# A new PDE approach for pricing arithmetic average Asian options*

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**Abstract.** In this paper, arithmetic average Asian options are studied. It is observed that the Asian option is a special case of the option on a traded account. The price of the Asian option is characterized by a simple one-dimensional partial differential equation which could be applied to both continuous and discrete average Asian option. The article also provides numerical implementation of the pricing equation. The implementation is fast and accurate even for low volatility and/or short maturity cases.

**Key words:** Asian options, Options on a traded account, Brownian motion, fixed strike, floating strike.

## 1 Introduction

Asian options are securities with payoff which depends on the average of the underlying stock price over certain time interval. Since no general analytical solution for the price of the Asian option is known, a variety of techniques have been developed to analyze arithmetic average Asian options. A number of approximations that produce closed form expressions have appeared, see Turnbull and Wakeman [18], Vorst [19], Levy [13], Levy and Turnbull [14]. Geman and Yor [8] computed the Laplace transform of the Asian option price, but numerical inversion remains problematic for low volatility and/or short maturity cases (see Geman and Eydeland [6] or Fu, Madan and Wang [5]). Monte Carlo simulation works well, but it can be computationally expensive without the enhancement of variance reduction techniques and one must account for the inherent discretization bias resulting from the approximation of continuous time processes through discrete sampling (see Broadie and Glasserman [3], Broadie, Glasserman and Kou [4] and Kemma and Vorst [12]).

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In general, the price of an Asian option can be found by solving a PDE in two space dimensions (see Ingersoll [10]), which is prone to oscillatory solutions. Ingersoll [10] also observed that the two-dimensional PDE for a floating strike Asian option can be reduced to a one-dimensional PDE. Rogers and Shi [17] have formulated a one-dimensional PDE that can model both floating and fixed strike Asian options. They reduced the dimension of the problem by dividing $K-\bar{S}_t$ ($K$ is the strike, $\bar{S}_t$ is the average stock price over $[0,t]$) by the stock price $S_t$. However this one-dimensional PDE is difficult to solve numerically since the diffusion term is very small for values of interest on the finite difference grid. The dirac delta function also appears as a coefficient of the PDE in the case of the floating strike option. Zvan, Forsyth and Vetzal [21] were able to improve the numerical accuracy of this method by using computational fluid dynamics techniques. Andreasen [2] applied Rogers and Shi's reduction to discretely sampled Asian option. More recently, Lipton [15] noticed similarity of pricing equations for the passport and the Asian option, again using Rogers and Shi's reduction.

In this article, an alternative one-dimensional PDE is derived by a similar space reduction. It is noted that the arithmetic average Asian option (both floating and fixed strike) is a special case of an option on a traded account. See Shreve and Večeř [16] and [20] for a detailed discussion about options on a traded account. Options on a traded account generalize the concept of many options (passport, European, American, vacation) and the same pricing techniques could be applied to price the Asian option. The resulting one-dimensional PDE for the price of the Asian option is simple enough to be easily implemented to give very fast and accurate results.

Section 2 of the article briefly describes options on a traded account. It is shown in section 3 that the Asian option is a special case of the option on a traded account. The one-dimensional PDE for the price of the Asian option is given. Section 4 describes the numerical implementation and compares results with results of other methods. Section 5 concludes the paper.

## 2 Options on a traded account

An option on a traded account is a contract which allows the holder of the option to switch during the life of the option among various positions in an underlying asset (stock). The holder accumulates gains and losses resulting from this trading, and at the expiration of the option he gets the call option payoff with strike 0 on his final account value, i.e., he keeps any gain from trading and is forgiven any loss.

Suppose that the stock evolves under the risk neutral measure according to the equation

$$dS_t = S_t(rdt + \sigma dW_t), \tag{2.1}$$

where $r$ is the interest rate and $\sigma$ is the volatility of the stock. Denote the option holder's trading strategy by $q_t$, the number of shares held at time $t$. The strategy $q_t$ is subject to the contractual constraint $q_t \in [\alpha_t, \beta_t]$, where $\alpha_t \le \beta_t$. It turns out that the holder of the option should never take an intermediate position, i.e, at any time he should hold either $\alpha_t$ shares of stock or $\beta_t$ shares. In the case of Asian options, $\alpha_t = \beta_t$, so option holder's trading strategy is a priori given to him.

In our model the value of the option holder's account corresponding to the strategy $q_t$ satisfies

$$\begin{aligned} dX_t^q &= q_t dS_t + \mu(X_t^q - q_t S_t)dt \\ X_0^q &= X_0. \end{aligned} \tag{2.2}$$

This represents a trading strategy in the money market and the underlying asset, where $X_0$ is the initial wealth and $\mu$ is the interest rate corresponding to reinvesting the cash position $X_t^q - q_t S_t$ (possibly different from the risk-neutral interest rate $r$). The trading strategy is self-financing when $\mu = r$. The holder of the option will receive at time $T$ the payoff $[X_T^q]^+$. The objective of the seller of the option, who makes this payment, is to be prepared to hedge against all possible strategies of the holder of the option. Therefore the price of this contract at time $t$ should be the maximum over all possible strategies $q_u$ of the discounted expected value under the risk-neutral probability $\mathbb{P}$ of the payoff of the option, i.e.,

$$V^{[\alpha,\beta]}(t, S_t, X_t) = \max_{q_u \in [\alpha,\beta]} e^{-r(T-t)} \mathbb{E}[[X_T^q]^+ | \mathcal{F}_t], \quad t \in [0, T]. \tag{2.3}$$

Computation of the expression in (2.3) is a problem of stochastic optimal control, and the function $V^{[\alpha,\beta]}(t, s, x)$ is characterized by the corresponding Hamilton–Jacobi–Bellman (HJB) equation

$$\begin{aligned} -rV + V_t + rsV_s + \max_{q \in [\alpha,\beta]} [(\mu x + q(r-\mu))V_x \\ + \tfrac{1}{2}\sigma^2 s^2 (V_{ss} + 2qV_{sx} + q^2 V_{xx})] = 0 \end{aligned} \tag{2.4}$$

with the boundary condition

$$V(T, s, x) = x^+. \tag{2.5}$$

The maximum in (2.4) is attained by the optimal strategy $q_t^{opt}$.

The case $\alpha_t = \beta_t = 1$ reduces to the European call, the case $\alpha_t = \beta_t = -1$ reduces to the European put. The American call and put give the holder of the option the right to switch at most once during the life of the option to zero position (i.e., exercise the option), but it does not pay interest on the traded account while the holder has a position in the stock market. These can be modelled by setting $\mu = 0$ in (2.2) and allowing only one switch in $q_t$, either from 1 to 0 (American call) or from $-1$ to 0 (American put). The passport option has contractual conditions $\alpha_t = -1$, $\beta_t = 1$, the so-called vacation call has $\alpha_t = 0$, $\beta_t = 1$ and the so-called vacation put has $\alpha_t = -1$, $\beta_t = 0$.

By the change of variable

$$Z_t^q = \tfrac{X_t^q}{S_t}, \tag{2.6}$$

we can reduce the dimensionality of the problem (2.3), as we show below. The same change of variable was used in Hyer, Lipton-Lifschitz and Pugachevsky [9] and in Andersen, Andreasen and Brotherton-Ratcliffe [1] to price passport options and in Shreve and Večeř [16] to price options on a traded account. Applying Itô's formula to the process $Z_t^q$, we get

$$dZ_t^q = (q_t - Z_t^q)\,(r - \mu - \sigma^2)dt + (q_t - Z_t^q)\,\sigma dW_t. \tag{2.7}$$

We next define a new probability measure $\widetilde{\mathbb{P}}$ by $\widetilde{\mathbb{P}}(A) = \int_A D_T d\mathbb{P},\ A \in \mathcal{F}$, where

$$D_T = e^{-rT} \cdot \tfrac{S_T}{S_0} = \exp\left(\sigma W_T - \tfrac{1}{2}\sigma^2 T\right). \tag{2.8}$$

Under $\widetilde{\mathbb{P}}$, $\widetilde{W}_t = -\sigma t + W_t$ is a Brownian motion, according to Girsanov's theorem. Notice that

$$e^{-rT}\mathbb{E}[X_T^q]^+ = e^{-rT}\widetilde{\mathbb{E}}\left[\tfrac{X_T^q}{D_T}\right]^+ = S_0 \cdot \widetilde{\mathbb{E}}\left[\tfrac{X_T^q}{S_T}\right]^+ = S_0 \cdot \widetilde{\mathbb{E}}\,[Z_T^q]^+ \tag{2.9}$$

and

$$dZ_t^q = (q_t - Z_t^q)\,(r - \mu)dt + (q_t - Z_t^q)\,\sigma d\widetilde{W}_t. \tag{2.10}$$

The corresponding reduced HJB equation becomes

$$u_t + \max_{q \in [\alpha,\beta]} \left((r-\mu)(q-z)u_z + \tfrac{1}{2}(q-z)^2\sigma^2 u_{zz}\right) = 0 \tag{2.11}$$

with the boundary condition

$$u(T,z) = z^+. \tag{2.12}$$

The relationship between $V$ and $u$ is

$$V(0, S_0, X_0) = S_0 \cdot u\left(0, \tfrac{X_0}{S_0}\right). \tag{2.13}$$

Closed form solutions and optimal strategies are provided in Shreve and Večeř [16] for the prices of the option on a traded account for any general constraints of the type $\alpha_t \equiv \alpha$ and $\beta_t \equiv \beta$ when $\mu = r$.

# 3 Asian option as an option on a traded account

Options on a traded account also represent Asian options. Notice that $d(tS_t) = tdS_t + S_t dt$, or equivalently,

$$TS_T = \int_0^T t dS_t + \int_0^T S_t dt. \tag{3.1}$$

After dividing by the maturity time $T$ and rearranging the terms we get

$$\tfrac{1}{T}\int_0^T S_t dt = \int_0^T \left(1 - \tfrac{t}{T}\right) dS_t + S_0. \tag{3.2}$$

In the terminology of the option on a traded account, the Asian fixed strike call payoff $(\bar{S}_T - K)^+$ is achieved by taking $q_t = 1 - \frac{t}{T}$ and $X_0 = S_0 - K$ and where the traded account evolves according to the equation

$$dX_t = \left(1 - \tfrac{t}{T}\right) dS_t, \tag{3.3}$$

i.e., when $\mu = 0$ so no interest is added or charged to the traded account. We have then

$$X_T = \int_0^T (1 - \tfrac{t}{T}) dS_t + S_0 - K = \bar{S}_T - K. \tag{3.4}$$

Thus the average of the stock price could be achieved by a selling off one share of stock at the constant rate $\frac{1}{T}$ shares per unit time.

Similarly, the Asian fixed strike put payoff $(K - \bar{S}_T)^+$ is achieved by taking $q_t = \frac{t}{T} - 1$ and $X_0 = K - S_0$. For the Asian floating strike call with payoff $(KS_T - \bar{S}_T)^+$ we take simply $q_t = \frac{t}{T} - 1 + K$ and $X_0 = S_0(K-1)$, for the Asian floating strike put with payoff $(\bar{S}_T - KS_T)^+$ we take $q_t = -\frac{t}{T} + 1 - K$ and $X_0 = S_0(1-K)$.

The discrete average Asian option payoff could be achieved by taking a step function approximation of the stock position $q_t$ of its continous average option counterpart. Let us take for example the case of the Asian fixed strike call when $q_t = 1 - \frac{t}{T}$ and $X_0 = S_0 - K$. A step function approximation of $1 - \frac{t}{T}$ is

$$q_t = 1 - \tfrac{1}{n} \left[ n \tfrac{t}{T} \right], \tag{3.5}$$

where $[\ \cdot\ ]$ denotes the integer part function. If we look directly at the Asian option traded account equation

$$dX_t = q_t dS_t, \tag{3.6}$$

we get for the stock position $q_t$ given by (3.5)

$$X_T = \tfrac{1}{n} \sum_{k=1}^{n} S_{(\frac{k}{n}) \cdot T} - S_0 + X_0. \tag{3.7}$$

Thus we get the discrete average Asian fixed strike call payoff

$$\left( \tfrac{1}{n} \sum_{k=1}^{n} S_{(\frac{k}{n}) \cdot T} - K \right)^+ \tag{3.8}$$

by taking $X_0 = S_0 - K$ and $q_t = 1 - \frac{1}{n} \left[ n \frac{t}{T} \right]$. We get analogous results for other Asian option types.

Since we showed that Asian options are options on a traded account, we can apply the same pricing techniques to determine the price of Asian options. In particular, we can use the HJB equation (2.11), which becomes for the case of Asian options just a simple PDE

$$u_t + r(q_t - z)u_z + \tfrac{1}{2}(q_t - z)^2 \sigma^2 u_{zz} = 0 \tag{3.9}$$

| Asian option type | Payoff | Stock position $q_t$ | Initial wealth $X_0$ |
|---|---|---|---|
| Fixed strike call | $(\bar{S}_T - K)^+$ | $1 - \frac{t}{T}$ | $S_0 - K$ |
| Fixed strike put | $(K - \bar{S}_T)^+$ | $\frac{t}{T} - 1$ | $K - S_0$ |
| Floating strike call | $(KS_T - \bar{S}_T)^+$ | $\frac{t}{T} - 1 + K$ | $S_0(K-1)$ |
| Floating strike put | $(\bar{S}_T - KS_T)^+$ | $-\frac{t}{T} + 1 - K$ | $S_0(1-K)$ |

Table 1: Asian options as options on a traded account

with the boundary condition

$$u(T, z) = z^+. \tag{3.10}$$

The price of the Asian option is then given in terms of $u$ by (2.13).

The relationship between different kinds of Asian options and options on a traded account is summarized in Table 1.

# 4 Numerical examples

Since there is very little hope that the partial differential equation (3.9) with the boundary condition (3.10) admits a closed form solution, one must compute the price of the Asian option numerically. Equation (3.9) is on the other hand very easy to implement and since it is an equation of the Black-Scholes type, it is also very stable and fast to compute. Numerical implementation of this PDE gives answers within very tight analytical bounds even for low volatility or short maturity contracts. The numerical implementation of the Asian option PDE (3.9) is similar to the numerical implementation for the passport option as described in Andersen, Andreasen and Brotherton-Ratcliffe [1] because of the above mentioned similarity in the pricing equation for both options. The Asian option pricing is even simpler compared to the passport option pricing. The reason is that the position in the stock $q_t$ is deterministically given for the Asian option, while the optimal position $q_t$ must be computed for the case of the passport option. Results obtained in Andersen et al. [1] for the case of the passport option show that the numerical implementation gave almost indistinguishable results from the analytical solution (less than 0.01% off) within less than a second of CPU time on 166 MHz Pentium.

Let us consider a finite difference discretization of PDE (3.9) with a uniform mesh

$$z_i = z_0 + i \cdot dz, \quad t_j = j \cdot dt$$

for $0 \leq i \leq M$, $0 \leq j \leq N$, and $t_N = T$, where $z_0$ and $z_M$ represent $-\infty$ and $\infty$. Reasonable choices are $z_0 = -1$ and $z_M = 1$. One point represents the Asian option with strike equal to zero, the other represents the Asian option with strike equal to double of the stock price. Using the short notation $u_{i,j} = u(t_j, z_i)$ and

$q_j = q(t_j)$, a mixed implicit/explicit finite discretization scheme for (3.9) is given by

$$\begin{aligned}
&\theta[\sigma^2(q_j - z_i)^2 - dz \cdot r(q_j - z_i)]u_{i-1,j} - 2[\theta\sigma^2(q_j - z_i)^2 + \nu]u_{i,j} \\
&\qquad + \theta[\sigma^2(q_j - z_i)^2 + dz \cdot r(q_j - z_i)]u_{i+1,j} = \\
&- (1-\theta)[\sigma^2(q_j - z_i)^2 - dz \cdot r(q_j - z_i)]u_{i-1,j+1} + 2[(1-\theta)\sigma^2(q_j - z_i)^2 - \nu]u_{i,j+1} \\
&\qquad - (1-\theta)[\sigma^2(q_j - z_i)^2 + dz \cdot r(q_j - z_i)]u_{i+1,j+1}, \qquad (4.1)
\end{aligned}$$

where $0 \leq \theta \leq 1$ and $\nu = \frac{dz^2}{dt}$. The boundary condition for this system of equations is

$$u_{i,N} = z_i^+. \qquad (4.2)$$

Solving for $u_{i,j}$ is done in the usual way by solving the corresponding tridiagonal system of equations in (4.1). For the boundary conditions at $z_0$ and $z_M$ we can take

$$u_{0,j} = 0 \quad \text{and} \quad u_{M,j} = 2u_{M-1,j} - u_{M-2,j} \text{ (linear interpolation).}$$

The parameter $\theta$ in (4.1) determines at what time point the partial derivatives with respect to $z$ are evaluated. If $\theta = 0$, the $z$ derivatives are evaluated at $t_{j+1}$ and the scheme is known as the explicit finite difference method or as a trinomial tree. If $\theta = 1$, the $z$ derivatives are evaluated at $t_j$ and the scheme becomes fully implicit finite difference method. The average of these two methods, i.e., when $\theta = \frac{1}{2}$, is known as a Crank-Nicolson method. Crank-Nicolson method is usually preferred, because it has the highest convergence order in $dt$. This method was used to get numerical results in this article.

Table 2 compares results of the above described method with results of Rogers and Shi [17], Zvan, Forsyth and Vetzal [21] and with Monte Carlo methods. The comparison for the fixed strike Asian call when $r = 0.15$, $S_0 = 100$ and $T = 1$, which they considered as the most difficult case is reported. Zvan et al. improved the accuracy of the method of Rogers and Shi by using a nonuniform spatial grid and techniques of computational fluid dynamics. To be consistent with the result of Zvan et al., same number of points of space and time grid (200 space points, 400 time points) are used in (4.1).

The Monte Carlo method used here as a comparison uses techniques from Glasserman, Heidelberger and Shahabuddin [7] together with Sobol numbers and geometric Asian call option as control variate, which both reduces the variance and the bias from the discretization (see Fu, Madan and Wang [5]).

The lower and upper analytical bounds mentioned here are according to Rogers and Shi. As seen from the table, the accuracy of the method suggested in this article is very good; it always gives prices within analytical bounds. It is stable for low volatilities and short maturities contrary to numerical inversion of the Laplace transform of the Asian option price or to other PDE methods for the Asian option.

This implementation, done in MATLAB, gave accurate results in a few seconds.

| $\sigma$ | $K$ | Večeř | Zvan et al. | Monte Carlo | Lower | Upper |
|---|---|---|---|---|---|---|
| 0.05 | 95 | 11.094 | 11.094 | 11.094 | 11.094 | 11.114 |
| | 100 | 6.795 | 6.793 | 6.795 | 6.794 | 6.810 |
| | 105 | 2.744 | 2.744 | 2.745 | 2.744 | 2.761 |
| 0.10 | 90 | 15.399 | 15.399 | 15.399 | 15.399 | 15.445 |
| | 100 | 7.029 | 7.030 | 7.028 | 7.028 | 7.066 |
| | 110 | 1.415 | 1.410 | 1.418 | 1.413 | 1.451 |
| 0.20 | 90 | 15.643 | 15.643 | 15.642 | 15.641 | 15.748 |
| | 100 | 8.412 | 8.409 | 8.409 | 8.408 | 8.515 |
| | 110 | 3.560 | 3.554 | 3.556 | 3.554 | 3.661 |
| 0.30 | 90 | 16.516 | 16.514 | 16.516 | 16.512 | 16.732 |
| | 100 | 10.215 | 10.210 | 10.210 | 10.208 | 10.429 |
| | 110 | 5.736 | 5.729 | 5.731 | 5.728 | 5.948 |

Table 2: Comparison of results of different methods for fixed strike Asian call when $r = 0.15$, $S_0 = 100$ and $T = 1$. The upper and lower bounds were obtained from Rogers and Shi [17].

# 5 Conclusion

The pricing method for Asian options suggested in this article connects pricing of Asian options and options on a traded account. Options on a traded account (passport, European, American, vacation, Asian) satisfy the same type of one-dimensional PDE. The method suggested here has a simple form, is easy to implement, has stable performance for all volatilities, is fast and accurate, and is applicable for both continuous and discrete average Asian options.

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