Pricing Asian Options in a Semimartingale Model*

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Abstract. In this article we study arithmetic Asian options when the underlying stock is driven by special semimartingale processes. We show that the inherently path dependent problem of pricing Asian options can be transformed into a problem without path dependency in the payoff function. We also show that the price satisfies a simpler integro-differential equation in the case the stock price is driven by a process with independent increments, Lévy process being a special case.

Key words: Asian options, Special semimartingales, Lévy processes, Integro-differential equations.

1 Introduction

Asian options are securities with payoff which depends on the average of the underlying stock price S over a certain time interval. If we denote by λ the averaging factor of the option, we can write the general Asian option payoff as

$$\left(\xi \cdot \left(\int_0^T S_t d\lambda(t) - K_1 S_T - K_2\right)\right)^+. \tag{1.1}$$

When $K_1 = 0$, we have fixed strike option; when $K_2 = 0$, we have floating strike option. The constant $\xi = \pm 1$ determines whether the option is call or put. The averaging factor λ has finite variation and is typically taken to be

$$\lambda(t) = \tfrac{t}{T}$$

for the case of continuously sampled Asian options, or

$$\lambda(t) = \tfrac{1}{n} \cdot \left\lfloor \tfrac{nt}{T} \right\rfloor$$

for the case of discretely sampled Asian options. Other averaging is also possible (exponential, etc.), but less frequently used in practice. Notice that European type options are just a special case of Asian option for the following choice of parameters: $\lambda(t) = 1_{\{T\}}(t)$ and $K_1 = 0$.

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There has been a growing concern in the literature on the lognormality assumption of the underlying stock price, and a number of alternative approaches have been suggested. One of the most studied situations is the case when the stock is driven by a particular Lévy process. Carr, Geman, Madan and Yor (2000) have recently suggested the so-called CGMY model for the stock price, which shows a good match with empirical data. Another alternative approach, namely the general hyperbolic model, is discussed in Eberlein and Prause (1998).

The problem of pricing Asian options is already complicated when the underlying stock is a geometric Brownian motion. Most of the literature we know studies only this type of model. A number of approximations that produce closed form expressions have appeared, most recently in Thompson (1999), who provides tight analytical bounds for the Asian option price. Geman and Yor (1993) computed the Laplace transform of the Asian option price, and Eydeland and Geman (1995) showed how it can be related to the Fast Fourier Transform. More recently, Donato-Martin, Ghomrasni and Yor (2001) generalized Laplace transform approach for the case of continuously sampled Asian option where the underlying asset is driven by a Lévy process. This method uses equivalence of law of certain exponentials of Lévy processes. Exponentials of Lévy processes have been previously studied for instance by Carmona, Petit, Yor (2001).

Rogers and Shi (1995) have formulated a one-dimensional PDE that can model both floating and fixed strike continuous average Asian options. They apply the technique of change of numeraire introduced in Geman, El Karoui and Rochet (1995) to reduce the dimensionality of the pricing problem. Andreasen (1998) has extended this approach for pricing discretely sampled Asian option. Linetsky (2002) computed the price of continuously sampled Asian option using analytical expansion method, however this technique is limited to diffusions.

Monte Carlo methods seem to work well, but sampling the entire path of the underlying asset greatly reduces competitiveness of this approach, even with the help of variation reduction techniques (Fu, Madan, Wang (1998/99)).

In the recent paper of Večeř (2002), it was shown that one can reformulate the problem of pricing Asian options in a way which removes the inherent path dependency of the contract. This paper applies the techniques developed in Shreve and Večeř (2000) for pricing options on a traded account. The model discussed there assumes that the underlying stock is a geometric Brownian motion, in which case one can obtain a simple one-dimensional partial differential equation for the price which is easy to solve numerically. A similar formulation of the pricing partial differential equation appears in an independent work of Hoogland and Neumann (2001).

We show in this article that the approach of removing the path dependency in the formulation of the Asian option pricing problem can be generalized to the case when the underlying asset is driven by a special semimartingale process. We also show that the price satisfies an integro-differential equation in the case the stock price is driven by a process with independent increments, Lévy processes being a special case. Integro-differential equations have been previously applied in a different context for modelling of perpetuities or in the risk theory; see for instance Paulsen (1998).

2 Pricing Formula for Asian Options

Let S be a real-valued, strictly positive semimartingale on the stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ that satisfies the usual conditions. We will from now on assume $\mathbb{P}$ to be a risk-neutral measure and the interest rate to be a constant r. In particular, we assume that $e^{-rt}S_t$ is a martingale under $\mathbb{P}^{\dagger}$.

In order to reformulate the pricing problem and to remove the path dependency in Asian option valuation, we will define the Asian forward contract and use the following procedure of replicating the Asian forward payoff. Without loss of generality, we assume $\xi = 1$.

†Here we will not discuss in detail how to choose this equivalent martingale measure for pricing purpose. Interested readers are referred to Föllmer and Schweizer (1991) for the Föllmer-Schweizer minimal measure; Miyahara (2001) and Frittelli (2000) for the minimal entropy martingale measure; Bellini and Frittelli (1998) for the minimax measure; Goll and Rüschendorf (2001) for the minimal distance martingale measures; Elliott, Hunter, Kopp and Madan (1995) for the equivalent martingale measure resulting from multiplicative decomposition; Gerber and Shiu (1994) for Esscher transform. A nice presentation of earlier results for geometric Lévy processes can be found in Chan (1999).

Definition 2.1 (Asian forward contract) *An* Asian forward contract *has the following payoff at maturity date T:*

$$\int_0^T S_t d\lambda(t) - K, \tag{2.1}$$

where S_t is the underlying process, $\lambda(t)$ is the averaging factor, and K is a constant.

The difference between Asian forward and Asian option is that the payoff of Asian option (with fixed strike) is the positive part of the Asian forward payoff. An important feature of forward contracts is that their price does not depend on the choice of the risk neutral measure. Thus the price of forwards is model independent and there is a uniquely defined hedge q_t. This fact is shown in the following proposition. By simple observation that options and forwards differ only in the payoff while having the same pricing equation, we will be able to characterize the Asian option price in the next section.

Proposition 2.2 (Replication of the Asian forward contract) *Suppose that we have a self-financing portfolio X evolving as*

$$dX_t = q_{t-}dS_t + r(X_{t-} - q_{t-}S_{t-})dt, \tag{2.2}$$

where S_t is a semimartingale. If we set the shares invested in the stock to be

$$q_t = e^{-rT}\int_t^T e^{rs} d\lambda(s) \tag{2.3}$$

where $\lambda(t)$ is a deterministic function with finite variation, and start with the initial wealth

$$X_0 = q_0 S_0 - e^{-rT}K_2, \tag{2.4}$$

then we will have

$$X_T = \int_0^T S_t d\lambda(t) - K_2. \tag{2.5}$$

Proof. For notational purpose, let $B_t = e^{-rt}S_t$. By the definition of quadratic variation and (2.2),

$$e^{-rT}X_T - X_0 = \int_0^T q_{t-}dB_t = q_T B_T - q_0 S_0 - \int_0^T B_{t-}dq_t - [q,B]_T. \tag{2.6}$$

Since q_t is of finite variation,

$$\int_0^T B_{t-}dq_t + [q,B]_t = \int_0^T B_{t-}dq_t + \sum_{0<u\le t}\Delta q_u \Delta B_u = \int_0^T B_t dq_t = \int_0^T e^{-rt}S_t dq_t.$$

Given the formula (2.3) for q_t (note that $q_T = 0$), and the formula (2.4) for X_0, (2.6) simplifies to

$$\begin{aligned} X_T &= e^{rT}X_0 - e^{rT}q_0S_0 - \int_0^T e^{r(T-t)}S_t dq_t \\ &= \int_0^T S_t d\lambda(t) - K_2. \end{aligned}$$

$\diamond$

For pricing Asian options, we can apply the change of numeraire technique introduced in Geman, El Karoui and Rochet (1995). Let us define a new measure $\mathbb{Q}$ by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_t = \frac{S_t}{S_0 e^{rt}}, \tag{2.7}$$

and a numeraire process $Z_t = \frac{X_t}{S_t}$.

Theorem 2.3 (Pricing Formula) *Let $V^\lambda(0,S_0,K_1,K_2)$, the price of the Asian option with the payoff (1.1) when $\xi=1$, be defined as*

$$V^\lambda(0,S_0,K_1,K_2) \triangleq E^{\mathbb{P}}\left[e^{-rT}\left(\int_0^T S_t d\lambda(t) - K_1 S_T - K_2\right)^+\right]. \tag{2.8}$$

Then we have the following relationship

$$V^\lambda(0,S_0,K_1,K_2) = S_0\cdot\mathbb{E}^{\mathbb{Q}}[(Z_T-K_1)^+] \tag{2.9}$$

where $\mathbb{Q}$ is defined by (2.7), X_t is the self-financing portfolio (2.2) with the initial condition X_0 and trading strategy q_t defined in (2.4) and (2.3), and $Z_t=\frac{X_t}{S_t}$.

Proof. An easy consequence of proposition 2.2 is

$$\begin{aligned}
V^\lambda(0,S_0,K_1,K_2) &= e^{-rT}\cdot\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T S_t d\lambda(t) - K_1 S_T - K_2\right)^+\right] \\
&= e^{-rT}\cdot\mathbb{E}^{\mathbb{P}}\left[(X_T-K_1S_T)^+\right] \\
&= e^{-rT}\cdot\mathbb{E}^{\mathbb{Q}}\left[(X_T-K_1S_T)^+\frac{S_0e^{rT}}{S_T}\right] \\
&= S_0\cdot\mathbb{E}^{\mathbb{Q}}[(Z_T-K_1)^+].
\end{aligned}$$

$\diamond$

3 Integro-Differential Equation

For our next analysis, we need the following result:

Lemma 3.1 *$Z_t=\frac{X_t}{S_t}$ is a local martingale under $\mathbb{Q}$.*

Proof. Recall that $\mathbb{P}$ is a risk-neutral measure. Equation (2.2) and the fact that q_t is deterministic ensure that $e^{-rt}X_t$ is a martingale. For $0\le u\le t$,

$$\begin{aligned}
E^{\mathbb{Q}}[Z_t|\mathcal{F}_u] &= \frac{S_0e^{ru}}{S_u}E^{\mathbb{P}}\left[\frac{S_tZ_t}{S_0e^{rt}}\mid\mathcal{F}_u\right] \\
&= \frac{e^{ru}}{S_u}E^{\mathbb{P}}\left[e^{-rt}X_t\mid\mathcal{F}_u\right] \\
&= \frac{e^{ru}}{S_u}e^{-ru}X_u = Z_u.
\end{aligned}$$

$\diamond$

To derive the integro-differential equation, we need to impose more restrictions on the structure of the stock price to get the Markovian property. Let H be a semimartingale on the same stochastic basis $(\Omega,\mathcal{F},\mathbf{F}=(\mathcal{F}_t)_{t\in\mathbb{R}_+},\mathbb{P})$, with values in $\mathbb{R}$ and $H_0=0$. Suppose the stock price has the following dynamics:

$$dS_t = S_{t-}dH_t, \tag{3.1}$$

Since we have assumed $e^{-rt}S_t$ to be a martingale under $\mathbb{P}$, H is necessarily a special semimartingale. Following the notation in Jacod and Shiryaev (2002), H has the canonical decomposition:

$$H_t = rt + H_t^c + \int_0^t\int_{-\infty}^{\infty} x\left(\mu(ds,dx)-\nu(ds,dx)\right), \tag{3.2}$$

where H_t^c is the continuous martingale part, $\mu(dt,dx)$ is the random measure associated with the jumps of H and $\nu(dt,dx)$ is the compensator. According to II.2.9 and II.2.29 in Jacod and Shiryaev (2002), we can always choose a good version of ν, i.e., $\nu(\{t\},\mathbb{R}) \le 1$, $\nu(\mathbb{R}_+,\{0\}) = 0$ and $\int_0^t \int_{-\infty}^{\infty} (|x|^2 \wedge |x|)\nu(dt,dx)$ is a process with locally integrable variation. The assumption of the strict positiveness of S translates to the following assumption

Assumption 3.2 *$\mu([0,t],(-\infty,-1]) = 0$ for all $t \ge 0$.*

The Doléans-Dade formula gives

$$S_t = S_0\mathcal{E}(H) = S_0 \exp\left(H_t - \tfrac{1}{2}\langle H^c\rangle_t\right) \prod_{0<s\le t}(1+\Delta H_s)e^{-\Delta H_s}. \tag{3.3}$$

If H is a PII (process with independent increments) with decomposition (3.2), then we can find a deterministic function c_t, a deterministic measure-valued function K_t and a deterministic increasing function A_t such that

$$\begin{cases} d\langle H^c\rangle_t(\omega) = c_t dA_t(\omega), \\ \nu(dt,dx) = K_t(dx)dA_t(\omega). \end{cases}$$

Further if H is a Lévy process, then we can take $A_t = t$, c to be a constant, and $K(dx)$ (the Lévy measure) to be independent of t, to integrate $|x|^2 \wedge |x|$, and to satisfy $K(\{0\}) = 0$.

Theorem 3.3 (Integro-differential equation for Asian options) *Suppose that H is PII with canonical decomposition (3.2). The value of the Asian option is a function of t and Z_t, denoted by $v(t,Z_t)$, such that $V^\lambda(0,S_0,K_1,K_2) = S_0\, v(0,Z_0)$. Assume v_t, v_z and v_{zz} exist and are continuous. Then the following integro-differential equation holds:*

$$\begin{aligned} \int_0^t \Bigg[& v_s(s,Z_{s-})ds + \tfrac{1}{2}v_{zz}(s,Z_{s-})(q_{s-}-Z_{s-})^2 d\langle H^c\rangle_s \\ & + \int_{-\infty}^{\infty} \left\{ v\left(s, Z_{s-} + (q_{s-}-Z_{s-})\tfrac{x}{1+x}\right) - v(s,Z_{s-}) - v_z(s,Z_{s-})(q_{s-}-Z_{s-})\tfrac{x}{1+x} \right\} \nu(ds,dx) \Bigg] = 0 \end{aligned} \tag{3.4}$$

for $0 \le t \le T$.

Proof. Apply Ito's formula to get

$$\begin{aligned} \tfrac{X_T}{S_T} &= \tfrac{X_0}{S_0} + \int_0^T \tfrac{1}{S_{t-}} dX_t - \int_0^T \tfrac{X_{t-}}{S_{t-}^2} dS_t - \int_0^T \tfrac{1}{S_{t-}^2} d\langle X^c, S^c\rangle_t \\ &\quad + \int_0^T \tfrac{X_{t-}}{S_{t-}^3} d\langle S^c\rangle_t + \sum_{0<t\le T}\left(\tfrac{X_t}{S_t} - \tfrac{X_{t-}}{S_{t-}} - \tfrac{1}{S_{t-}}\Delta X_t + \tfrac{X_{t-}}{S_{t-}^2}\Delta S_t\right) \\ &= \tfrac{X_0}{S_0} + \int_0^T \tfrac{1}{S_{t-}}\left(q_{t-}S_{t-}dH_t + r(X_{t-} - q_{t-}S_{t-})dt\right) \\ &\quad - \int_0^T \tfrac{X_{t-}}{S_{t-}} dH_t - \int_0^T q_{t-} d\langle H^c\rangle_t + \int_0^T \tfrac{X_{t-}}{S_{t-}} d\langle H^c\rangle_t \\ &\quad + \sum_{0<t\le T}\left(\tfrac{X_t}{S_t} - \tfrac{X_{t-}}{S_{t-}} - \tfrac{1}{S_{t-}}\Delta X_t + \tfrac{X_{t-}}{S_{t-}^2}\Delta S_t\right). \end{aligned}$$

Note that

$$\Delta S_t = S_{t-}\Delta H_t, \qquad \Delta X_t = q_{t-}S_{t-}\Delta H_t,$$

$$\tfrac{X_t}{S_t} - \tfrac{X_{t-}}{S_{t-}} = \left(q_{t-} - \tfrac{X_{t-}}{S_{t-}}\right)\left(1 - \tfrac{1}{1+\Delta H_t}\right) = \left(q_{t-} - \tfrac{X_{t-}}{S_{t-}}\right)\left(\tfrac{\Delta H_t}{1+\Delta H_t}\right).$$

We can write

$$\begin{aligned}
d\left(\tfrac{X_t}{S_t}\right) &= \left(q_{t-} - \tfrac{X_{t-}}{S_{t-}}\right)\left(dH_t - rdt - d\langle H^c\rangle_t\right) + \left(q_{t-} - \tfrac{X_{t-}}{S_{t-}}\right)\left(\tfrac{\Delta H_t}{1+\Delta H_t} - \Delta H_t\right). \\
&= \left(q_{t-} - \tfrac{X_{t-}}{S_{t-}}\right)\left(dH_t - rdt - d\langle H^c\rangle_t - \tfrac{\Delta H_t^2}{1+\Delta H_t}\right) \\
&= \left(q_{t-} - \tfrac{X_{t-}}{S_{t-}}\right)\left(dH_t^c - d\langle H^c\rangle_t + \int_{-\infty}^{\infty} x\left(\mu(dt,dx) - \nu(dt,dx)\right) - \int_{-\infty}^{\infty} \tfrac{x^2}{1+x}\,\mu(dt,dx)\right)
\end{aligned}$$

or

$$dZ_t = (q_{t-} - Z_{t-})\left(dH_t^c - d\langle H^c\rangle_t + \int_{-\infty}^{\infty} x\left(\mu(dt,dx) - \nu(dt,dx)\right) - \int_{-\infty}^{\infty} \tfrac{x^2}{1+x}\,\mu(dt,dx)\right).$$

Observe that Z_t is a Markovian process under $\mathbb{Q}$. Theorem 2.3 and the Markovian property give us the value process

$$v(t, Z_t) = \mathbb{E}^{\mathbb{Q}}[(Z_T - K_1)^+ | \mathcal{F}_t],$$

which is a martingale by definition.

Note $d\langle Z^c\rangle_t = (q_{t-} - Z_{t-})^2 d\langle H^c\rangle_t$ and thus

$$\begin{aligned}
dv(t, Z_t) &= v_t(t, Z_{t-})dt + v_z(t, Z_{t-})dZ + \tfrac{1}{2}v_{zz}(t, Z_{t-})d\langle Z^c\rangle_t \\
&\quad + v(t, Z_t) - v(t, Z_{t-}) - v_z(t, Z_{t-})\Delta Z_t \\
&= v_t(t, Z_{t-})dt + v_z(t, Z_{t-})dZ + \tfrac{1}{2}v_{zz}(t, Z_{t-})(q_{t-} - Z_{t-})^2 d\langle H^c\rangle_t \\
&\quad + v\left(t, Z_{t-} + (q_{t-} - Z_{t-})\tfrac{\Delta H}{1+\Delta H}\right) - v(t, Z_{t-}) - v_z(t, Z_{t-})(q_{t-} - Z_{t-})\tfrac{\Delta H}{1+\Delta H} \\
&= \text{Local Martingale} + v_t(t, Z_{t-})dt + \tfrac{1}{2}v_{zz}(t, Z_{t-})(q_{t-} - Z_{t-})^2 d\langle H^c\rangle_t \\
&\quad + \int_{-\infty}^{\infty}\Big\{v\left(t, Z_{t-} + (q_{t-} - Z_{t-})\tfrac{x}{1+x}\right) \\
&\qquad\qquad - v(t, Z_{t-}) - v_z(t, Z_{t-})(q_{t-} - Z_{t-})\tfrac{x}{1+x}\Big\}\nu(dt, dx).
\end{aligned}$$

The fact that a predictable local martingale with finite variation starting at zero is zero concludes the proof. ◇

Corollary 3.4 *In the case when H is a Lévy process, the integro-differential equation simplifies to*

$$\begin{aligned}
(3.5)\quad & v_t(t,z) + \tfrac{c}{2}v_{zz}(t,z)(q_{t-} - z)^2 \\
&\quad + \int_{-\infty}^{\infty}\left\{v\left(t, z + (q_{t-} - z)\tfrac{x}{1+x}\right) - v(t,z) - v_z(t,z)(q_{t-} - z)\tfrac{x}{1+x}\right\}K(dx) = 0
\end{aligned}$$

for $0 \le t \le T$ and $z \in \mathbb{R}$.

Proof. The canonical decomposition of H is

$$H_t = rt + \int_0^t \sqrt{c}\,dW_s + \int_0^t \int_{-\infty}^{\infty} x\left(\mu(ds,dx) - K(dx)dt\right)$$

where W_t is a standard Brownian Motion. Applying theorem 3.3, we get

$$\begin{aligned}
(3.6)\quad & v_t(t, Z_{t-}) + \tfrac{c}{2}v_{zz}(t, Z_{t-})(q_{t-} - Z_{t-})^2 \\
&\quad + \int_{-\infty}^{\infty}\left\{v\left(t, Z_{t-} + (q_{t-} - Z_{t-})\tfrac{x}{1+x}\right) - v(t, Z_{t-}) - v_z(t, Z_{t-})(q_{t-} - Z_{t-})\tfrac{x}{1+x}\right\}K(dx) = 0.
\end{aligned}$$

Since the support for Z_{t-} is $\mathbb{R}$, we get the above equation. ◇

4 Applications to Different Lévy Models

1. Geometric Brownian Motion with Poisson Jump

Let us start with a model similar to the one suggested in Andreasen (1998). Suppose that the stock price process evolves as

$$dS_t = S_{t-}dH_t = S_{t-}\left(rdt + \sigma dW_t + (e^{\phi_t} - 1)dM_t\right),$$

where W_t is a standard Brownian motion, and M_t is a compensated Poisson process, i.e., $M_t = N_t - \lambda t$. Let ϕ_t be a Gaussian process with independent increments, and be independent of both W_t and N_t, such that $\mathbb{E}[\phi_t] = \mu$ and $\mathrm{Var}[\phi_t] = \gamma^2$. Assume that $\gamma > 0, \sigma > 0, \mu$ are constants. In this case,

$$\langle H\rangle_t = \sigma^2 t, \qquad K(x) = \tfrac{\lambda}{\sqrt{2\pi}\gamma}\cdot\exp\left\{-\tfrac{(\ln(x+1)-\mu)^2}{\gamma^2}\right\},$$

and $v(t,z)$ satisfies (3.4). If $\gamma = 0$, then the jump size reduces to a constant $e^{\mu} - 1$, i.e., $K(x) = \lambda\,\delta(\{e^{\mu}-1\})$ and (3.4) simplifies to:

$$\begin{aligned} v_t(t,z) &+ \tfrac{\sigma^2}{2}v_{zz}(t,z)(q_{t-}-z)^2 \\ &+ \left[v\left(t, z + (q_{t-}-z)\tfrac{\phi}{1+\phi}\right) - v(t,z) - (q_{t-}-z)v_z(t,z)\right]\tfrac{\phi}{1+\phi}\lambda = 0, \end{aligned} \tag{4.1}$$

for $0 \le t \le T$. In the geometric Brownian model, $dS_t = S_{t-}dH_t = S_{t-}\left(rdt + \sigma dW_t\right)$, $\phi = 0$, and we simply have

$$v_t(t,z) + \tfrac{\sigma^2}{2}(q_t - z)^2 v_{zz}(t,z) = 0, \tag{4.2}$$

as shown in Večeř (2001).

2. Pure Jump Processes Models: CGMY and General Hyperbolic

In our model (3.1),

$$dS_t = S_{t-}dH_t, \tag{4.3}$$

the stock price is a stochastic exponential of H. Another usual approach in the literature is to let the stock price to be a geometric exponential of the underlying:

$$S_t = S_0 e^{\widehat{H}_t}. \tag{4.4}$$

Applying Ito's lemma and rewriting (3.3):

$$dS_t = e^{\widehat{H}_{t-}}\left(d\widehat{H}_t + \frac{1}{2}d\langle\widehat{H}^c\rangle_t + e^{\Delta\widehat{H}_t} - 1 - \Delta\widehat{H}_t\right), \tag{4.5}$$

$$S_t = S_0\exp\left\{H_t - \frac{1}{2}\langle H^c\rangle_t + \sum_{0<s\le t}\left(\ln(1+\Delta H_s) - \Delta H_s\right)\right\}. \tag{4.6}$$

We can easily find the relationship between H and $\widehat{H}$:

$$H_t = \widehat{H}_t + \frac{1}{2}\langle\widehat{H}^c\rangle_t + \sum_{0<s\le t}\left(e^{\Delta\widehat{H}_s} - 1 - \Delta\widehat{H}_s\right), \tag{4.7}$$

$$\widehat{H}_t = H_t - \frac{1}{2}\langle H^c\rangle_t + \sum_{0<s\le t}\left(\ln(1+\Delta H_s) - \Delta H_s\right). \tag{4.8}$$

Therefore the two ways of modelling are equivalent with Assumption 3.2. If we are given the compensator $\widehat{\mu}(dt, dx)$ for the model (4.4), then the IDE in corollary (3.4) becomes

$$
\begin{aligned}
(4.9)\quad & \widehat{v}_t(t,z) + \tfrac{c}{2}\widehat{v}_{zz}(t,z)(q_{t-}-z)^2 + \\
& \int_{-\infty}^{\infty} \left\{ \widehat{v}\left(t, z + (q_{t-}-z)\tfrac{e^{\widehat{x}}-1}{e^{\widehat{x}}}\right) - \widehat{v}(t,z) - \widehat{v}_z(t,z)(q_{t-}-z)\tfrac{e^{\widehat{x}}-1}{e^{\widehat{x}}} \right\} \widehat{K}(d\widehat{x}) = 0,
\end{aligned}
$$

because $\Delta H_t = e^{\Delta \widehat{H}_t} - 1$.

We mention here two geometric exponential models with pure jump processes. One is CGMY in Carr, Geman, Madan, Yor (2000) with Lévy measure:

$$
\widehat{k}_{CGMY} = \begin{cases} C\frac{\exp(-G|\widehat{x}|)}{|\widehat{x}|^{1+Y}}, & \text{for } \widehat{x} < 0; \\ C\frac{\exp(-M|\widehat{x}|)}{|\widehat{x}|^{1+Y}}, & \text{for } \widehat{x} > 0. \end{cases}
$$

The other is the General Hyperbolic Model in Eberlein and Prause (1998) with Lévy measure:

$$
\widehat{k}_{EP} = \begin{cases} \frac{e^{\beta\widehat{x}}}{|\widehat{x}|}\left(\int_0^\infty \frac{\exp\left(-\sqrt{2y+\alpha^2}|\widehat{x}|\right)}{\pi^2 y (J_\lambda^2(\delta\sqrt{2y}) + Y_\lambda^2(\delta\sqrt{2y}))} dy + \lambda e^{-\alpha|\widehat{x}|}\right), & \text{if } \lambda \geq 0; \\ \frac{e^{\beta\widehat{x}}}{|\widehat{x}|}\left(\int_0^\infty \frac{\exp\left(-\sqrt{2y+\alpha^2}|\widehat{x}|\right)}{\pi^2 y (J_{-\lambda}^2(\delta\sqrt{2y}) + Y_{-\lambda}^2(\delta\sqrt{2y}))} dy\right), & \text{if } \lambda < 0; \end{cases}
$$

where J_λ and Y_λ are the Bessel functions of the first and second kind respectively. In both models the value of the Asian option satisfy:

$$
(4.10)\qquad \widehat{v}_t(t,z) + \int_{-\infty}^{\infty} \left\{ \widehat{v}\left(t, z + (q_{t-}-z)\tfrac{e^{\widehat{x}}-1}{e^{\widehat{x}}}\right) - \widehat{v}(t,z) - \widehat{v}_z(t,z)(q_{t-}-z)\tfrac{e^{\widehat{x}}-1}{e^{\widehat{x}}} \right\} \widehat{K}(d\widehat{x}) = 0,
$$

for $0 \leq t \leq T$.

3. Numerical Issues

Integro-differential equations can be solved numerically. The numerical procedures for solving integro-differential equations in option pricing have been recently studied for instance in Hirsa and Madan (2001). They have developed discretization scheme which applies to the case of Asian option as well.

There are several alternative methods for pricing Asian options in the geometric Brownian motion model, especially for continuously averaged Asian. In this case our pricing equation becomes a two-term partial differential equation which is simple to implement. Extensive comparisons of different methods are documented in Večeř (2002), with the conclusion that several alternative approaches obtain prices with arbitrary precision. These methods include Večeř's partial differential equation, Geman-Yor (1993) Laplace transform and Linetsky's (2002) analytical expansion method. The pricing partial differential equation presented here is numerically stable and the convergence of the numerical scheme is not affected by the choice of underlying parameters.

5 Conclusion

We have shown in this paper that we can remove the path dependency in the payoff function of all kinds of Asian options regardless of the dynamics of the underlying asset. This reformulation of the problem gives us an integro-differential equation for the price of the option when the stock is driven by an exponential Lévy process. This equation simplifies even more if we assume a particular stock price model, such as Geometric Brownian Motion with Poisson Jump model, the Carr, Geman, Madan, Yor model, or a general hyperbolic model. In the case of Black-Scholes model, we obtain a one-dimensional PDE which is simple and robust to implement.

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