

# Active Management of the Maximum Drawdown

Jan Vecer, Columbia University, Department of Statistics, New York, NY 10027, USA

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## Abstract

Risk management of drawdowns and portfolio optimization with drawdown constraints is becoming increasingly important among practitioners. In this paper, we introduce new techniques how to control the maximum drawdown (MDD). One can view the maximum drawdown as a contingent claim, and price and hedge it accordingly as a derivative contract. Trading drawdown contracts or replicating them by hedging would directly address the concerns of portfolio managers who would like to control them. Similar contracts can be written on the maximum drawup (MDU). We show that buying a contract on MDD or MDU is equivalent to adopting a momentum trading strategy, while selling it corresponds to contrarian trading. We also discuss more complex products, such as plain vanilla options on the maximum drawdown or drawup, or related barrier options on MDD and MDU which we call crash and rally options, respectively.

## 1 Drawdowns and Drawups

Suppose that we have an underlying asset whose price process at time  $t$  is given by  $S_t$ . For example, the price process could be a stock price, index, interest rate or exchange rate. Denote by  $M_t$  its running maximum up to time  $t$ :

$$M_t = \max_{u \in [0, t]} S_u.$$

*Drawdown*  $D_t$  is defined as the drop of the asset price from its running maximum:

$$(1) \quad D_t = M_t - S_t.$$

*Maximum drawdown*  $MDD_t$  is defined as the maximal drop of the asset price from its running maximum over a given period of time:

$$(2) \quad MDD_t = \max_{u \in [0, t]} D_u.$$

Assume that you have an investor who enters the market at a certain point and leave it at some following point within a given time period. Maximum drawdown measures the worst loss of such an investor, meaning that he buys the asset at a local maximum and sells it at the subsequent lowest point, and this drop is the largest in the given time period. This represents the worst period for holding this asset and could be written mathematically as:

$$(3) \quad MDD_t = \sup_{\substack{q_u \in \{0, 1\} \\ \text{one switch } (0 \rightarrow 1) \text{ and } (1 \rightarrow 0)}} \int_0^t (-q_u) dS_u.$$

Similarly, we can define the concepts of drawup, and maximum drawup. *Drawup*  $U_t$  is defined as the increase of the asset price from its running minimum:

$$(4) \quad U_t = S_t - m_t,$$

where

$$m_t = \min_{u \in [0, t]} S_u.$$

Maximum drawup  $MDU_t$  is given by

$$(5) \quad MDU_t = \max_{u \in [0, t]} U_u.$$

Figure 1 illustrates the concepts of drawdown and drawup on data taken from S&P500 for the year 2005 with daily monitoring of the closing values. More frequent monitoring is also possible.

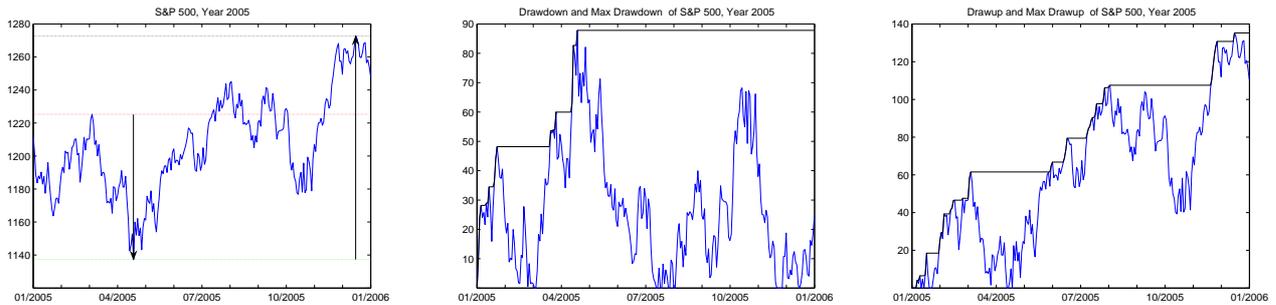


Figure 1: Left: S&P500 daily closing values in 2005. The downward arrow marks the maximum drawdown, the upward arrow marks the maximum drawup. Center: Drawdown (blue) and maximum drawdown (black) of S&P500 in 2005. Right: Drawup (blue) and maximum drawup (black) of S&P500 in 2005.

Maximum drawdown has been extensively studied in the recent literature. Risk measures based on the maximum drawdown can serve as an alternative to the commonly used Value-at-Risk. Portfolio optimization using the drawdown is considered in Chekhlov, Uryasev and Zabarkin (2005). Analytical results linking the maximum drawdown to the mean return appeared in the paper of Magdon-Ismail and Atiya (2004).

It is important to stress that the risk measures which depend on the maximum drawdown are directly observable at the end of the given period because they are functions of the data, and thus could be traded. In contrast, Value at Risk, or recently proposed Coherent Risk Measures (Artzner, Delbaen, Eber, Heath (1999)) do not give quantities which could be traded.

The price of a futures contract on maximum drawdown can serve as an important risk measure indicator which could be quoted by the market (rather than determined internally). When the market is in a bubble, it is reasonable to expect that the prices of drawdown contracts would be significantly higher. On the other hand, when the market is stable, or when it exhibits mean reversion behavior, the prices of drawdown contracts would become cheaper.

The volume of drawdown trading can be relatively small (or the contract could be replicated by adopting the corresponding hedging strategy), but it would still carry important information about the perception of drawdown risk associated with the given asset or index. If the market enters a bubble, the price of the drawdown contract would increase, indicating the increased risk associated with the given asset. Higher risk leads to a limited exposure to such an asset, and possibly to a fast price correction. Transparent quoting of maximum drawdown prices could potentially lead to more stable markets without long periods of bubbles followed by significant crashes because the market would adapt through small and short corrections rather than large and extended drops.

Currently traded contracts have only limited ability to insure the market drawdowns. For instance, deep out of the money puts are optimal to buy when the market reaches its maximum, otherwise they lose their

value. This happens when the market goes up before it drops down. Thus timing is very important for put options. On the other hand, maximum drawdown is not sensitive to timing, the contract delivers the largest drop no matter when this drop happens.

## 2 Pricing and Hedging

As for the pricing, the value  $v(t, S_t, M_t, MDD_t)$  of any type of contract that depends on the maximum drawdown is given by taking the conditional expectation of the discounted payoff under the risk neutral measure:

$$v(t, S_t, M_t, MDD_t) = \mathbb{E}[e^{-(T-t)} f(\{MDD_u\}_{u=0}^T) | S_t, M_t, MDD_t].$$

A similar formula applies for the pricing of contracts that depend on the maximum drawup,  $MDU_t$ . Here, the function  $f$  determines the type of payoff defined by the contract (for instance,  $f(\{MDD_t\}_{t=0}^T) = MDD_T - K$  for the forward contract, etc.). For the evolution of the underlying asset under the risk neutral measure, we may assume that

$$dS_t = rS_t dt + g(t, S_t) dN_t,$$

for a general martingale  $N_t$  (diffusion or jump type process).

The price can be computed by using Monte Carlo simulations. The pricing via conditional expectations can be linked to partial differential equations through the Feynman-Kac theorem (see for instance Shreve (2004)). However, the resulting PDE, which has 3 spatial dimensions is difficult to implement and slow to compute. Therefore Monte Carlo methods would be more efficient and easy to implement in this situation. Monte Carlo also allows for more complicated dynamics of the asset price, such as stochastic volatility or jumps.

The hedging strategy is given by a standard delta hedge:

$$(6) \quad \Delta(t) = v_s(t, S_t, M_t, MDD_t).$$

This computation is also straightforward if one adopts Monte Carlo simulation (see Glasserman (2004)). Figure 2 shows the rescaled price of the forward contract on the maximum drawdown  $MDD_T$  as a function of the ratio of the drawdown and the stock and the difference of the MDD and the drawdown divided by the stock, assuming the geometric Brownian motion model. It also shows the hedging strategy, which is quite intuitive. When the stock is near its maximum (drawdown is close to zero), or when the current drawdown is far from the level of the maximum drawdown, the hedging position is close to zero (limited exposure to the market). However, when the current drawdown is close to the maximum drawdown, there is the potential that a new level of the maximum drawdown will be set, and thus the hedging position is to short some fraction of the index. Numerically it seems that the hedge never gets below -1.

Notice that the hedging strategy has some of the same features of *momentum trading*. The hedger should be short when the market is setting new record lows. Similarly, the momentum trader assumes that the given trend will continue and thus takes also a short position in the falling market. Thus momentum traders are natural buyers of this contract. On the other hand, selling the (unhedged) contract is equivalent to taking the opposite strategy, namely buying the asset when it is setting its new low. This is known as the *contrarian trading*. The contrarian traders are natural sellers of this contract.

Table 1 lists realized values and theoretical prices of contracts depending on the maximum drawdown or drawup written on S&P500 in the year 2005. Notice that in that period, theoretical prices of 1 year contracts written on both drawdown and drawup exceeded the realized values. This might be due to the effect of mean reversion – stable real markets could stay within a more conservative range in the long run if compared to models based on geometric Brownian motion model. We assumed a rather small estimate of volatility

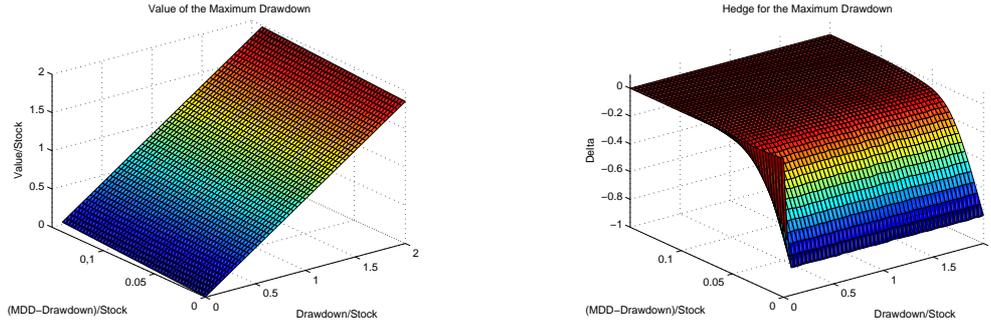


Figure 2: Left: The price of the contract with the payoff  $MDD_T$  (maximum drawdown) as a function of the ratio of the drawdown and the stock and the difference of the MDD and the drawdown divided by the stock, assuming the geometric Brownian motion model, using interest rate  $r = 0.03$ , volatility  $\sigma = 11\%$ , time to maturity  $T = 1/4$ . Right: The corresponding hedging strategy.

$\sigma = 11\%$  implied from one year options on S&P 500 at that time. This period of time would have led to significant profits for the sellers of the MDD contract and the contrarian traders in general.

| Contracts Expiring<br>on 12/30/2005, S&P500 | Realized Values |        | Theoretical Prices |        |
|---|-----------------|--------|--------------------|--------|
|   | 3 months        | 1 year | 3 months           | 1 year |
| Forward on MDD                              | 51.97           | 87.81  | 71.14              | 141.25 |
| Forward on MDU                              | 95.90           | 135.24 | 78.80              | 171.19 |

Table 1: The realized values and theoretical prices of forward contracts on the maximum drawdown, and, the maximum drawup, where the underlying index is S&P500. We assume that the 3 month contract starts on Oct 3, 2005 and expires on Dec 30, 2005; the 1 year contract starts on Jan 3, 2005 and expires on Dec 30, 2005. We use opening value 1,211.92 of the index for the one year contract, opening value 1228.81 of the index for the 3 months contract, interest rate  $r = 0.03$ , volatility  $\sigma = 11\%$ . Theoretical prices are obtained from Monte Carlo simulation, standard deviations do not exceed 0.01.

Figure 3 shows the evolution of the price of the MDD contract (daily monitoring) in year 2005 and its corresponding hedging strategy. The green line is the theoretical price at each given time, converging to the realized MDD at the end. This could be regarded as a futures contract on MDD. The red line represents the value of the hedging portfolio which was constructed by daily rebalancing using the delta hedge computed by Monte Carlo simulation. We assumed a simple geometric Brownian motion model and a constant volatility of 11%. Notice that the value of the MDD contract was declining over that period of time, giving potentially substantial returns to the sellers (contrarian traders). The hedging portfolio replicates the MDD contract remarkably closely, suggesting that MDD is quite a robust statistic which is not affected too much by fluctuations of the parameters (such as the stochastic volatility). The MDD contract is increasing as a function of volatility as are most derivative contracts.

Figure 4 shows the opposite situation, namely the largest 3 months drop of S&P 500 in the history (May – July 2002; 308.89 points dropping from 1106.59 to 797.70). The market was also more volatile, with a volatility estimate of 25%. The hedging portfolio almost perfectly replicates this event as well. Notice that the price of the MDD contract more than doubled in 3 months, giving a remarkable return to the buyer (or the momentum trader). The hedging strategy stays close to a unit short position, realizing this significant return in the falling market.

### 3 Calls, Puts and Barriers (Crash and Rally Options)

One natural step is to extend the universe of Maximum drawdown (MDD) or Maximum drawup (MDU) contracts to the usual derivative contracts, such as plain vanilla or barrier options. We may consider call or

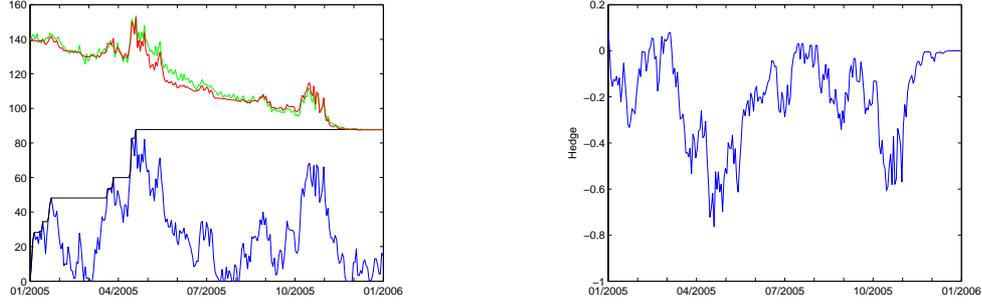


Figure 3: Left: Drawdown (blue), Maximum drawdown (black), price of the MDD contract (green) and the value of the hedging portfolio (red). The underlying asset is the index S&P500 in 2005 with opening value 1,211.92, interest rate  $r = 0.03$ , volatility  $\sigma = 11\%$ . Right: The corresponding hedge.

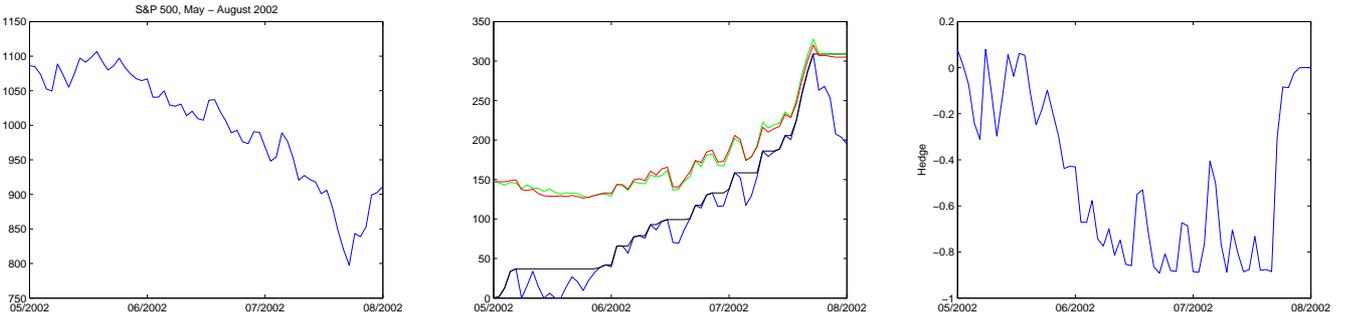


Figure 4: The worst 3 months drawdown in S&P 500 history (May – July 2002). Left: S&P 500 in the given period. Center: Drawdown (blue), Maximum drawdown (black), price of the MDD contract (green) and the value of the hedging portfolio (red). Opening value 1086.46, interest rate  $r = 0.03$ , volatility  $\sigma = 25\%$ . Right: The corresponding hedge.

put options on the MDD (MDU) with payoffs

$$(7) \quad (MDD_T - K)^+, \quad \text{or} \quad (MDU_T - K)^+$$

for the *call options*, and

$$(8) \quad (K - MDD_T)^+, \quad \text{or} \quad (K - MDU_T)^+$$

for the *put options*. Figure 5 gives the value of the one year option on the MDD as a function of the strike.

Figure 6 gives the hedging strategy for the call option on the Maximum drawdown as a function of the ratio of the drawdown and the stock and the difference of the MDD and the drawdown divided by the stock, assuming the geometric Brownian motion model. Notice that the hedge for the call is very similar to hedging the maximum drawdown itself, but it is close to zero in the region when the drawdown is below the strike. Thus all the significant hedging happens once the MDD exceeds the strike. MDD is a nondecreasing process and when the call option gets in the money, it will expire in the money.

Similarly, we can consider knock-in barrier option on MDD and MDU respectively. Let us define the event of the market crash as the first time the stock price  $S_t$  drops by a constant  $a$  from its running maximum, i.e.,

$$T_a = \inf\{t \geq 0 : D_t = M_t - S_t \geq a\} = \inf\{t \geq 0 : MDD_t \geq a\},$$

which is obviously the same event as the first time the maximum drawdown exceeds level  $a$ .

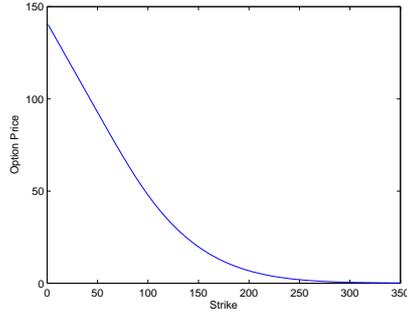


Figure 5: The price of the call option  $(MDD_1 - K)^+$  as a function of the strike  $K$ , same underlying index S&P500, the value of the option on 01/03/2005 with maturity 12/30/2005.

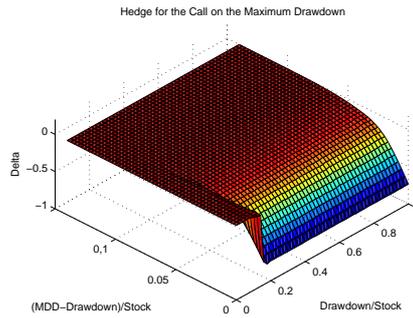


Figure 6: The hedge of the call option on MDD as a function of the ratio of the drawdown and the stock and the difference of the MDD and the drawdown divided by the stock, assuming the geometric Brownian motion model, using interest rate  $r = 0.03$ , volatility  $\sigma = 11\%$ , time to maturity  $T = 1/4$ , strike  $K = 0.15S_0$ .

A *crash option* is defined as a contract which pays off  $a$  at the time of the market crash  $T_a$  if  $T_a < T$ , where  $T$  is the maturity. Other payoffs, such as  $M_{T_a} - S_{T_a}$  ( $M_{T_a} - S_{T_a}$  could be greater than  $a$  if the market exhibits price discontinuity at the time of the crash), are possible to consider. The crash option can be regarded as a barrier option on the maximum drawdown.

The closest contract to the crash option that has been studied in the literature is the Russian option. It is a perpetual option that, at the time of the exercise, pays off the running maximum (discounted by a rate  $\alpha > r$ ) of the asset price. It turns out that in the Black-Scholes model, the optimal exercise time is the time of a drawdown of a certain size  $a$ . See, for instance, Shepp and Shiryaev (1993).

The crash option resets its holder to the historical maximum of the asset price during the lifetime of the contract at the time of the market crash. If a crash does not happen up to the time of the maturity of the contract, the option expires worthless. Since the market crash is a rather extreme event, most of the time the option will not end up in the money if we are considering a large drop. This feature will make the contract cheap. It is possible to construct a contract which would set the wealth of its holder to a different level than the running maximum at the time of the crash, for instance to the running average, or to some intermediate point between the running maximum and the crash value. However, hedging of the barrier contracts might lead to huge delta positions when the option is about to expire, and the underlying process is close to knock in. This is a general feature of similar barrier contracts.

Figure 7 plots the theoretical price of the crash option as a function of the barrier level. Notice that small drawdowns typically happen within the given time framework simply from the random nature of the price

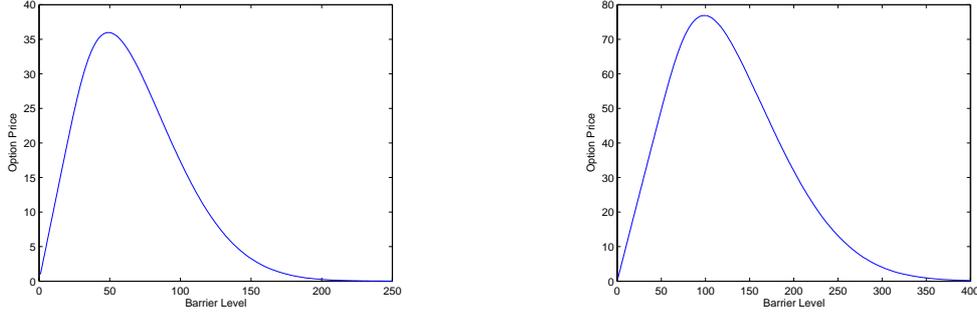


Figure 7: Price of the crash option (with daily monitoring) as a function of barrier level and maturities 3 months (left) and 1 year (right). We assume that the 3 month contract starts on Oct 3, 2005 and expires on Dec 30, 2005; the 1 year contract starts on Jan 3, 2005 and expires on Dec 30, 2005. We use opening value 1,211.92 of the index for the one year contract, opening value 1228.81 of the index for the 3 months contract, interest rate  $r = 0.03$ , volatility  $\sigma = 11\%$ .

process, making the price of crash option increase almost linearly around zero. However, drops of larger sizes become increasingly unlikely, making the corresponding option price cheaper.

A *rally option* insures the opposite event of a crash, the case of a market rally. Define the rally event in the absolute value as the first time the stock price raises by a constant  $b$  from its running minimum, i.e.,

$$T_b = \inf\{t \geq 0 : S_t - m_t \geq b\} = \inf\{t \geq 0 : MDU_t \geq b\}.$$

The rally option is defined as a contract which pays off  $b$  at the time of the rally  $T_b$  if  $T_b < T$ , where  $T$  is the maturity. Other payoffs, such as  $S_{T_b} - m_{T_b}$ , are possible to consider.

All contracts described in this paper can be also defined using the percentage (relative value) drawdown or drawup as opposed to the absolute values. For instance, we can define the *maximum relative drawdown*  $MRDD_t$  as the largest relative drop of the asset with respect to its running maximum up to time  $t$ :

$$MRDD_t = \max_{u \in [0, t]} \left(1 - \frac{S_u}{M_u}\right).$$

One can trade various options with maximum relative drawdown as the underlying process. For example, the barrier option on  $MRDD_t$  represents the relative value crash option with the payoff  $a^* \cdot M_{T_{a^*}}$  at the first time  $T_{a^*}$  when the relative drawdown exceeds  $100 \cdot a^*$  percent:

$$T_{a^*} = \inf\{t \geq 0 : MRDD_t \geq a^*\}.$$

Similarly, we can define the *maximum relative drawup* as

$$MRDU_t = \max_{u \in [0, t]} \frac{S_u}{m_u} - 1,$$

and the corresponding contracts written on  $MRDU_t$  as the underlying process.

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