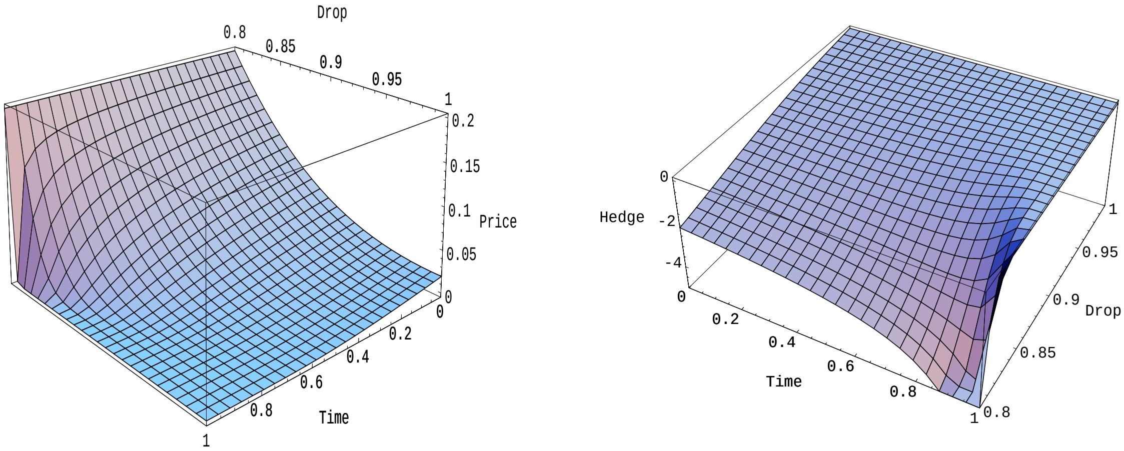

Figure 6: Left: The value of the crash option price $u\,(t, z)$ divided by the value of the maximum of the asset price M_t for $T = 1$, $r = 3\%$, $\sigma = 12\%$ and $x = 20\%$. Right: The hedge of the option given by $v_s(t, s, m) = u_z \;\; t, \frac{s}{m} \;\;$ as a function of time and the ratio $z = \frac{s}{m}$.

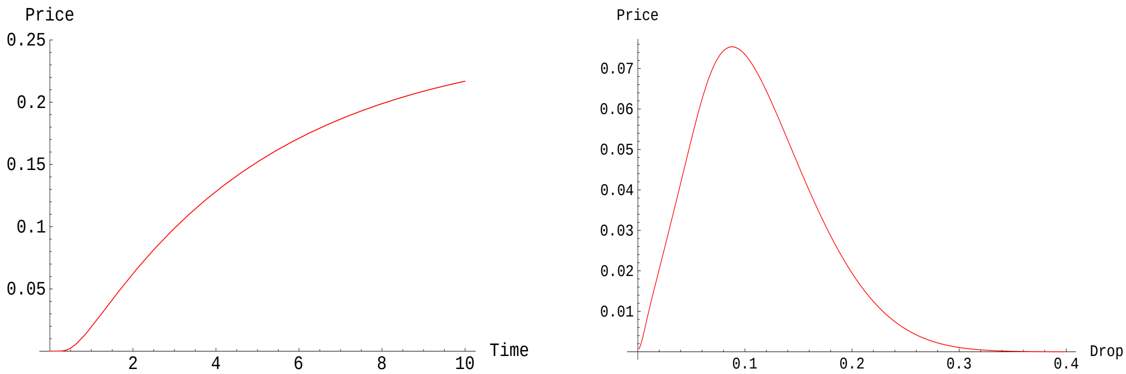

Figure 7: Left: The initial price of the crash option as a function of time to maturity T for $S_0 = 1$, $r = 3\%$, $\sigma = 12\%$ and $x = 20\%$. Right: The initial price of the crash option as a function of the relative drop x.

4 Appendix

Proof of Theorem 2.1: Fix a number $x \in (0, 1)$ and express the distribution function of D_T as

$$\mathbb{P}(D_T \leq x) = 1 - \mathbb{P}(D_T > x) = 1 - \mathbb{E}I_{\{D_T > x\}}, \tag{22}$$

where the symbol I represents the indicator function.

The first time when the relative drop $\left(1 - \frac{S_t}{M_t}\right)$ reaches the level x is denoted by T_x:

$$T_x = \inf\left\{u \geq 0 \quad \left(1 - \frac{S_u}{M_u}\right) \geq x\right\} \tag{23}$$

If the difference does not exceed x, we set T_x to be equal to infinity. The events $\{D_T > x\}$ and $\{T_x \leq T\}$ are identical. Therefore $\mathbb{P}(D_T \leq x) = 1 - \mathbb{E}I_{\{T_x \leq T\}}$.

Let us define the process V_t

$$V_t = \mathbb{E}[I_{\{T_x \leq T\}} | \mathcal{F}_t] \tag{24}$$

Drop Percentage	Maturity						
	1M	**3M**	**6M**	**1Y**	**5Y**	**25Y**	∞
5%	1.34%	3.83%	4.94%	5.25%	5.26%	5.26%	5.26%
10%	0.04%	1.42%	3.99%	7.35%	11.07%	11.11%	11.11%
15%	0.00%	0.16%	1.38%	4.60%	15.65%	17.65%	17.65%
20%	0.00%	0.01%	0.25%	1.95%	15.21%	24.87%	25.00%
25%	0.00%	0.00%	0.02%	0.56%	11.69%	31.02%	33.33%

Table 2: The price of the Percentage Crash Option for selected drop levels and selected maturities. The price of the option is given as a percentage of the initial asset price using the parameters $r = 3\%$, $\sigma = 12\%$. The perpetual option has the price $\frac{x}{1-x}$. The percentage drop of any level happens in finite time, the payoff of the option being xM_{T_x}, which is a value exceeding xS_0.

The following notation will be used: $T_{x,t} = \inf\left\{u \geq t \quad \left(1 - \frac{S_u}{M_u}\right) \geq x\right\}$. On the set $\{t < T_x\}$, it holds $I_{\{T_x \leq T\}} = I_{\{T_{x,t} \leq T\}}$. Hence, we can write:

$$V_t = \mathbb{E}[I_{\{T_{x,t} \leq T\}} | \mathcal{F}_t] \quad \text{on} \quad \{t < T_x\} \tag{25}$$

Indicator $I_{\{T_{x,t} \leq T\}}$ depends on $(S_s,\ t \leq s \leq T)$ and $(M_s,\ t \leq s \leq T)$. By using the Markov property of (S_t, M_t), it is possible to express V_t as a function of time and (S_t, M_t):

$$V_t = v(t, S_t, M_t) \quad \text{on the set} \quad \{t < T_x\} \tag{26}$$

We can apply the Itô Formula to the process $v(t, S_t, M_t)$:

$$\begin{aligned} dv(t, S_t, M_t) &= v_t\ dt + v_s\ dS_t + v_m\ dM_t + \tfrac{1}{2} v_{ss}\ d\langle S_t \rangle \\ &= \left(v_t + rS_t v_s + \tfrac{1}{2}\sigma^2 S_t^2 v_{ss}\right) dt + v_m dM_t + \sigma S_t v_s dW_t. \end{aligned} \tag{27}$$

The definition of V_t implies that this process is a $\mathcal{F}_t$-martingale and $v(t, S_t, M_t)$ is a $\mathcal{F}_t$-martingale for $t < T_x$. According to this fact, the function $v(t, s, m)$ must satisfy the following partial differential equation:

$$\begin{gathered} v_t(t, s, m) + rsv_s(t, s, m) + \tfrac{1}{2}\sigma^2 s^2 v_{ss}(t, s, m) = 0 \\ \text{on} \quad (0, T) \times \left\{(s, m);\ s > 0\ \&\ s < m < \tfrac{s}{1-x}\right\} \end{gathered} \tag{28}$$

$$\begin{aligned} v(T, s, m) &= 0 \text{ for } s \leq m < \tfrac{s}{1-x} \\ v_m(t, s, s) &= 0 \text{ for } t \in [0, T] \\ v\left(t, s, \tfrac{s}{1-x}\right) &= 1 \text{ for } t \in [0, T] \end{aligned} \tag{29}$$

The terminal condition reflects the fact that if $\left(1 - \frac{S_t}{M_t}\right) < x$ for $t \in [0, T]$, then $I_{\{T_x \leq T\}} = 0$ and $V_T = 0$. The first boundary condition ensures that $v_m dM_u = 0$ (see Shreve (2004), section 7.4.2, for details). The other boundary condition corresponds to the following equality: $v(T_x, S_{T_x}, M_{T_x}) = V_{T_x} = 1$.

If v satisfies the above PDE, then $dv(t, S_t, M_t) = \sigma v_s dW_t$, and the process $v(t, S_t, M_t)$ is a $\mathcal{F}_t$-martingale.

Process V_t depends on values S_t and M_t only through the ratio $\frac{S_t}{M_t}$ (on the set $\{t < T_x\}$). Consequently, function v satisfies the equality below:

$$v(t, s, m) = v(t, \lambda s, \lambda m). \tag{30}$$

Such a property allows us to define a function $u(t, z)$ on $[0, T] \times [1 - x, 1]$:

$$u\left(t, \frac{s}{m}\right) = v(t, s, m) \tag{31}$$

It holds that $v_t = u_t$, $v_s = \frac{1}{m}u_z$, $v_{ss} = \frac{1}{m^2}u_{zz}$, $v_m = -\frac{s}{m^2}u_z$.

Hence, the partial differential equation can be expressed in terms of u :

$$u_t(t,z) + rzu_z(t,z) + \tfrac{1}{2}\sigma^2 z^2 u_{zz}(t,z) = 0 \quad \text{on } (0,T)\times(1-x,1) \tag{32}$$

$$\begin{aligned} u(T,z) &= 0 \text{ for } 1-x < z \le 1 \\ u_z(t,1) &= 0 \text{ for } t \in [0,T] \\ u(t,1-x) &= 1 \text{ for } t \in [0,T] \end{aligned} \tag{33}$$

Variable z stands for the ratio $\frac{s}{m}$.

Let u be the solution of the above equation. Then $v(0,S_0,S_0) = u(0,1)$. For the given $x \in (0,1)$, the following holds:

$$\mathbb{P}(D_T \le x) = 1 - \mathbb{E}I_{(T_x \le T)} = 1 - V_0 = 1 - v(0,S_0,S_0) = 1 - u(0,1). \tag{34}$$

$\diamond$

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