The Cost of Negative Returns

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Abstract

We study the impact of negative returns on the health of a given financial portfolio. It is often the case that a series of significant negative returns trigger a credit event such as a downgrade in rating, or even a default of the portfolio owner. We focus our attention on a Weighted Average of Ordered Returns, which is a statistic that allows us to weight returns according to their relative adverse impact. We use an option pricing approach to derive the theoretical price and properties of a forward, a swap contract, a call and a put option written on the Weighted Average of Ordered Returns under different assumptions about the distribution of returns. The models of returns considered in this paper are defined by the following underlying price processes: geometric Brownian motion, Merton model with Poisson jumps, and GARCH model. We present a convergence result which states that the price of a forward on the Weighted Average of Ordered Returns converges to the theoretical law invariant coherent risk measure. Finally, we show that the forward price process itself satisfies the axioms of a dynamic coherent risk measures.

Keywords: negative returns, price of return risk, risk measures

1 Introduction

The main idea of this paper is to study the impact of negative returns on the health of a given financial portfolio. It is often the case that a series of significant negative returns trigger a credit event such as a downgrade in rating, or even a default of the portfolios owner. We introduce the concept of a *Realized Return Risk* which is defined as a directly observable function of realized returns on standardized portfolios. A *Weighted Average of Ordered Returns*, a special case of the Realized Return Risk, will be the main focus of this paper. Predictive assessment of the future risk is given by the *Price of Return Risk* – the price of a theoretical contract which pays its holder the Realized Return Risk for a certain period. In particular, a high Price of Return Risk on a portfolio implies a high risk of its mispricing, and an excessively high Price of Return Risk could indicate the imminency of the next market crisis. The Price of Return Risk depends on the risk-neutral distribution of negative returns and is strongly affected by the following features of the distribution: no jumps in the underlying asset prices or presence of jumps, dependent or independent returns. We study models based on geometric Brownian motion, Merton model with Poisson jumps, and GARCH models.

In practice, market participants use pricing models for their trading. Thus, we do provide simple benchmark dynamic pricing formulas for a Price of Return Risk in Section 3, where we made the assumption of independent and identically distributed returns, but left the choice of the distribution of returns open.

Let us put the concept of Realized Return Risk in the context of past research. Within the areas of mathematical finance and mathematical insurance, there has been almost simultaneous development of an axiomatic approach of measuring risk. Arztner et al. [3] and [4] established the representation theorem of a Coherent Risk Measure as a supremum of expectations under the axioms of monotonicity, subadditivity, positive homogeneity and translation invariance in a finite probability space. Wang et al. [28] independently deduced the

Choquet integral representation of the distributional property of risk measures based on the work of Yaari [30] with additional assumptions of law invariance and comonotonicity. Kusuoka [19] developed equivalent representations to Wang et al. [28] in the form of Weighted Value-at-Risk. Recent research has focused on extending to general spaces where the representation theorem applies (Delbaen [12], Cherny [9]), attempting to develop a dynamic version of coherent risk measures (Artzner et al. [5], Riedel [23], Cheridito et al. [8], Frittelli and Scandolo [15], Kloppel and Schweizer [18], Weber [29]), or relaxing the axioms to convex risk measures (Föllmer and Schied [13]). The universal industry approach is represented by the latest Basel II framework which adopts Value-at-Risk (VaR) as a minimal capital reserve requirement for market risks. Though the fact that VaR is not a Coherent Measure of Risk motivated the original work of Artzner et al. [3] and [4], it has nevertheless remained as the industry standard up to date and its adequacy has been a topic of research in Peura and Jokivuolle [22]. On the risk transfer side, there already exists significant volume in trading non-coherent based risks in today's market. For example, volatility swaps provide a way to trade and hedge realized volatility. For options on realized variance, see Carr et al. [7]. Jarrow [17] also studied put option premium as a risk measure. Our choice of a Realized Return Risk will focus on theoretical contracts that satisfy axioms for coherent based risk measure.

Recall that a Realized Return Risk is a directly observable function of realized returns and a contract with the Realized Return Risk payoff is called a Price of Return Risk. In this sense, we show in Section 2 that most of the currently traded contracts, such as Total Return Swap and Variance Swap, are in fact special cases of a Price of Return Risk. However, neither are they estimators of the popular Value-at-Risk (VaR), nor do they satisfy axioms of a Coherent Measure of Risk. Thus, we focus on a forward written on a *Weighted Average of Ordered Returns*. It is proposed in Heyde et al. [16] as a risk measure itself. We show that it serves both as an estimator of a Weighted Value-at-Risk from the distribution of returns (Theorem 4.1), and demonstrate that under certain conditions, its dynamic forward price satisfies generalized axioms for dynamic Coherent Measures (Theorem 5.2). Other important examples of the contracts with Realized Return Risk include *VaR Swap, Worst Return Swap,* and *Shortfall Swap*, where we emphasize their similarity to the existing Variance Swap or Total Return Swap.

This paper is organized as follows. In Section 2, we define a Realized Return Risks an provide several examples. This allows us to introduce a Price of Return Risk – the price of a contract with a Realized Return Risk payoff. Section 3 contains pricing formulas for a special case of a Realized Return Risk, Weighted Average of Ordered Returns, with the assumption of independent and identically distributed returns. Section 4 shows that the price of a Weighted Average of Ordered Returns converges to a weighted average of quantiles from the distribution of returns, and thus it can serve as an estimator to popular risk measures such as VaR or Expected Shortfall. Interpretation of Price of Return Risk as a dynamic risk measure is discussed in Section 5. Section 6 concludes the paper.

2 Realized Return Risk and Price of Return Risk

Let T be a finite time horizon and $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N \leq T$ a partition of the interval $[0, T]$. Let S_{t_i} be the market price, at time t_i, of a traded asset or a portfolio of assets. Random variable X_i represents a return on the asset over period $[t_{i-1}, t_i]$ - either the dollar return, the percentage return, or the log return:

$$X_i = S_{t_i} - S_{t_{i-1}}, \quad X_i = \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}, \quad \text{or} \quad X_i = \log \frac{S_{t_i}}{S_{t_{i-1}}}, \quad i = 1, \ldots, N.$$

Let us consider a general function of returns $X_1, \ldots, X_N$: $RR_N((X_i)_{i=1}^N)$. One can think of the function RR_N as the payoff of a contract. As an example, assume that RR_N is given as follows:

$$RR_N((X_i)_{i=1}^N) = -X_{(1)}, \quad \text{where } X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(N)} \text{ are ordered returns,}$$

An investor holding a contract paying such RR_N at time T will be compensated for the largest daily loss over $[0,T]$. Note that the value of $RR_N((X_i)_{i=1}^N)$ is known only at time T, after all the returns are observed. Therefore, $RR_N((X_i)_{i=1}^N)$ will be called the *Realized Return Risk.*

Definition 2.1 *Let $0 = t_0 < t_1 < \ldots < t_N \leq T$ be a partition of interval $[0,T]$ and X_i the return for period $[t_{i-1}, t_i]$, $1 \leq i \leq N$. The Realized Return Risk RR_N is defined as a function of realized returns $X_1, \ldots, X_N$:*

$$RR_N((X_i)_{i=1}^N).$$

In effect, the Realized Risk is a mapping $RR_N : \mathbb{R}^N \longrightarrow \mathbb{R}$.

The concept of a Realized Return Risk allows us to introduce the following definition of a Market Crash.

Definition 2.2 *Let $[0,T]$ be a finite time horizon and RR_n a Realized Return Risk over n periods. A market is said to have experienced a crash, MC, if the Realized Return Risk $RR_n((X_i)_{i=1}^n)$ exceeded a threshold C for at least one $n = 1, ..., N$. The time of a Market Crash, TMC, is defined as the first moment when RR_n exceeds C :*

$$TMC = \inf \left\{ n \geq 1 \,|\, RR_n((X_i)_{i=1}^n) > C \right\}.$$

If $RR_n((X_i)_{i=1}^n) \leq C$ for all $n = 1, ..., N$, no crash has occurred.

Remark 2.3 *The event of a Market Crash defined above may result in a downgrade of credit rating, or even a default if the underlying asset is a stock.*

Let us assume that S is a tradeable asset and that there exists a risk-neutral measure $\mathbb{Q}$. The no-arbitrage value, at time $t \in [0,T]$, of a contract with the payoff $RR_N((X_i)_{i=1}^N)$ at T is given as

$$e^{-r(T-t)} \, \mathbb{E}^{\mathbb{Q}} \left[RR_N((X_i)_{i=1}^N) \,|X_1, ..., X_n \right],$$

where $\mathbb{E}^Q$ is the expected value under $\mathbb{Q}$, time $t \in [t_n, t_{n+1})$, and r is a constant risk-free interest rate. We will call this value the *Price of Return Risk RR*. Realistically speaking, there will be no good hedging possibility for for some of the contracts, therefore this is an incomplete market pricing problem. However, let us assume there is a risk-neutral pricing measure $\mathbb{Q}$ for practical purposes.

Definition 2.4 *Let $[0,T]$ be a finite time horizon and RR_N a Realized Return Risk over this horizon. The Price of Return Risk associated with RR, $\rho(t,T)$, is the price, at time t, of a financial contract with the payoff $RR_N((X_i)_{i=1}^N)$ at T :*

$$\rho(t, T, (X_i)_{t_i \leq t}) \; = \; e^{-r(T-t)} \, \mathbb{E}^{\mathbb{Q}} \left[RR_N((X_i)_{i=1}^N) \,|X_1, ..., X_n \right], \tag{1}$$

where $t \in [t_n, t_{n+1})$, $\mathbb{E}^{\mathbb{Q}}$ is the risk-neutral expected value, and $X_1, ..., X_n$ are the observed past returns.

The definition of a Price of Return Risk can be justified in the following way. If there existed market quotes for $\rho(t, T, , (X_i)_{t_i \leq t})$, they would provide investors an indication of how the market views the risks associated with RR. "High" ("low") values of $\rho(t, T, , (X_i)_{t_i \leq t})$ would imply that market participants expect the Realized Return Risk to be "high" ("low").

A call option and a put option on a Realized Return Risk RR_N can be defined as contracts with payoffs $(RR_N((X_i)_{i=1}^N) - K)^+$ and $(K - RR_N((X_i)_{i=1}^N))^+$, respectively:

$$c(t, T, (X_i)_{t_i \leq t}) \; = \; e^{-r(T-t)} \, \mathbb{E}^{\mathbb{Q}} \left[(RR_N((X_i)_{i=1}^N) - K)^+ \,|X_1, ..., X_n \right], \tag{2}$$

$$p(t, T, (X_i)_{t_i \leq t}) \; = \; e^{-r(T-t)} \, \mathbb{E}^{\mathbb{Q}} \left[(K - RR_N((X_i)_{i=1}^N))^+ \,|X_1, ..., X_n \right]. \tag{3}$$

Note that both the call and the put options are special cases of a Price of Return Risk.

Examples of Realized Return Risk:

1. The Asset Itself. The most trivial example of a Realized Return Risk is the underlying asset S_T itself. Assume that $t = 0$ and $\{X_1, \ldots, X_N\}$ are future percentage returns: $X_i = \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}$. If we set

$$RR_N((X_i)_{i=1}^N) = S_0 \prod_{i=1}^{N}(1 + X_i) = S_T,$$

then $\rho(0, T)$ coincides with a forward on the underlying asset and $c(0, T)$ and $p(0, T)$ are respectively European call and put options on that asset.

2. Weighted Average of Ordered Returns. This paper will focus on the Realized Return Risk RR_N defined as a Weighted Average of Ordered Returns:

$$RR_N((X_i)_{i=1}^N) = -\sum_{i=1}^{N} w_i X_{(i)}, \quad \text{where } w_i \geq 0 \text{ and } \sum_{i=1}^{N} w_i = 1. \tag{4}$$

Random variable $X_{(i)}$ is the $i-$th order statistic: $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(N)}$. If the weights in (4) are decreasing, $w_1 \geq \ldots \geq w_N$, then $RR_N((X_i)_{i=1}^N)$ is a statistical approximation of the class of law invariant convex comonotonic risk measures, called Weighted VaR, that is based on probability distortion of Conditional VaR and is equivalent to the Choquet integral representation (see Kusuoka [19] and Wang et al. [28]). The details are given in Appendix A, in order not to deviate from the current presentation. For additional justification from an axiomatic approach in finite probability space, see Heyde et al. [16]. The method in Appendix A is closer to the idea as in Acerbi [1], while we provide convergence results in Section 4. Important special cases of Weighted Average of Ordered Returns include: the Worst Return, the $j-$th Worst Return, the Empirical Value-at-Risk (VaR), and the Empirical Expected Shortfall.

2a. The Worst Return. The Worst Return is a special case of the Weighted Average of Ordered Returns with

$$RR_N((X_i)_{i=1}^N) = -X_{(1)}, \quad \rho_{(1)}(t, T, (X_i)_{t_i \leq t}) = -e^{-r(T-t)}\, \mathbb{E}^{\mathbb{Q}}[X_{(1)}|\mathcal{F}_t].$$

The weights are given by $w_i = 1$ if $i = 1$, and $w_i = 0$ if $i > 1$.

2b. The j−th Worst Return. Let j be an integer between 1 and N. The **j**−th Worst Return is defined as

$$RR_N((X_i)_{i=1}^N) = -X_{(j)}, \quad \rho_{(j)}(t, T, (X_i)_{t_i \leq t}) = -e^{-r(T-t)}\, \mathbb{E}^{\mathbb{Q}}[X_{(j)}|\mathcal{F}_t].$$

The weights are given by $w_i = 1$ if $i = j$ and $w_i = 0$ if $i \neq j$.

2c. Empirical Value-at-Risk at the $(1 - \lambda)100\%$ level. Let $\lambda \in (0, 1)$ such that $N\lambda \geq 1$.

$$RR_N((X_i)_{i=1}^N) = -X_{(\lfloor N\lambda \rfloor)}, \quad \rho_\lambda(t, T, (X_i)_{t_i \leq t}) = -e^{-r(T-t)}\, \mathbb{E}^{\mathbb{Q}}[X_{(\lfloor N\lambda \rfloor)}|\mathcal{F}_t].$$

Weights: $w_i = 1$ if $i = \lfloor N\lambda \rfloor$ and $w_i = 0$ if $i \neq \lfloor N\lambda \rfloor$, where $\lfloor N\lambda \rfloor$ denotes the largest integer less than or equal to $N\lambda$, $0 < \lambda < 1$.

2d. Empirical Expected Shortfall at the $(1 - \lambda)100\%$ level. Let $\lambda \in (0, 1)$ such that $N\lambda \geq 1$.

$$RR_N((X_i)_{i=1}^N) = -\frac{1}{\lfloor N\lambda \rfloor} \sum_{i=1}^{\lfloor N\lambda \rfloor} X_{(i)}, \quad \rho(t, T, (X_i)_{t_i \leq t}) = -e^{-r(T-t)} \frac{1}{\lfloor N\lambda \rfloor} \sum_{i=1}^{\lfloor N\lambda \rfloor} \mathbb{E}^{\mathbb{Q}}[X_{(i)}|\mathcal{F}_t].$$

Weights: $w_i = \frac{1}{\lfloor N\lambda \rfloor}$ if $i \leq \lfloor N\lambda \rfloor$ and $w_i = 0$ if $i > \lfloor N\lambda \rfloor$.

2e. Crash Option. A crash option is a contract the payoff of which is triggered by a drop in S by more than $K\%$ before the time of expiration T. The holder of the option receives the compensation equal to the difference between the drop and level K. Thus, the option pays $(RR_{TMC\wedge T}((X_i)_{i=1}^{TMC\wedge T}) - K)^+$ at time $TMC \wedge T$, where $\wedge$ denotes the minimum, $RR_N((X_i)_{i=1}^N) = -X_{(1)}$ is the maximum loss, and $TMC = \inf\{n \geq 1 | RR_n((X_i)_{i=1}^n) \geq K\}$. The price of this option at time $t \leq TMC \wedge T$ is:

$$CO(t, T, (X_i)_{t_i \leq t}) = \mathbb{E}^{\mathbb{Q}} \left[e^{-r(TMC\wedge T - t)} (RR_{TMC\wedge T}((X_i)_{i=1}^{TMC\wedge T}) - K)^+ | \mathcal{F}_t \right].$$

A crash option of this kind was studied by Tankov [26]. Another definition of a crash option was introduced by Longin [20]. Its holder receives amount $(-X_{(1)} - K)^+$ at time T. The price of such an option is:

$$CO(t, T, (X_i)_{t_i \leq t}) = -e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} \left[(-X_{(1)} - K)^+ | \mathcal{F}_t \right]$$

Note that this contract can be considered a European plain-vanilla option on the Worst Return over the period $[0, T]$.

3. Maximum Drawdown. Another example of Realized Return Risk is a discretely monitored maximum drawdown of the price process S_t, which can be defined as follows:

$$RR_N((X_i)_{i=1}^N) = \max_{0 \leq k \leq N} \left(\max_{0 \leq l \leq k} S_{t_l} - S_{t_k} \right) = \max_{0 \leq k < l \leq N} (S_{t_k} - S_{t_l}) = - \min_{1 \leq k < l \leq N} \sum_{i=k+1}^{l} X_i.$$

Random variables X are the dollar returns on $S : X_i = S_{t_i} - S_{t_{i-1}}$ for $i = 1, \ldots, N$. For a more systematic treatment of Maximum Drawdown, see Vecer [27].

4. Realized Variance. As mentioned earlier, contracts on realized variance are already traded. In fact, such contracts are a Price of Return Risk with the payoff:

$$RR_N((X_i)_{i=1}^N) = \frac{1}{N} \sum_{i=1}^{N} \left(X_i - \bar{X} \right)^2,$$

where $X_1, ..., X_N$ are percentage returns on an asset. See Carr et al. [7] for details.

A particular feature of Price of Return Risk is that the past is seamlessly connected to the future in a non-parametric way. If the current time is t and the time of maturity T, the realized returns $(X_i)_{t_i \leq t}$ have an impact on $\rho(t, T, (X_i)_{t_i \leq t})$ through its reflection in the payoff function $RR_N((X_i)_{i=1}^N)$.

Remark 2.5 (Worst Return Swap, VaR Swap, Shortfall Swap) Worst Return, Empirical VaR and Empirical Shortfall can be traded as Forwards (or Swaps) so that no initial change of capital is required to enter such contract. In analogy to the existing Total Return Swap contract, they can be written on returns of two markets.

Let $X_1, X_2, \ldots, X_N$ and $Y_1, Y_2, \ldots, Y_N$ be realized returns in two different markets. *Total Return Swap* is defined as a contract which pays off $X_i - Y_i$ at times t_i, $i = 1, \ldots, N$. We can define *Worst Return Swap* as a contract which pays off $X_{(1)} - Y_{(1)}$ at time T, *VaR Swap* as a contract which pays off $X_{(k)} - Y_{(k)}$ for $k = \lfloor N\lambda \rfloor$ at time T, and *Shortfall Swap* as a contract which pays off $\frac{1}{\lfloor N\lambda \rfloor} \sum_{i=1}^{\lfloor N\lambda \rfloor} (X_{(i)} - Y_{(i)})$ at time T.

When we fix the second market to be a constant, we can introduce *Worst Return Rate*, *VaR Rate*, and *Shortfall Rate* for a single market.

Worst Return Rate is given by

$$R_{(1)} = e^{-r(T-t)}\ \mathbb{E}^{\mathbb{Q}}[X_{(1)}|\mathcal{F}_t],$$

VaR Rate is given by

$$R_{(k)} = e^{-r(T-t)}\ \mathbb{E}^{\mathbb{Q}}[X_{(k)}|\mathcal{F}_t],$$

for $k = \lfloor N\lambda \rfloor$, and

Shortfall Rate is given by

$$\bar{R}_{(k)} = e^{-r(T-t)} \frac{1}{\lfloor N\lambda \rfloor} \sum_{i=1}^{\lfloor N\lambda \rfloor} \mathbb{E}^{\mathbb{Q}}[X_{(i)}|\mathcal{F}_t],$$

for $k = \lfloor N\lambda \rfloor$.

The above rates carry potentially important information about the expected stability of a given market, and these rates can be possibly influenced by a central bank analogously to interest rates.

3 Price of Weighted Average of Ordered Returns

In this section, we will limit our focus to a special case of a Realized Return Risk, namely on the Weighted Average of Ordered Returns. Miura [21] set up a similar pricing problem of a lookback option on order statistics. Since it was applied directly to the asset price instead of the returns, a closed-form solution was not obtained. Assume that the financial returns $\{X_1, X_2, ..., X_N\}$ are independent and identically distributed with cumulative distribution function $F_X(x)$. The cumulative distribution function of the ith order statistic $X_{(i)}$ is:

$$F_{X_{(i)}}(x) = \sum_{k=i}^{N} \binom{N}{k} [F_X(x)]^k [1 - F_X(x)]^{N-k}. \tag{5}$$

When the returns have a continuous distribution with probability density function $f_X(x)$, the density of $X_{(i)}$ is given by:

$$f_{X_{(i)}}(x) = \frac{N!}{(i-1)!(N-i)!} f_X(x)[F_X(x)]^{i-1}[1 - F_X(x)]^{N-i}. \tag{6}$$

If we assume the information available at any given time comprises the observed returns X_i's up to that time, meaning $\mathcal{F}_t = \sigma\left((X_i)_{t_i \le t}\right)$, then we can rewrite the dynamic forward price (1) as:

$$\begin{aligned} \rho(t, T, (X_i)_{t_i \le t}) &= -e^{-r(T-t)} \sum_{i=1}^{N} w_i \mathbb{E}^{\mathbb{Q}}[X_{(i)}|\mathcal{F}_t] \\ &= -e^{-r(T-t)} \sum_{i=1}^{N} w_i \mathbb{E}^{\mathbb{Q}}[X_{(i)}|X_1, ..., X_n], \quad \text{where } t_n \le t < t_{n+1}. \end{aligned} \tag{7}$$

In general, we need to find the conditional distributions of order statistics $X_{(i)}$'s based on the first n observations $X_1, X_2, ..., X_n$. Let us formulate the question in the following way: suppose we have an ordered set of real numbers $-\infty = x_0 < x_1 < x_2 < ... < x_n < x_{n+1} = \infty$ and a set of random variables $Y_1, Y_2, ..., Y_m$ which are independent draws from a common distribution with cumulative distribution function $F(x)$. Let us mix the samples $x_1, x_2, ..., x_n, Y_1, Y_2, ..., Y_m$ and call them $Z_1, Z_2, ..., Z_{n+m}$. We need to compute the conditional distribution of $Z_{(i)}$ as a function of $x_1, x_2, ..., x_n$: $F_{Z_{(i)}}(z|x_1, ..., x_n)$.

Lemma 3.1 *The conditional cumulative distribution function has the following representation:*

$$F_{Z_{(i)}}(z|x_1,...,x_n) = \sum_{k=0\vee(i-m)}^{n\wedge i} F_{Y_{(i-k)}}(z)\mathbb{I}_{[x_k,x_{k+1})}(z) + \mathbb{I}_{\{i<n\}}\mathbb{I}_{[x_{i+1},\infty)}(z), \quad \textit{for } 1 \le i \le n+m, \tag{8}$$

where $\mathbb{I}_A(x)$ is the indicator function of set A.

Proof. The order statistics of Y_i's are written as $Y_{(0)}, Y_{(1)}, ..., Y_{(m)}$, where we have added an extra variable $Y_{(0)} = -\infty$. The x_i's divide the real line into $n+1$ intervals, and we need to keep track of which interval each $Y_{(i)}$ falls in. When $z \in [x_k, x_{k+1})$, the event $\{Z_{(i)} \le z\}$ is equivalent to $\{Y_{(i-k)} \le z\}$. Therefore, we can write:

$$\mathbb{I}_{\{Z_{(i)}\le z\}} = \sum_{k=0\vee(i-m)}^{n\wedge i} \mathbb{I}_{\{Y_{(i-k)}\le z\}}\mathbb{I}_{[x_k,x_{k+1})}(z) + \mathbb{I}_{\{i<n\}}\mathbb{I}_{[x_{i+1},\infty)}(z), \quad \text{for } 1 \le i \le n+m.$$

Note that the constraints $0 \le k \le n$ and $0 \le i-k \le m$ give the range of summation $0 \vee (i-m) \le k \le n \wedge i$ in the above equation. The conditional cumulative distribution function is therefore given by the following expression:

$$\begin{aligned}
F_{Z_{(i)}}(z|x_1,...,x_n) &= \mathbb{Q}(Z_{(i)} \le z|x_1,...,x_n) \\
&= \sum_{k=0\vee(i-m)}^{n\wedge i} \mathbb{Q}(Y_{(i-k)} \le z)\mathbb{I}_{[x_k,x_{k+1})}(z) + \mathbb{I}_{\{i<n\}}\mathbb{I}_{[x_{i+1},\infty)}(z) \\
&= \sum_{k=0\vee(i-m)}^{n\wedge i} F_{Y_{(i-k)}}(z)\mathbb{I}_{[x_k,x_{k+1})}(z) + \mathbb{I}_{\{i<n\}}\mathbb{I}_{[x_{i+1},\infty)}(z), \quad \text{for } 1 \le i \le n+m. \qquad (9)
\end{aligned}$$

◇

Note that the conditional cumulative distribution function, $F_{Z_{(i)}}(z|x_1,...,x_n)$, is equal to $F_{Y_{(i-k+1)}}(z)$ on the interval $z \in [x_{k-1}, x_k)$, and $F_{Y_{(i-k)}}(z)$ on the interval $z \in [x_k, x_{k+1})$. Therefore, there is a discrete probability mass at every $z = x_k$, whenever $1 \le k \le i$:

$$\begin{aligned}
\mathbb{Q}(Z_{(i)} = x_k|x_1,...,x_n) &= F_{Z_{(i)}}(x_k|x_1,...,x_n) - F_{Z_{(i)}}(x_k - |x_1,...,x_n) \\
&= F_{Y_{(i-k)}}(x_k) - F_{Y_{(i-k+1)}}(x_k) \\
&= \sum_{j=i-k}^{m} \binom{m}{j} [F(x_k)]^j[1-F(x_k)]^{m-j} - \sum_{j=i-k+1}^{m} \binom{m}{j} [F(x_k)]^j[1-F(x_k)]^{m-j} \\
&= \binom{m}{i-k} [F(x_k)]^{i-k}[1-F(x_k)]^{m-i+k}. \qquad (10)
\end{aligned}$$

If the Y_i's have a continuous distribution with density function $f(x)$, we can write the conditional probability density (mass) function of $Z_{(i)}$, with the help of the Dirac delta function $\delta_a(x)$:

$$\begin{aligned}
f_{Z_{(i)}}(z|x_1,...,x_n) &= \sum_{k=0\vee(i-m)}^{n\wedge(i-1)} f_{Y_{(i-k)}}(z)\mathbb{I}_{(x_k,x_{k+1})}(z) + \sum_{k=1}^{i} \mathbb{Q}(Z_{(i)} = x_k|x_1,...,x_n)\delta_{x_k}(z) \\
&= \sum_{k=0\vee(i-m)}^{n\wedge(i-1)} \frac{m!}{(i-k-1)!(m-i+k)!} f(z)[F(z)]^{i-k-1}[1-F(z)]^{m-i+k}\mathbb{I}_{(x_k,x_{k+1})}(z) \\
&\quad + \sum_{k=1}^{i} \binom{m}{i-k} [F(x_k)]^{i-k}[1-F(x_k)]^{m-i+k}\delta_{x_k}(z), \qquad (11)
\end{aligned}$$

for $1 \le i \le n+m$.

Theorem 3.2 *Suppose the returns $\{X_1, X_2, ..., X_N\}$ are independent and identically distributed with cumulative distribution function $F_X(x)$ under $\mathbb{Q}$, and $t \in [t_n, t_{n+1})$. Let $\hat{X}_{(1)}, \ldots, \hat{X}_{(n)}$ statistics of observed returns $X_1, \ldots, X_n$, and $\tilde{X}_{(1)}, \ldots, \tilde{X}_{(N-n)}$ the order statistics of future returns $X_{n+1}, \ldots, X_N$. The dynamic forward price process defined in (7) is:*

$$\rho(t,T,(X_i)_{t_i\le t}) = -\, e^{-r(T-t)}\sum_{i=1}^{N} w_i \mathbb{E}^{\mathbb{Q}}[X_{(i)}|X_1,...,X_n] \tag{12}$$

$$= -\, e^{-r(T-t)}\sum_{i=1}^{N} w_i \sum_{k=0\vee(i-N+n)}^{n\wedge i}\int_{(\hat{X}_{(k)},\hat{X}_{(k+1)})} xF_{\tilde{X}_{(i-k)}}(dx) \tag{13}$$

$$-\, e^{-r(T-t)}\sum_{i=1}^{N} w_i \sum_{k=1}^{i}\hat{X}_{(k)}\begin{pmatrix} N-n \\ i-k \end{pmatrix}[F_X(\hat{X}_{(k)})]^{i-k}[1-F_X(\hat{X}_{(k)})]^{N-n-i+k},$$

where

$$F_{\tilde{X}_{(i-k)}}(x) = \sum_{j=i-k}^{N-n}\begin{pmatrix} N-n \\ j \end{pmatrix}[F_X(x)]^j[1-F_X(x)]^{N-n-j}.$$

Furthermore, when the distribution of X_i is continuous with probability density function $f_X(x)$, we can write:

$$\rho(t,T,(X_i)_{t_i\le t}) = -\, e^{-r(T-t)}\sum_{i=1}^{N} w_i \sum_{k=0\vee(i-N+n)}^{n\wedge(i-1)}\int_{(\hat{X}_{(k)},\hat{X}_{(k+1)})} xf_{\tilde{X}_{(i-k)}}(x)dx \tag{14}$$

$$-\, e^{-r(T-t)}\sum_{i=1}^{N} w_i \sum_{k=1}^{i}\hat{X}_{(k)}\begin{pmatrix} N-n \\ i-k \end{pmatrix}[F_X(\hat{X}_{(k)})]^{i-k}[1-F_X(\hat{X}_{(k)})]^{N-n-i+k},$$

where:

$$f_{\tilde{X}_{(i-k)}}(x) = \frac{(N-n)!}{(i-k-1)!(N-n-i+k)!}f_X(x)[F_X(x)]^{i-k-1}[1-F_X(x)]^{N-n-i+k}.$$

Proof. The results follow directly from Lemma 3.1 and the comments afterwards, where we replace $x_1, ..., x_n$ with $X_1, ..., X_n$, and $Y_1, ..., Y_m$ with $X_{n+1}, ..., X_N$. ◇

The pricing formulas given in Theorem 3.2 are based on the distribution and density functions (5) and (6) defined at the beginning of this section. Here, we present a lemma that associates the distribution of order statistics to a Value-at-Risk (VaR) transformation of Beta distribution in a general continuous distribution case and provide an alternative pricing formula. As before, $X_{(1)}, X_{(2)}, ..., X_{(N)}$ are the order statistics of independent and identically distributed random variables $X_1, X_2, ..., X_N$ with cumulative distribution function $F_X(x)$ under $\mathbb{Q}$. Since $VaR_\lambda(X)$ is the negative value of the $\lambda-$quantile function of X, it is the negative value of an inverse function of $F_X(x)$. Note that a Beta(α, β) random variable Y has probability density function:

$$f_{B(\alpha,\beta)}(x) = \frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}, 0 < x < 1, \tag{15}$$

where $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$ and $\alpha > 0$, $\beta > 0$. Its expectation and variance are simply given by $\mathbb{E}Y = \frac{\alpha}{\alpha+\beta}$ and $Var(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

Lemma 3.3 *$F_X(x)$ is continuous. Suppose $\{X_1, X_2, ..., X_N\}$ are independent and identically distributed random variables with a continuous distribution. Then the j-th order statistics $X_{(j)}$ and $-VaR_Y(X)$ have the same law, where Y is a random variable with Beta(j, N-j+1) distribution.*

PROOF: The proof is very simple. It is well-known that the j-th order statistics from an independent identically distributed Uniform(0,1) random sample of size N has a Beta(j, N-j+1) distribution. Since $F_X(X_j) \sim$ Uniform(0,1) and F_X is an increasing function and therefore preserves the order of the statistics, $F_X(X_{(j)}) \sim$ Beta(j, N-j+1) and the result follows easily. A direct proof using the probability density function in (6) is also straightforward. Recall that:

$$f_{X_{(j)}}(x) = \frac{N!}{(j-1)!(N-j)!} f_X(x)[F_X(x)]^{j-1}[1-F_X(x)]^{N-j}.$$

We have:

$$\begin{aligned}
\mathbb{Q}(X_{(j)} \leq z) &= \int_{-\infty}^{z} \frac{N!}{(j-1)!(N-j)!} f_X(x)[F_X(x)]^{j-1}[1-F_X(x)]^{N-j} dx \\
&= \int_{-\infty}^{z} \frac{N!}{(j-1)!(N-j)!} [F_X(x)]^{j-1}[1-F_X(x)]^{N-j} dF_X(x) \\
&= \int_{0}^{F_X(z)} \frac{1}{B(j, N-j+1)} y^{j-1}(1-y)^{N-j} dy \\
&= \mathbb{Q}(Y \leq F_X(z)) = \mathbb{Q}(-VaR_Y(X) \leq z),
\end{aligned}$$

where $F_X^{-1}(\lambda) = -VaR_\lambda(X)$. ◇

Lemma 3.3 allows us to think of any order statistics as a transform of a Beta random variable with parameters depending only on the order of the statistics and the sample size. From a computational perspective, it also makes the expectation formula simpler to evaluate. Theorem 3.2 is based on the direct formula (6):

$$\mathbb{E}^{\mathbb{Q}} X_{(j)} = \int_{-\infty}^{\infty} x \frac{1}{B(j, N-j+1)} f_X(x)[F_X(x)]^{j-1}[1-F_X(x)]^{N-j} dx.$$

If we use Lemma 3.3 and (15) instead, we arrive to an alternative formula:

$$\mathbb{E}^{\mathbb{Q}} X_{(j)} = -\int_0^1 VaR_\lambda(X) f_{Beta(j,N-j+1)}(\lambda) d\lambda = -\int_0^1 VaR_\lambda(X) \frac{1}{B(j, N-j+1)} \lambda^{j-1}(1-\lambda)^{N-j} d\lambda. \tag{16}$$

The second approach is much more systematic for evaluation or simulation, and formula (14) in Theorem 3.2 can be simplified correspondingly in the unconditional case which we will state in the following Corollary.

Corollary 3.4 *Suppose $\{X_1, X_2, ..., X_N\}$ are independent and identically distributed random variables with a continuous distribution. The initial forward price, a special case of Theorem 3.2, can be calculated as*

$$\begin{aligned}
\rho(0,T) &= -\, e^{-rT} \sum_{i=1}^{N} w_i \mathbb{E}^{\mathbb{Q}}[X_{(i)}] \\
&= -\, e^{-rT} \int_0^1 VaR_\lambda(X) \left(\sum_{i=1}^{N} w_i f_{B(i,N-i+1)}(\lambda) \right) d\lambda.
\end{aligned}$$

Let us compare formula (16) to the equation (31) in Appendix A. The negative value of the expectation of a particular order statistic is not going to be a good risk measure that satisfies the usual axioms because the Beta density does not have the monotonicity property which function ψ possess in the representation (31) for Weighted VaR. On the other hand, the negative value of the expectation of the Weighted Average of Ordered Returns when the weights are decreasing will serve as a good risk measure. Its convergence to Weighted VaR is proved in Section 4, and its properties as a dynamic risk measure are discussed in Section 5.

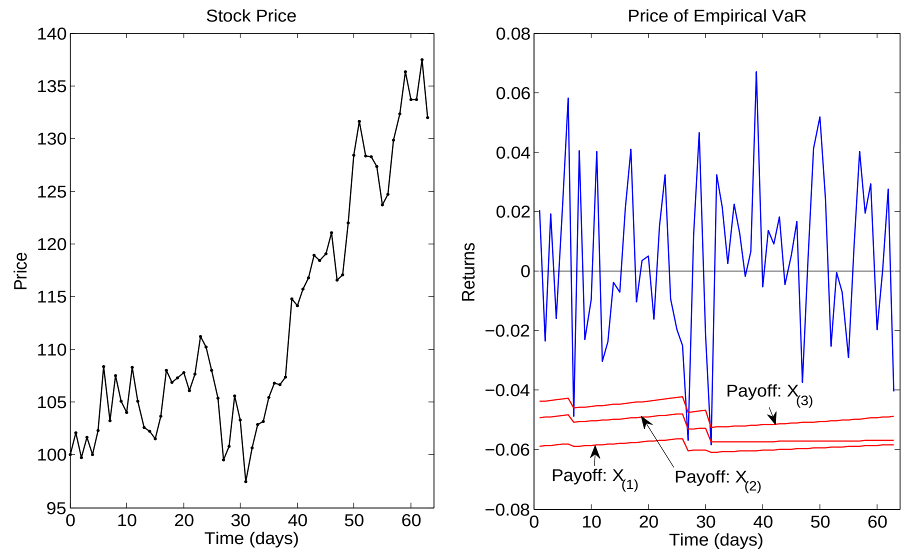

Figure 1: First graph: simulated stock price under the Black-Scholes model. Second graph: paths of $-\rho_{(j)}(t, T, (X_i)_{t_i \le t})$, where $j = 1, 2, 3$, $T = 3$ months, and $\Delta t = 1$ day. The daily returns, X, are assumed to follow the Black-Scholes model with interest rate $r = 1\%$ and annual volatility $\sigma = 40\%$ under the risk-neutral measure.

Now let us turn to the question of how to apply Theorem 3.2 to a portfolio process W_t, $0 \le t < \infty$, and give several numerical examples. The value of this process will be recorded at times $0 = t_0 < t_1 < \ldots < t_n = T$ to compute the log returns:

$$X_i = \ln\left(\frac{W_i}{W_{i-1}}\right), \quad i = 1, 2, ..., N,$$

where we denote $W_i = W_{t_i}$. Without the loss of generality, we will choose uniform time intervals with length $\Delta t = t_i - t_{i-1}$. If W_t follows the Black-Scholes model, then:

$$dW_t = W_t(rdt + \sigma dB_t),$$

under the risk-neutral measure, where r is a risk-free interest rate, $\sigma > 0$ annual volatility, and B_t a standard Brownian motion. Discrete returns in the Black-Scholes model,

$$X_i = (r - \tfrac{\sigma^2}{2})\Delta t + \sigma(B_{t_i} - B_{t_{i-1}}), \tag{17}$$

are independent and identically distributed with $N\left((r - \frac{\sigma^2}{2})\Delta t, \sigma^2 \Delta t\right)$ distribution. Thus, Theorem 3.2 applies, and the dynamic forward price can be explicitly calculated as:

$$\begin{aligned}\rho(t,T,(X_i)_{t_i \le t}) = & -e^{-r(T-t)} \sum_{i=1}^{N} w_i \sum_{k=0 \vee (i-N+n)}^{n \wedge (i-1)} \int_{(\hat{X}_{(k)}, \hat{X}_{(k+1)})} x f_{\tilde{X}_{(i-k)}}(x) dx \\ & - e^{-r(T-t)} \sum_{i=1}^{N} w_i \sum_{k=1}^{i} \hat{X}_{(k)} \begin{pmatrix} N-n \\ i-k \end{pmatrix} \left[\Phi\left(\frac{\hat{X}_{(k)} - m\Delta t}{\sigma\sqrt{\Delta t}}\right)\right]^{i-k} \left[1 - \Phi\left(\frac{\hat{X}_{(k)} - m\Delta t}{\sigma\sqrt{\Delta t}}\right)\right]^{N-n-i+k},\end{aligned}$$

where $m = r - \frac{\sigma^2}{2}$, Φ is the cumulative distribution function of the standard normal distribution, and:

$$f_{\tilde{X}_{(i-k)}}(x) = \frac{(N-n)!}{(i-k-1)!(N-n-i+k)!} \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} e^{-\frac{(x-m)^2}{2\sigma^2\Delta t}} \left[\Phi\left(\frac{x - m\Delta t}{\sigma\sqrt{\Delta t}}\right)\right]^{i-k-1} \left[1 - \Phi\left(\frac{x - m\Delta t}{\sigma\sqrt{\Delta t}}\right)\right]^{N-n-i+k}.$$

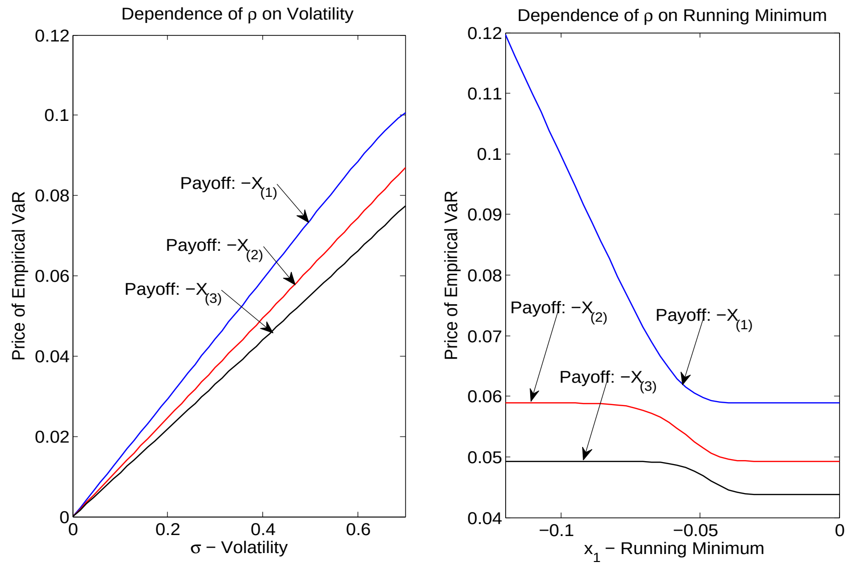

Figure 2: First graph: dependence of $\rho_{(j)}(0,T)$, where $j = 1,2,3$, on volatility σ. Second graph: dependence of $-\rho_{(j)}(t_1, T, (X_1))$, where $j = 1,2,3$, on the running minimum return X_1. The daily returns, X, are assumed to follow the Black-Scholes model with interest rate $r = 1\%$ and annual volatility $\sigma = 40\%$ under the risk-neutral measure.

Let us assume that $T = 3$ months, $\Delta t = 1$ day, interest rate $r = 1\%$, and annual volatility $\sigma = 40\%$. We will study properties of contracts on the the first, the second, and the third worst returns: $\rho_{(1)}(t, T, (X_i)_{t_i \le t})$, $\rho_{(2)}(t, T, (X_i)_{t_i \le t})$, and $\rho_{(3)}(t, T, (X_i)_{t_i \le t})$. Note that in this case, $\lfloor N5\% \rfloor = 3$, therefore the third worst return coincides with the 95% Empirical VaR: $\rho_{(3)}(t, T, (X_i)_{t_i \le t}) = \rho_{5\%}(t, T, (X_i)_{t_i \le t})$.

Figure 1 shows a simulated stock price under the Black-Scholes model and the corresponding processes $\rho_{(j)}(t, T, (X_i)_{t_i \le t})$, $j = 1,2,3$. The graph illustrates how the values of the ρ's react to newly observed returns X_i. If a new return is large negative, all three processes drop.

The first graph in Figure 2 depicts the dependence of values $\rho_{(j)}(0,T)$, $j = 1,2,3$, on the stock volatility. As one can expect, the risk increases with increasing volatility and moreover, the gap between $\rho_{(j)}(0,T)$ and $\rho_{(j-1)}(0,T)$ widens. The second graph shows how the values $\rho_{(j)}(t_1, T, (X_1))$, $j = 1,2,3$, change with X_1, the running minimum. Clearly, if X_1 is close to 0, it does not affect the Price of Return Risk. On the other hand, if X_1 is significantly lower than the expected future minimum, -0.059, then $\rho_{(1)}(t_1, T, (X_1)) \approx -X_1$ and $\rho_{(2)}(t_1, T, (X_1)) \approx \rho_{(1)}(t_1, T, 0)$ and $\rho_{(3)}(t_1, T, (X_1)) \approx \rho_{(2)}(t_1, T, 0)$.

In Figure 3, we present the term structure of the Empirical VaR, $\rho_{5\%}(0,T)$, for different values of T (ranging from 6 months to 12 years). The prices with longer time to maturity tend to be closer to the theoretical value of quantile $q_{5\%}(X)$, which is consistent with results we prove in Section 4.

Note that in the simple Black-Scholes model (17), the real expected value and the risk-neutral expected value of the Weighted Average of Ordered Returns differ by the risk premium. Let μ be the expected annualized return on portfolio W under the real probability measure $\mathbb{P}$. Thus, $X_i \sim \mathcal{N}((\mu - \sigma^2/2)dt, \sigma\sqrt{dt})$ under $\mathbb{P}$ and $X_i \sim \mathcal{N}((r - \sigma^2/2)dt, \sigma\sqrt{dt})$ under $\mathbb{Q}$ for all $i = 1, ..., N$, which implies that $\mathbb{E}^{\mathbb{P}}[X_{(i)}] =$

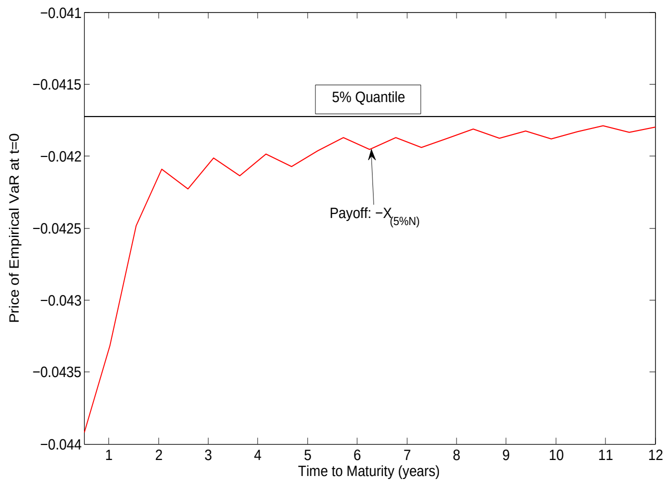

Figure 3: Term structure of $-e^{rT}\rho_{5\%}(0,T)$, where $T \in [0.5, 12]$ years, Δt= one day, and $N = T/\Delta t$. Daily returns X are assumed to follow the Black-Scholes model with interest rate $r = 1\%$ and annual volatility $\sigma = 40\%$. The horizontal line represents the theoretical 5% quantile, $q_{5\%}(X) = F_X^{-1}(0.05)$.

$\mathbb{E}^{\mathbb{Q}}[X_{(i)}] + (\mu - r)dt$ and

$$\begin{aligned} \rho(t, T, (X_i)_{t_i \le t}) &= -e^{-r(T-t)} \sum_{i=1}^{N} w_i \mathbb{E}^{\mathbb{Q}}[X_{(i)} | X_1, \ldots, X_n] \\ &= e^{-r(T-t)}(\mu - r)dt - e^{-r(T-t)} \sum_{i=1}^{N} w_i \mathbb{E}^{\mathbb{P}}[X_{(i)} | X_1, \ldots, X_n]. \end{aligned} \tag{18}$$

Let us provide a numerical example to compare the Price of Return Risk when the returns are given by the Black-Scholes model and the Merton model with jumps while the total volatilities in both models are kept at the same level. The Merton model is defined as:

$$dW_t = W_{t-}(rdt + \tilde{\sigma} dB_t + Y_t dN_t),$$

where B_t is a standard Brownian motion, N_t is a standard Poisson process with intensity λ_{Po}, and Y_t are independent and identically distributed normal random variables with mean μ and standard deviation ν, $Y_t \sim N(\mu, \nu^2)$. The discrete return can be calculated from the Doléans-Dade exponential formula:

$$X_i = (r - \tfrac{\tilde{\sigma}^2}{2})\Delta t + \tilde{\sigma}(B_{t_i} - B_{t_{i-1}}) + \sum_{k=1}^{N_{t_i} - N_{t_{i-1}}} Y_k. \tag{19}$$

The density function of X_i has a series expansion:

$$f_{X_i} = e^{-\lambda_{Po}\Delta t} \sum_{k=0}^{\infty} \frac{(\lambda_{Po}\Delta t)^k \exp\left\{-\frac{\left(x - (r - \tilde{\sigma}^2/2)\Delta t - k\mu\right)^2}{2(\tilde{\sigma}^2 \Delta t + k\nu^2)}\right\}}{k!\sqrt{2\pi(\tilde{\sigma}^2\Delta t + k\nu^2)}}.$$

Figure 4 shows a simulated stock price process under the Merton model and the corresponding price processes of the worst, the second worst, and the third worst returns: $\rho_{(1)M}(t, T, (X_i)_{t_i \le t})$, $\rho_{(2)M}(t, T, (X_i)_{t_i \le t})$, and $\rho_{(3)M}(t, T, (X_i)_{t_i \le t})$. In Figure 5, we have plotted values of $\rho_{(j)M}(0, T)$, $j = 1, 2, 3$, where $T = 3$ months

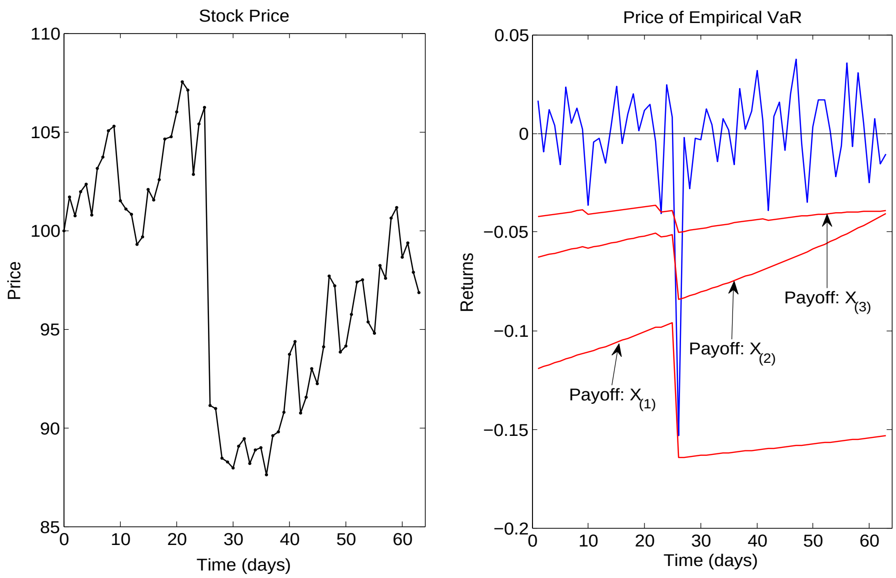

Figure 4: First graph: simulated stock price under the Merton model. Second graph: paths of $-\rho_{(j)}(t, T, (X_i)_{t_i \le t})$, where $j = 1, 2, 3$, $T = 3$ months, and $\Delta t = 1$ day. The daily returns, X, are assumed to follow the Merton model with interest rate $r = 1\%$, $\tilde{\sigma} = 40\%$, $\lambda = 10$, $\mu = -0.05$, and $\nu = 0.1$ under the risk-neutral measure.

and $\Delta t = 1$ day, obtained from the Merton model with different intensities λ_{Po}. The total volatility of the returns is kept at 30% by changing the diffusion volatility $\tilde{\sigma}$ along with λ_{Po} : $\sqrt{\tilde{\sigma}^2 + \lambda_{Po}(\nu^2 + \mu^2)} = 30\%$. For comparison purposes, we also include the Black-Scholes values of $\rho_{(j)BS}(0, T)$, $j = 1, 2, 3$, with $r = 1\%$ and $\sigma = 30\%$. Thus, the total stock volatilities in the two models are identical and moreover, when $\lambda_{Po} = 0$, the models coincide. As the intensity of the jumps λ_{Po} increases, $\rho_{(1)M}$ exceeds $\rho_{(1)BS}$ due to the jumps with standard deviation $\nu = 0.05$. On the other hand, $\rho_{(2)M}$ first decreases until it reaches a minimum because the jumps do not occur often enough and $\tilde{\sigma}$ decreases. Afterwards, $\rho_{(2)M}$ increases, as the effect of more frequent jumps begins to outweigh lower values of $\tilde{\sigma}$. Finally, $\rho_{(3)M}$ is lower than $\rho_{(3)BS}$ for the given range of intensities λ_{Po}.

An important assumption of Theorem 3.2 is the independence of the returns. Let us present numerical results for a GARCH model, in which the condition of independence does not hold (see Bollerslev [6] for details). In particular, we will consider the following GARCH(1,1) model for the returns that preserves the risk-neutral expected value of $(r - \sigma_a^2/2)dt$ with σ_a denoting the annual volatility:

$$\begin{aligned} X_i &= rdt - \sigma_i^2/2 + \epsilon_i, \quad i = 1, \ldots, N, \\ \epsilon_i &= \sigma_i e_i, \\ \sigma_i^2 &= \alpha_0 + \alpha_1 \epsilon_{i-1}^2 + \beta_1 \sigma_{i-1}^2. \end{aligned} \tag{20}$$

$\{e_i\}_{i=1}^N$ is a sequence of independent random variables with standard normal distribution, $\epsilon_0 = 0$ and σ_0 is given. Symbol σ_i denotes the daily volatility of the return X_i.

Since the formula in Theorem 3.2 does not apply to dependent returns, we need to use a Monte-Carlo simulation to estimate $\rho_{(j)}$ in the GARCH model. Figure 6 shows evolution of $\rho_{(j)}(t, T, (X_i)_{t_i \le t})$, $j = 1, 2, 3$, assuming the returns follow the GARCH model (20). Note that paths of $\rho_{(j)}(t, T, (X_i)_{t_i \le t})$ react to significant changes in the conditional volatility of the returns, σ_i. Subsequently, let us assume that $\beta_1 = 0$. Figure 7 illustrates the effect of the ARCH coefficient, α_1, on the Price of Return Risk.

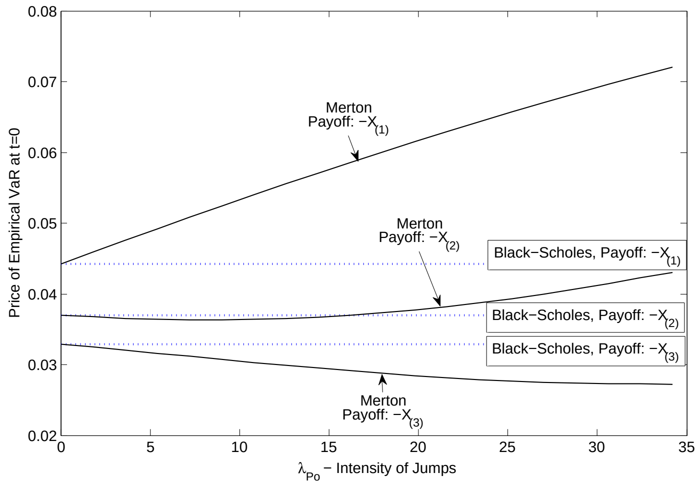

Figure 5: Comparison of $\rho_{(j)}(0,T)$, where $j = 1,2,3$, for the Black-Scholes and the Merton model of the daily returns, where $T = 3$ months and $\Delta t = 1$ day. The horizontal lines represent the price of the Empirical VaR in the Black-Scholes model with annual volatility $\sigma = 30\%$ and interest rate $r = 1\%$. The curves display the dependence of the price of the Empirical VaR on λ_{Po} in the Merton model. Other parameters are set to be: $\mu = 0$, $\nu = 0.05$, $r = 1\%$. Volatility of the diffusion component, $\tilde{\sigma}$, changed along with λ in order to preserve the total volatility of $30\% : \sqrt{\tilde{\sigma}^2 + \lambda_{Po}(\nu^2+\mu^2)} = 30\%$.

Now we will give formulas for the dynamic version of the call and put prices defined in (2) and (3). These formulas are derived only for the Empirical VaR and the $j-$the Worst Return because we have calculated the conditional distribution of one order statistic $X_{(i)}$ in Lemma 3.1, not of a linear combination $\sum_{i=1}^N w_i X_{(i)}$.

Theorem 3.5 *Suppose that returns $\{X_1, X_2, ..., X_N\}$ are independent and identically distributed with cumulative distribution function $F_X(x)$ under $\mathbb{Q}$. Let $\hat{X}_{(1)}, \ldots, \hat{X}_{(n)}$ be the order statistics of observed returns $X_1, \ldots, X_n$, and $\tilde{X}_{(1)}, \ldots, \tilde{X}_{(N-n)}$ the order statistics of future returns $X_{n+1}, \ldots, X_N$. If the Realized Return Risk is defined as the $j-$the Worst Return, $RR_N((X_i)_{1\le i\le N}) = -X_{(j)}$, then the dynamic call and put option price processes are equal to:*

$$\begin{aligned}
(21)\qquad c_j(t,T,(X_i)_{t_i\le t}) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\left(-X_{(j)} - K\right)^+ | X_1, ..., X_n] \\
&= e^{-r(T-t)} \sum_{k=0\vee(j-N+n)}^{n\wedge j} \int_{(\hat{X}_{(k)}, \hat{X}_{(k+1)})} (-x-K)^+ F_{\tilde{X}_{(j-k)}}(dx) \\
&+ e^{-r(T-t)} \sum_{k=1}^{j} (-\hat{X}_{(k)} - K)^+ \binom{N-n}{j-k} [F_X(\hat{X}_{(k)})]^{j-k}[1-F_X(\hat{X}_{(k)})]^{N-n-j+k},
\end{aligned}$$

$$\begin{aligned}
(22)\qquad p_\lambda(t,T,(X_i)_{t_i\le t}) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[(K+X_j)^+ | X_1, ..., X_n] \\
&= e^{-r(T-t)} \sum_{k=0\vee(j-N+n)}^{n\wedge j} \int_{[\hat{X}_{(k)}, \hat{X}_{(k+1)})} (K+x)^+ F_{\tilde{X}_{(j-k)}}(dx) \\
&+ e^{-r(T-t)} \sum_{k=1}^{j} (K+\hat{X}_{(k)})^+ \binom{N-n}{j-k} [F_X(\hat{X}_{(k)})]^{j-k}[1-F_X(\hat{X}_{(k)})]^{N-n-j+k},
\end{aligned}$$

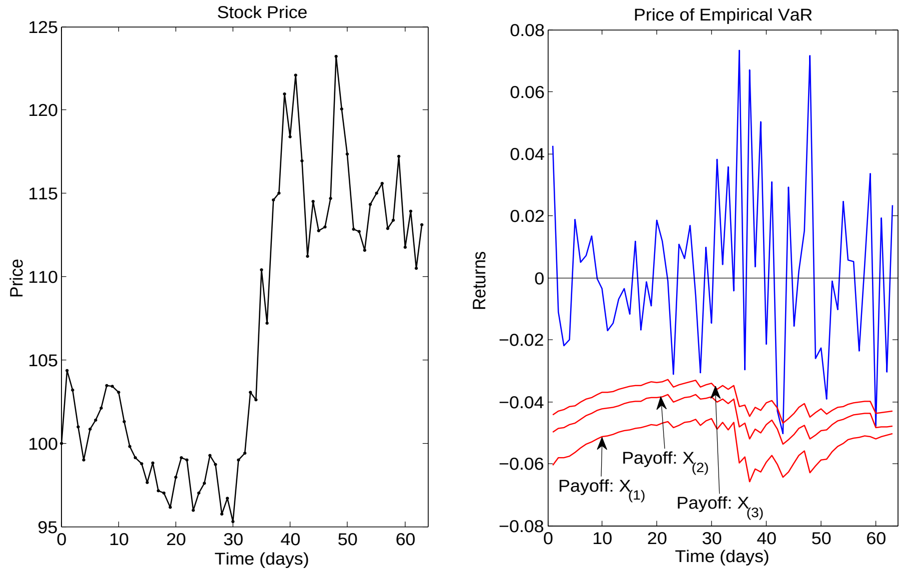

Figure 6: First graph: simulated stock price under the GARCH model. Second graph: paths of $-\rho_{(j)}(t, T, (X_i)_{t_i \le t})$, where $j = 1, 2, 3$, $T = 3$ months, and $\Delta t = 1$ day. The daily returns, X, are assumed to follow a GARCH model, $X_i = (rdt - \sigma_i^2/2) + \varepsilon_i$, $\varepsilon_i = \sigma_i e_i$, $\sigma_i^2 = \alpha_0 + \alpha_1 \varepsilon_{i-1}^2 + \beta_1 \sigma_{t_{i-1}}^2$. Parameters: interest rate $r = 1\%$, $\alpha_0 = 0.00005$, $\alpha_1 = 0.15$, $\beta_1 = 0.8$. Note the drop at day 35 of $-\rho_{(j)}(t, T, (X_i)_{t_i \le t})$, for $j = 1, 2, 3$, caused by an increased volatility while the returns are still positive.

where $t_N \le T < t_{N+1}$, $t_n \le t < t_{n+1}$, *and*

$$F_{\tilde{X}_{(j-k)}}(x) = \sum_{i=j-k}^{N-n} \binom{N-n}{i} [F_X(x)]^i [1 - F_X(x)]^{N-n-i}.$$

Furthermore, when the distribution of X_i *is continuous with probability density function* $f_X(x)$*, we can write:*

$$c_j(t, T, (X_i)_{t_i \le t}) = e^{-r(T-t)} \sum_{k=0 \vee (j-N+n)}^{n \wedge (j-1)} \int_{(\hat{X}_{(k)}, \hat{X}_{(k+1)})} (-x - K)^+ f_{\tilde{X}_{(j-k)}}(x) dx$$
$$+ e^{-r(T-t)} \sum_{k=1}^{j} (-\hat{X}_{(k)} - K)^+ \binom{N-n}{j-k} [F_X(\hat{X}_{(k)})]^{j-k} [1 - F_X(\hat{X}_{(k)})]^{N-n-j+k},$$

$$p_j(t, T, (X_i)_{t_i \le t}) = e^{-r(T-t)} \sum_{k=0 \vee (j-N+n)}^{n \wedge (j-1)} \int_{(\hat{X}_{(k)}, \hat{X}_{(k+1)})} (K + x)^+ f_{\tilde{X}_{(j-k)}}(x) dx$$
$$+ e^{-r(T-t)} \sum_{k=1}^{j} (K + \hat{X}_{(k)})^+ \binom{N-n}{j-k} [F_X(\hat{X}_{(k)})]^{j-k} [1 - F_X(\hat{X}_{(k)})]^{N-n-j+k},$$

where:

$$f_{\tilde{X}_{(j-k)}}(x) = \frac{(N-n)!}{(j-k-1)!(N-n-j+k)!} f_X(x) [F_X(x)]^{j-k-1} [1 - F_X(x)]^{N-n-j+k}.$$

Paths of processes $c_{(j)}(t, T, (X_i)_{t_i \le t})$, $j = 1, 2, 3$, where $T = 3$ months and $\Delta t = 1$ day, is shown in Figure 8.

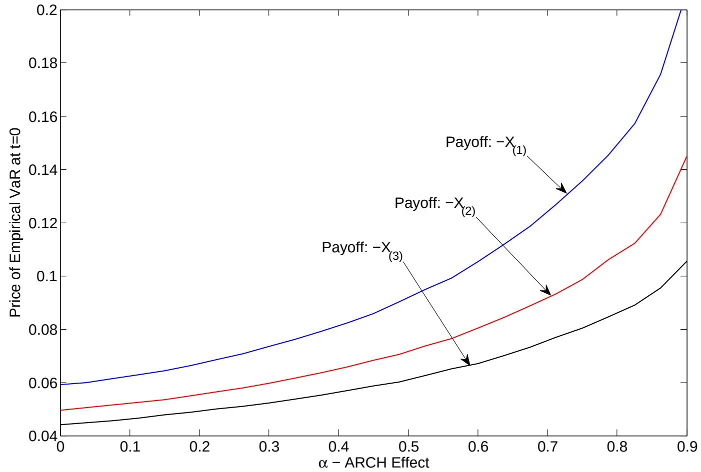

Figure 7: Dependence of $\rho_{(j)}(t,T,(X_i)_{t_i\le t})$, where $j=1,2,3$ and the returns are given by a GARCH-M model, on parameter α_1. Assumptions: $T=3$ months, and $\Delta t=1$ day. Model of the returns: $X_i=(rdt-\sigma_i^2/2)+\varepsilon_i$, $\varepsilon_i=\sigma_i e_i$, $\sigma_i^2=\alpha_0+\alpha_1\varepsilon_{i-1}^2+\beta_1\sigma_{t_{i-1}}^2$. Parameters: interest rate $r=1\%$ and $\alpha_0=0.00063$.

4 Convergence Theorems

Let us discuss the convergence of a Price of Return Risk, $\rho(t,T,(X_i)_{t_i\le t})$, if we let the time of maturity $T \to \infty$. The number of future returns increases and the expected discrete payoff, $\mathbb{E}^{\mathbb{Q}}\left[RR_N((X_i)_{0<i\le N})|X_1,...,X_n\right]$, approaches the continuous payoff of which $RR_N((X_i)_{0<i\le N})$ is an estimate. The continuous payoff is in fact the Weighted VaR (see Appendix A). We define the $\lambda-$quantile of a distribution with cumulative distribution function $F_X(x)$ as:

$$q_\lambda(X)=\sup\{x: F_X(x)<\lambda\},\quad 0<\lambda<1. \tag{23}$$

We see from (30) in Appendix A that the VaR is simply defined as the negative value of the above $\lambda-$quantile function:

$$VaR_\lambda(X)=-q_\lambda(X)=\inf\{\, m \,|\, \mathbb{P}(X+m\le 0)<\lambda\,\},$$

where X has distribution $F_X(x)$.

Theorem 4.1 *Suppose that returns $\{X_1,\dots,X_N\}$ are independent and identically distributed with a continuous and increasing cumulative distribution function $F_X(x)$ and a density function $f_X(x)$, where $0<f_X(x)<\infty$, $x\in\mathbb{R}$.*

(i) If $\lambda\in(0,1)$, $\int_0^1 VaR_\lambda(X)d\lambda<\infty$ and $RR_N((X_i)_{0<i\le N})=-X_{(\lfloor N\lambda\rfloor)}$, then:

$$e^{rT}\rho(t,T)\longrightarrow VaR_\lambda(X)\ \text{almost surely as}\ T\to\infty,\ \text{for any}\ t\ge 0. \tag{24}$$

(ii) Let $\psi(\lambda)$ be a nonnegative, bounded, and continuous function on $[0,1]$, such that $\int_0^1\psi(\lambda)d\lambda=1$ and $\int_0^1 VaR_\lambda(X)\psi(\lambda)d\lambda<\infty$. If $RR_N((X_i)_{0<i\le N})=-\sum_{i=1}^N w_iX_{(i)}$ and the weights are given as:

$$w_i=\frac{1}{s_N N}\psi\left(\frac{i-1}{N-1}\right)\quad \text{for}\quad i=1,\dots,N,\quad \text{where}\quad s_N=\frac{1}{N}\sum_{i=1}^N\psi\left(\frac{i-1}{N-1}\right),$$

then:

$$e^{rT}\rho(t,T)\longrightarrow\int_0^1 VaR_\lambda(X)\psi(\lambda)d\lambda\ \text{almost surely as}\ T\to\infty,\ \text{for any}\ t\ge 0. \tag{25}$$

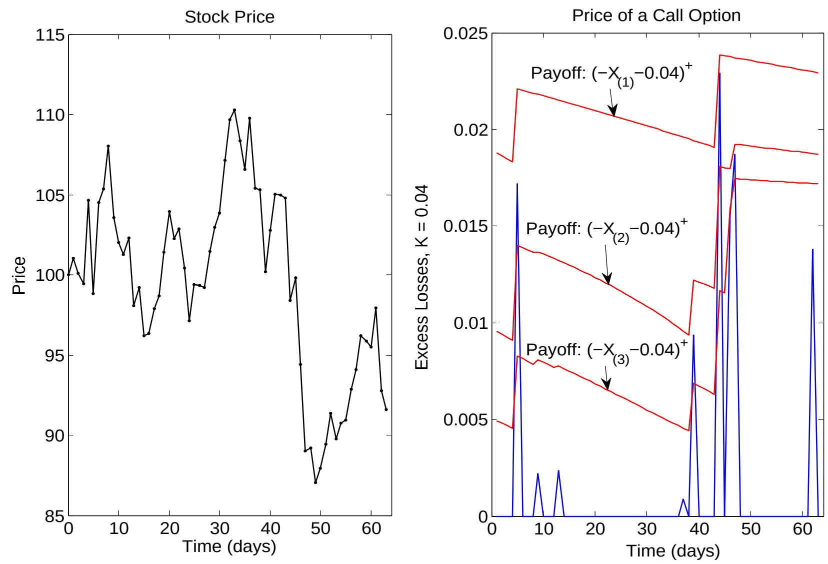

Figure 8: First graph: simulated stock price under the Black-Scholes model. Second graph: paths of $c_{(j)}(t, T, (X_i)_{t_i \le t})$, $j = 1, 2, 3$, where $T = 3$ months, $\Delta t = 1$ day, and $K = 4\%$. The daily returns, X, are assumed to follow the Black-Scholes model with interest rate $r = 1\%$ and annual volatility $\sigma = 40\%$.

PROOF:

(i) According to David and Nagaraja [11], result (10.2.1) on page 285,

$$X_{(\lfloor N\lambda \rfloor)} \to -VaR_\lambda(X) \text{ almost surely as } N \to \infty.$$

In order to prove $\mathbb{E}^{\mathbb{Q}}[X_{(\lfloor N\lambda \rfloor)}|\mathcal{F}_t] \to -VaR_\lambda(X)$, which is equivalet to formula (24), it suffices to show a conditional uniform integrability of $|X_{(\lfloor N\lambda \rfloor)}|$:

$$\lim_{\alpha\to\infty} \sup_N \mathbb{E}^{\mathbb{Q}}[\, |X_{(\lfloor N\lambda \rfloor)}| \, \mathbb{I}_{\{|X_{(\lfloor N\lambda \rfloor)}|>\alpha\}} \, |\mathcal{F}_t] \; = \; 0 \text{ almost surely} \tag{26}$$

Let $x_1, \ldots, x_n$ be the observed returns up to time t and $f_{X_{(\lfloor N\lambda \rfloor)}}(x|x_1, \ldots, x_n)$, $x \in \mathbb{R}$ the conditional density of the empirical quantile $X_{(\lfloor N\lambda \rfloor)}$ given $X_1, \ldots, X_n$. Since the returns are assumed to have a density f_X, the conditional density exists and is given by (11). Hence, the left-hand side of (26) can be rewritten in terms of the conditional density:

$$\lim_{\alpha\to\infty} \sup_N \int_{|x|>\alpha} |x| \; f_{X_{(\lfloor N\lambda \rfloor)}}(x|x_1, \ldots, x_n) \; dx.$$

Let $x_{min} = \min\{x_1, \ldots, x_n\}$ and $x_{max} = \max\{x_1, \ldots, x_n\}$. If $\alpha > \max\{|x_{min}|, |x_{max}|\}$ and $N > N_0$ for some N_0, then formula (11) implies that

$$\begin{aligned} f_{X_{(\lfloor N\lambda \rfloor)}}(x|x_1, \ldots, x_n) &= \frac{(N-n)!}{(\lfloor N\lambda \rfloor - n - 1)!(N - \lfloor N\lambda \rfloor)!} f_X(x)[F_X(x)]^{\lfloor N\lambda \rfloor - n - 1}[1 - F_X(x)]^{N - \lfloor N\lambda \rfloor} \\ &= f_{X_{(\lfloor N\lambda \rfloor)}}(x) \; \frac{(N-n)!(\lfloor N\lambda \rfloor - 1)!}{N!(\lfloor N\lambda \rfloor - n - 1)!} \; [F_X(x)]^{-n} \end{aligned}$$

for $x \in (\alpha, \infty)$ and

$$\begin{aligned} f_{X_{(\lfloor N\lambda \rfloor)}}(x|x_1,\dots,x_n) &= \frac{(N-n)!}{(\lfloor N\lambda \rfloor -1)!(N-n-\lfloor N\lambda \rfloor)!} f_X(x)[F_X(x)]^{\lfloor N\lambda \rfloor -1}[1-F_X(x)]^{N-n-\lfloor N\lambda \rfloor} \\ &= f_{X_{(\lfloor N\lambda \rfloor)}}(x)\ \frac{(N-n)!(N-\lfloor N\lambda \rfloor)!}{N!(N-n-\lfloor N\lambda \rfloor)!}\ [1-F_X(x)]^{-n} \end{aligned}$$

for $x \in (-\infty, -\alpha)$. Note that since $F_X(x)$ is increasing and sequences $\frac{(N-n)!(\lfloor N\lambda \rfloor -1)!}{N!(\lfloor N\lambda \rfloor -n-1)!}$ and $\frac{(N-n)!(N-\lfloor N\lambda \rfloor)!}{N!(N-n-\lfloor N\lambda \rfloor)!}$ converge to a finite number, one can write:

$$f_{X_{(\lfloor N\lambda \rfloor)}}(x|x_1,\dots,x_n) \le C\ f_{X_{(\lfloor N\lambda \rfloor)}}(x)$$

for $|x| > \alpha$, where C is a constant. According to David and Nagaraja [11], formula (10.2.6) on page 288, $E\left[X_{(\lfloor N\lambda \rfloor)} - VaR_\lambda(X)\right]^2 \longrightarrow 0$ as $N \to \infty$, therefore random variables $\{X_{(\lfloor N\lambda \rfloor)}\}_{N=1}^{\infty}$ with densities $\{f_{X_{(\lfloor N\lambda \rfloor)}}(x)\}_{N=1}^{\infty}$ are uniformly integrable. Hence, we have:

$$\lim_{\alpha\to\infty} \sup_N \int_{|x|>\alpha} |x|\ f_{X_{(\lfloor N\lambda \rfloor)}}(x|x_1,\dots,x_n)\ dx \ \le\ C \lim_{\alpha\to\infty} \sup_N \int_{|x|>\alpha} |x|\ f_{X_{(\lfloor N\lambda \rfloor)}}(x)\ dx\ =\ 0$$

for any real $x_1, ..., x_n$. This result concludes the proof.

(ii) Stigler [25] presented a proof of (25) for $t = 0$, the unconditional case. The conditional version of the formula will be proved in a similar way. Let $X_i = x_i$, $i = 1, \dots, n$, be the past returns observed at times $t_1, \dots, t_n$ where $t_n \le t < t_{n+1}$. Assume that $n \ge 1$ because $n = 0$ represents the unconditional case proved in Stigler [25]. Let us order the values $x_1, \dots, x_n$ and denote $x_0 = -\infty$ and $x_{n+1} = \infty$. Without loss of generality, it can be assumed that one of the past returns is $0 : x_j = 0$. If this is not the case, the distribution F_X and the observed returns can be shifted by $(0 - x_j)$ for some j. At the end of the calculation, the inverse transformation is applied and the result still hold true. The order statistics of the future returns $X_{n+1}, ..., X_N$ will be denoted by $\tilde{X}_{(1)}, ..., \tilde{X}_{(N-n)}$.

Term $-e^{rT}\rho(t,T)$ can be written in the following way:

$$\begin{aligned} -e^{rT}\rho(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{N} w_i X_{(i)} \middle| \mathcal{F}_t\right] = \\ &= -\int_{-\infty}^{0} \left\{\sum_{i=1}^{N} w_i \mathbb{Q}[X_{(i)} \le x|\mathcal{F}_t]\right\} dx + \int_0^{\infty} \left\{\sum_{i=1}^{N} w_i \mathbb{Q}[X_{(i)} > x|\mathcal{F}_t]\right\} dx = \\ &= -\sum_{k=0}^{j-1} \int_{x_k}^{x_{k+1}} \left\{\sum_{i=1}^{N} w_i \mathbb{Q}[X_{(i)} \le x|\mathcal{F}_t]\right\} dx + \sum_{k=j}^{n} \int_{x_k}^{x_{k+1}} \left\{\sum_{i=1}^{N} w_i \mathbb{Q}[X_{(i)} > x|\mathcal{F}_t]\right\} dx \end{aligned}$$

In order to calculate the limit $\lim_{N\to\infty} \sum_{i=1}^{N} w_i \mathbb{Q}[X_{(i)} > x|\mathcal{F}_t]$ for $x \in [x_k, x_{k+1})$, where $k = 0, \dots, n$, we consider a sequence of random variables $\{Y_N\}_{N=1}^{\infty}$ with the following conditional distribution on $\{\frac{0}{N-1}, \frac{1}{N-1}, ..., \frac{N-1}{N-1}\}_{N=1}^{\infty}$:

$$\mathbb{Q}\left[Y_N = \frac{i-1}{N-1} \middle| \mathcal{F}_t\right] = \frac{1}{c_N} \mathbb{Q}[X_{(i)} > x|\mathcal{F}_t], \quad i = 1, ..., N.$$

Constant c_N ensures that $\sum_{i=1}^{N} \mathbb{Q}\left[Y_N = \frac{i-1}{N-1}\middle| \mathcal{F}_t\right] = 1$:

$$\begin{aligned}
\sum_{i=1}^{N} \mathbb{Q}\left[Y_N = \tfrac{i-1}{N-1}|\mathcal{F}_t\right] &= \frac{1}{c_N}\sum_{i=1}^{N} \mathbb{Q}[X_{(i)} > x|\mathcal{F}_t] = \frac{1}{c_N}\mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{N} \mathbb{I}_{\{X_{(i)}>x\}}\middle| \mathcal{F}_t\right] \\
&= \frac{1}{c_N}\mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{N} \mathbb{I}_{\{X_i>x\}}\middle| \mathcal{F}_t\right] = \frac{1}{c_N}\left((n-k) + \sum_{i=n+1}^{N} \mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{\{\tilde{X}_i>x\}}\right]\right) \\
&= \frac{1}{c_N}((n-k) + (N-n)(1-F_X(x)))
\end{aligned}$$

Thus,

$$c_N = ((n-k) + (N-n)(1-F_X(x))) \quad \text{and} \quad \lim_{N\to\infty} \frac{N}{c_N} = \frac{1}{1-F_X(x)}.$$

Formula (9) implies that $\mathbb{Q}[X_{(i)} > x|\mathcal{F}_t]$ has three possible values:

$$\begin{aligned}
\mathbb{Q}[X_{(i)} > x|\mathcal{F}_t] &= 0 \quad \text{for} \quad i = 1, ..., k, \\
\mathbb{Q}[X_{(i)} > x|\mathcal{F}_t] &= 1 - F_{\tilde{X}_{(i-k)}}(x) \quad \text{for} \quad i = k, ..., N-n+k, \\
\mathbb{Q}[X_{(i)} > x|\mathcal{F}_t] &= 1 \quad \text{for} \quad i = N-n+k+1, ..., N.
\end{aligned}$$

Let $\theta \in (0,1)$ and N large enough, so that $\lceil\theta(N-1)\rceil + 1 > k$ where $\lceil r\rceil$ is the lowest integer greater than r. Then,

$$\mathbb{Q}\left[Y_N > \theta|\mathcal{F}_t\right] = \frac{1}{c_N}\sum_{i=\lceil\theta(N-1)\rceil+1}^{N} \mathbb{Q}[X_{(i)} > x|\mathcal{F}_t] = \frac{1}{c_N}\sum_{i=\lceil\theta(N-1)\rceil+1}^{N-n+k} \left(1 - F_{\tilde{X}_{(i-k)}}(x)\right) + \frac{n-k}{c_N}$$

According to (5), $1-F_{\tilde{X}_{(i-k)}}(x)$ is the probability that a Binomial random variable, $Bin(N-n, F_X(x))$, is less then or equal to $i-k-1$, which leads to the following convergence result:

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{Q}\left[Y_N > \theta|\mathcal{F}_t\right] &= \lim_{N\to\infty} \frac{N}{c_N}\frac{1}{N}\sum_{i=\lceil\theta(N-1)\rceil+1}^{N-n+k} \left(1 - F_{\tilde{X}_{(i-k)}}(x)\right) + \lim_{N\to\infty}\frac{n-k}{c_N} \\
&= \frac{1}{1-F_X(x)}\lim_{N\to\infty}\frac{1}{N}\sum_{i=\lceil\theta(N-1)\rceil+1}^{N-n+k} \mathbb{Q}\left[Bin(N-n, F_X(x)) \le i-k-1\right] + 0 \\
&= \frac{1-\max\{F_X(x), \theta\}}{1-F_X(x)}.
\end{aligned}$$

Hence, Y_N converges weakly to a uniform distribution on $(F_X(x), 1)$ even if we condition on $\mathcal{F}_t$. Since $w_i = \frac{1}{s_N N}\psi\left(\frac{i-1}{N-1}\right)$, we have $\sum_{i=1}^{N} w_i\mathbb{Q}[X_{(i)} > x|\mathcal{F}_t] = \frac{c_N}{Ns_N}\mathbb{E}^{\mathbb{Q}}\left[\psi(Y_N)|\, \mathcal{F}_t\right]$. Due to the assumption that ψ is continuous and bounded, the limit of this term is :

$$\begin{aligned}
\lim_{N\to\infty}\sum_{i=1}^{N} w_i\mathbb{Q}[X_{(i)} > x|\mathcal{F}_t] &= \lim_{N\to\infty}\frac{c_N}{Ns_N}\lim_{N\to\infty}\mathbb{E}^{\mathbb{Q}}\left[\psi(Y_N)|\, \mathcal{F}_t\right] \\
&= (1-F_X(x))\int_{F_X(x)}^{1} \psi(y)\frac{1}{(1-F_X(x))}dy = \int_{F_X(x)}^{1}\psi(y)dy.
\end{aligned}$$

Moreover,

$$\lim_{N\to\infty}\sum_{i=1}^{N} w_i\mathbb{Q}[X_{(i)} \le x|\mathcal{F}_t] = 1 - \lim_{N\to\infty}\sum_{i=1}^{N} w_i\mathbb{Q}[X_{(i)} > x|\mathcal{F}_t] = \int_0^{F_X(x)}\psi(y)dy.$$

Thus, the limit of the Price of Return Risk, $\rho(t,T)$, can be expressed as follows:

$$\begin{aligned}
&\lim_{T\to\infty} -e^{rT}\rho(t,T) = \\
&= -\lim_{N\to\infty}\sum_{k=0}^{j-1}\int_{x_k}^{x_{k+1}}\left\{\sum_{i=1}^{N} w_i\mathbb{Q}[X_{(i)}\le x|\mathcal{F}_t]\right\}dx + \lim_{N\to\infty}\sum_{k=j}^{n}\int_{x_k}^{x_{k+1}}\left\{\sum_{i=1}^{N} w_i\mathbb{Q}[X_{(i)}> x|\mathcal{F}_t]\right\}dx \\
&= -\int_{-\infty}^{0}\int_{0}^{F_X(x)}\psi(y)dydx + \int_{0}^{\infty}\int_{F_X(x)}^{1}\psi(y)dydx = \int_0^1 F_X^{-1}(y)\psi(y)dy,
\end{aligned}$$

where $F_X^{-1}(y) = -VaR_y(X)$, which completes the proof.

$\diamond$

Another feature of a Price of Return Risk is that $e^{r(T-t)}\rho(t,T,(X_i)_{t_i\le t})$ becomes a better estimate of payoff $RR_N((X_i)_{0<i\le N})$ as t converges to the time of maturity T. This observation is stated and proved in the following theorem.

Theorem 4.2 *Suppose that $\{X_1, X_2, ..., X_N\}$ are independent and identically distributed random variables with finite variance. Let us define function $h(t)$, $t\in[0,T]$ as follows:*

$$h(t) = \mathbb{E}^{\mathbb{Q}}\left[e^{r(T-t)}\rho(t,T,(X_i)_{t_i\le t}) - RR_N((X_i)_{0<i\le N})\right]^2, \tag{27}$$

where $RR_N((X_i)_{0<i\le N}) = -\sum_{i=1}^N w_iX_{(i)}$. Then $h(t)$ is a nonnegative decreasing function of t and $h(T)=0$.

Proof. Conditional expectations are L^2 projections of a random variable and therefore the result is not hard to understand. For completeness, we provide a proof below. Conditional expectation $\mathbb{E}^{\mathbb{Q}}[.|X_1,\dots,X_n]$ will be denoted by $\mathbb{E}_n^{\mathbb{Q}}[.]$ in this proof. Note that if $t_n\le t<t_{n+1}$, then $e^{r(T-t)}\rho(t,T,(X_i)_{t_i\le t}) = -\mathbb{E}_n^{\mathbb{Q}}\left[\sum_{i=1}^N w_iX_{(i)}\right]$. The following inequality proves claim (27):

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[-\mathbb{E}_n^{\mathbb{Q}}\left[\sum_{i=1}^N w_iX_{(i)}\right]+\sum_{i=1}^N w_iX_{(i)}\right]^2 &= \mathbb{E}^{\mathbb{Q}}\left[-\mathbb{E}_n^{\mathbb{Q}}\left[\sum_{i=1}^N w_iX_{(i)}\right]+\mathbb{E}_{n+1}^{\mathbb{Q}}\left[\sum_{i=1}^N w_iX_{(i)}\right]\right]^2 \\
&\quad +\mathbb{E}^{\mathbb{Q}}\left[-\mathbb{E}_{n+1}^{\mathbb{Q}}\left[\sum_{i=1}^N w_iX_{(i)}\right]+\sum_{i=1}^N w_iX_{(i)}\right]^2 \\
&\ge \mathbb{E}^{\mathbb{Q}}\left[-\mathbb{E}_{n+1}^{\mathbb{Q}}\left[\sum_{i=1}^N w_iX_{(i)}\right]+\sum_{i=1}^N w_iX_{(i)}\right]^2.
\end{aligned}$$

$\diamond$

5 Price of Return Risk as a Dynamic Risk Measure

In Section 3, we studied the properties of a Price of Return Risk $\rho(t,T,(X_i)_{t_i\le t})$ as the forward price process on the Weighted Average of Ordered Returns. As t changes, the definition of the conditional expectation in (1) makes the Price of Return Risk, $\rho(t,T,(X_i)_{t_i\le t})$, particularly well suited to serve as a dynamic measure of risk. In fact, the adoption of the option pricing approach implies its discounted value is naturally a martingale. Representation theorems for Dynamic Risk Measures that satisfy either coherent or convex principles have been proven in Artzner et al. [5], Riedel [23], Cheridito et al. [8], Frittelli and Scandolo [15], Klöppel and Schweizer [18], and Weber [29]. We will focus on listing some usual desirable properties of a Dynamic Risk Measure in Definition 5.1 and stating the properties which $\rho(t,T,(X_i)_{t_i\le t})$ satisfies in Theorem 5.2. For some of the properties, we will need to assume that the weights are decreasing: $w_1\ge\ldots\ge w_N$.

Definition 5.1 (Coherence, Relevance, and Time Consistency) *Suppose X_t and Y_t are stochastic processes for $t \in [0,T]$, and $\rho(t,T,X)$ and $\rho(t,T,Y)$ are dynamic risk measures on X_t and Y_t. We define the following axioms for ρ:*

1. *Monotonicity: $X_t \leq Y_t$ for all $t \in [0,T]$ implies $\rho(t,T,X) \geq \rho(t,T,Y)$ for all $t \in [0,T]$.*
2. *Subadditivity: $\rho(t,T,X+Y) \leq \rho(t,T,X) + \rho(t,T,Y)$ for all $t \in [0,T]$.*
3. *Positive Homogeneity: $\rho(t,T,\lambda X) = \lambda \rho(t,T,X)$ for any real number $\lambda \geq 0$ and for all $t \in [0,T]$.*
4. *Predictable Translation Invariance: fix $t \in [0,T]$, $\rho(t,T,X+Z) = \rho(t,T,X) - Z$ for any $\mathcal{F}_t$-measurable random variable Z.*
5. *Relevance: fix $u \in [0,T]$, if $X_s(\omega) = -\epsilon \mathbb{I}_A(\omega)\mathbb{I}_{[u,T]}(s)$ for some $A \in \mathcal{F}_u$ where $\mathbb{Q}(A) > 0$, and all $\epsilon > 0$, then $\rho(t,T,X) > 0$ for all $t \in [0,u)$.*
6. *Time Consistency: $e^{-rs}\rho(s,T,X) = \mathbb{E}^{\mathbb{Q}}\left[e^{-rt}\rho(t,T,X)|\mathcal{F}_s\right]$ for $0 \leq s \leq t \leq T$.*

Properties 1-4 are axioms of Coherent Measures of Risk; Relevance is related to the no-arbitrage principle; Time Consistency makes it convenient to implement dynamic programming methods.

Theorem 5.2 (Coherence, Relevance, and Time Consistency) *Let $\rho(t,T,(X_i)_{t_i \leq t})$ be a Price of Return Risk defined by*

$$\rho(t,T,(X_i)_{t_i \leq t}) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[RR_N((X_i)_{1\leq i \leq N})|\mathcal{F}_t] \tag{28}$$

with

$$RR_N((X_i)_{1\leq i\leq N}) = -\sum_{i=1}^{N} w_i X_{(i)}, \quad w_1 \geq w_2 \geq ... \geq w_N \geq 0, \text{ and } \sum_{i=1}^{N} w_i = 1. \tag{29}$$

Then $\rho(t,T,(X_i)_{t_i \leq t})$ satisfies the axioms of Monotonicity, Subadditivity, Positive homogeneity, Predictable Translation Invariance, Relevance, and Time Consistency.

Proof. The results are straightforward consequences of the definition of $\rho(t,T,(X_i)_{t_i \leq t})$ as the conditional expectation of weighted order statistics. The assumption about decreasing weights is a sufficient condition for axioms of Relevance, and Subadditivity (the proof can be found in Heyde et al. [16]). We choose to show only Relevance here (according to item 5 in Definition 5.1, $X \leq 0$):

$$\begin{aligned}\rho(t,T,(X_i)_{t_i \leq t}) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[RR_N((X_i)_{1\leq i\leq N})|\mathcal{F}_t] = -e^{-r(T-t)}\sum_{i=1}^{N} w_i \mathbb{E}^{\mathbb{Q}}[X_{(i)}|\mathcal{F}_t] \\ &\geq -e^{-r(T-t)} w_1 \mathbb{E}^{\mathbb{Q}}[X_{(1)}|\mathcal{F}_t] = e^{-r(T-t)} w_1 \epsilon \mathbb{Q}(A) > 0.\end{aligned}$$

◇

Note that $\rho(t,T,(X_i)_{t_i \leq t})$ satisfies all the properties mentioned in Definition 5.1.

6 Conclusion

We have considered the effect of significant negative returns on a given portfolio. Series of significant negative returns may cause a downgrade in credit rating, or even a default of the portfolio owner. We studied the price of Weighted Average of Ordered Returns, and computed it under different assumptions. Specifically, we have considered models with both dependent and independent returns, and with and without jumps in the price of the underlying asset.

A Realized Return Risk Measure Based on Weighted VaR

In this Section, we will assume that Realized Return Risk $g((X_i)_{1\leq i\leq N})$ is given by Weighted Average of Ordered Returns with decreasing weights: $w_1 \geq \ldots \geq w_N$. Let us relate formula (4) for g to Weighted VaR, noting that the former is the discrete approximation of the latter. Weighted VaR (see Wang et al. [28], Föllmer and Schied [14], Cherny [9]) is in fact a probabilistically distorted Conditional VaR (see Acerbi et al. [2], Rockafellar and Uryasev [24]), and it is the only possible form of law-invariant, comonotonic, convex risk measures on L^∞ on an atomless probability space (see Kusuoka [19]). $VaR_\lambda(X)$ is by definition the negative value of the λ-quantile of a random variable X defined in (23):

$$VaR_\lambda(X) = -q_X(\lambda) = \inf\{\, m \,|\, \mathbb{P}(X+m\leq 0) < \lambda\,\}. \tag{30}$$

Averaging VaR, we get Conditional VaR:

$$CVaR_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda VaR_\gamma(X)d\gamma, \quad \text{for } \lambda\in(0,1].$$

Suppose μ is a probability measure on $(0,1]$ and the probability space $(\Omega,\mathcal{F},\mathbb{P})$ is atomless. Weighted VaR is a distortion of Conditional VaR by probability measure μ:

$$WVaR_\mu(X) = \int_{(0,1]} CVaR_\gamma(X)\mu(d\gamma).$$

We can define $CVaR_0(X) = -\text{essinf } X$ to extend the definition of μ on the closed interval $[0,1]$. However, the essinf X will be captured if μ assigns positive probability on the point 0. In case X is unbounded below, this point will overweight the rest of the distribution of X. Therefore, we avoid this situation here by excluding point 0. Fubini's theorem gives:

$$WVaR_\mu(X) = \int_0^1 VaR_\gamma(X)\psi(\gamma)d\gamma, \quad \text{where } \psi(\gamma) = \int_{(\gamma,1]}\frac{1}{\alpha}\mu(d\alpha). \tag{31}$$

It is obvious that $\psi(\gamma)$ is a decreasing, right-continuous function. If we define $\nu(\lambda) = \int_0^\lambda \psi(\gamma)d\gamma$, then $\nu(\lambda)$ is increasing and concave with $\nu(0)=0$ and $\nu(1)=1$. Thus $\nu(\lambda)$ can be viewed as a distribution function with density function $\psi(\gamma)$. The discrete approximation of the $WVaR_\mu(X)$ from N statistical observations becomes our payoff function (4) with decreasing weights:

$$RR_N((X_i)_{1\leq i\leq N}) = -\sum_{i=1}^N w_i X_{(i)}, \quad \text{where } w_1\geq w_2\geq \ldots \geq w_N \geq 0 \text{ and } \sum_{i=1}^N w_i = 1.$$

The convergence from the discrete to the continuous case is given in Theorem 4.1. For additional justification from an axiomatic approach in finite probability space see Heyde et al. [16].

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