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VALUING CALLABLE AND PUTABLE REVENUE-PERFORMANCE-LINKED PROJECT BACKED SECURITIES

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Public owners face a constant demand for developing new projects and for funding the renewal, maintenance and operation of existing infrastructure projects. One way to raise capitals to provide new financial resources to constrained budgets is to securitize a stream of revenue cash flows from a portfolio of mature infrastructure projects. We present a new type of PBS, the revenue performance-linked project backed securities (PBS), with embedded call and put options. In this new PBS setting, risks for issuers and buyers can be confined within a cut-off area. This risk hedging feature is expected to facilitate the trading of such products.

Keywords: Dynamic programming; simulation; real options; project finance.

1. Background

Infrastructure project expenditures in the world economy have been growing substantially in the last two decades. The World Bank estimated that 2% of world GDP is spent annually on infrastructure development and maintenance. Public developers/owners of infrastructure projects must face the daunting challenge of balancing the huge spending demand within a constrained budget. As a result, developers

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have welcomed private participation in infrastructure (PPI) as an opportunity for augmenting their infrastructural capital budgets [7, 13]. However, PPI is only a viable alternative for relieving the economic burdens of developers if investment risks are shared fairly between the developers and private investors.

Among all the alternative opportunities to raise capitals through PPI, project backed securities (PBSs) play an important role in supplying long term and low interest rate infrastructure funds [8]. A PBS is a synthetic, tradable debt asset created by securitizing cash flows generated from one project or a pool of projects [15]. PBSs are offered to investors in both public and private markets. Because PBSs are only backed by revenue cash flows, they are usually considered marketable only after the generating infrastructure projects are completed and operating at full capacity. In essence, marketable PBSs need to be backed by mature and profitable projects. Very few exceptions are high-profile projects, whose future cash flows can be securitized during the construction or initial operation phase as it occurred, for instance, in the expansion of London City Airport [7].

Different from regular, fixed income PBSs, which pay investors periodical fixed coupon payment, Yoshino [11] proposed a revenue performance-linked coupon PBS, whose floating coupon payment depends on the revenue level generated by the underlying infrastructure projects. While revenue performance-linked PBSs implement project finance "more efficiently and effectively than conventional PBSs" [11], they shift the revenue risk from developers to the investors. As a result, perspective investors would require a contingent claim to hedge the risk that the underlying projects may generate lower revenues than expected.

Such contingent claim can take the form of an American put option, which entitles investors to sell the PBS back to the developers. Thus, through exercising this option, investors can terminate the contract, collect the de-investment proceedings and invest them in more profitable assets. On the other hand, developers would require a different contingent claim to hedge the missed gains that occur when the underlying projects generates higher revenues than expected. Such a contingent claim can be modeled as an American call option, which entitles developers to buy back the PBS from the investors. Thus, through exercising this option, developers can terminate the contract and sell a new set of PBSs at a higher price, which is consistent with the updated expected revenues.

Furthermore, a complementary advantage for incorporating the call and put options in the performance-linked PBS lies in protecting both investors and issuers from potential losses due to interest rate fluctuations. Recognizing and pricing the economic benefits of these contractual options embedded in the performance-linked PBSs are critical in preserving a fair distribution of risks and rewards between developers and investors. However, due to the unpredictability of future cash flows and high volatility of risk-free interest rates, pricing PBSs embedded with call and put options is not a trivial task. Moreover, there are no analytical formulas even under very simplified assumptions. Brennan and Schwartz [3] are credited to have presented one of the first approaches to price bonds embedded options. Their approach was based on a conventional finite difference method. Hull and White [10] proposed a tree based method using trinomial trees in a generalized version with time-dependent parameters. Büttler and Waldvogel [4] priced callable bonds using Green's functions. d'Halluin *et al.* [6] presented a valuation method for callable bonds based on finite difference approach using finite elements, flux limiters and appropriate time stepping. Ben-Ameur *et al.* [2] proposed a dynamic programming procedure to price options embedded in bonds in which the possibility of default of the issuer was neglected.

All of the abovementioned methods were developed under the assumption that the bond value depends only on one state variable, the interest rate. For a callable and putable performance-linked PBS, one more uncertain variable, the generated revenue of the bundled projects, needs to be added into the valuation model. Consequently, the optimal exercise policy of the embedded options is affected by the resulting combination of the interest rate and revenue value, along with the different optimization objectives of the developers and private investors.

This paper presents a fast and easy-to-implement pricing method for performance-linked PBS with embedded options based on the multilinear regression method (MRM), a combined *approximate dynamic programming* and *Monte Carlo simulation* approach. The paper is organized as follows. Section 2 reviews the stochastic processes of the two state variables and their formulas for simulation representation. Section 3 introduces the mechanism of callable and putable performance-linked PBS. Section 4 illustrates the multilinear regression method for pricing the option embedded performance-linked PBS. Numerical analyses of a hypothetical example are presented in Secs. 5 and 6. Finally, Sec. 7 draws and highlights the conclusions.

2. Underlying Stochastic Processes

2.1. *Stochastic process for revenue*

A performance-linked PBS is linked to revenues of projects that have been in operation for several years. At this advanced operation stage, a steady revenue trend can be established. Under the assumption that the net revenue cash flows may be negative [1], we use an Ornstein-Uhlenbeck process to simulate the total revenue of the pool of projects

$$dR = k_R\,(u_R - R)dt + \sigma_R dZ_R \tag{2.1}$$

where k_R is the speed at which the revenue R reverts to the average revenue, u_R. The corresponding revenue volatility is σ_R, and Z_R is the standard Wiener process. Furthermore, the drift of Eq. (2.1) is $\alpha = k_R(u_R - R)$.

The revenue risk-adjusted discount rate can be dually defined as $u = r + \lambda\sigma_R$ (i.e. the sum of the risk-free rate r and the revenue market risk premium $\lambda\sigma_R$) or $u = \alpha + \delta$ (i.e. the sum of the process drift α and a yield δ). The market price of revenue risk, λ, can be estimated through the capital asset pricing model (CAPM) as follows

$$\lambda = \rho_{R,m}\left[\frac{m-r}{\sigma_m}\right] \tag{2.2}$$

where $\rho_{R,m}$ is the correlation coefficient between the market return and the revenues, m is the expected return of the market portfolio, σ_m is the standard deviation of the return on the market, and r is the risk-free interest rate. By comparing the dual definition of u, it yields

$$\alpha - \lambda\sigma_R = r - \delta \tag{2.3}$$

$$\delta = u - \alpha = u - k_R\left(u_R - R\right) \tag{2.4}$$

The risk-neutralized drift that changes the probability measure of the process from real to risk-neutral is [9]

$$\alpha' = \alpha - \lambda\sigma_R \tag{2.5}$$

Thus, after factoring Eqs. (2.3) and (2.4) into (2.5) we obtain the risk-neutral drift

$$\alpha' = k_R\left(u_R - \frac{u-r}{k_R} - R\right) \tag{2.6}$$

Replacing the process drift α with the risk neutral drift α' yields the risk neutral equivalent process

$$dR = k_R\left(u_R - \frac{u-r}{k_R} - R\right)dt + \sigma_R dZ'_R \tag{2.7}$$

It follows that the mean and variance of the distribution R_i are:

$$E[R_i] = R_0\exp(-k_R t_i) + \left(u_R - \frac{u-r}{k_R}\right)(1-\exp(-k_R t_i))$$

$$Var(R_i) = (1-\exp(-2k_R t_i))\frac{\sigma_R^2}{2k_R}$$

Under a risk-neutrality assumption, the equation to perform the Monte Carlo simulation of the $\omega(k)$−th path, for $k = 1, \ldots, N$, is given by

$$\begin{aligned} R_i^{\omega(k)} &= [(R_{i-1}^{\omega(k)})\cdot\exp(-k_R\Delta t)] + \left[\left(u_R - \frac{u-r}{k_R}\right)(1-\exp(-k_R\Delta t))\right] \\ &\quad + \left[\sigma_R\sqrt{\frac{1-\exp(-2k_R\Delta t)}{2k_R}}\varepsilon_R^{\omega(k)}\right] \end{aligned} \tag{2.8}$$

with $\varepsilon_R \sim N(0,1)$ and Δt is the discrete time unit.

2.2. *Stochastic process for interest rate*

We assume that the annualized risk-free interest rate follows the generic Vasicek model process [16]

$$dr = k_r(u_r - r)dt + \sigma_r dZ_r \tag{2.9}$$

where k_r is the speed at which the risk-free interest rate r reverts to the average of risk-free interest rate, u_r. The corresponding volatility is σ_r, and Z_r is the standardized Wiener process. Hereafter, we assume that R and r are statistically correlated with correlation coefficient ρ.

The mean and variance of the distribution r_i are:

$$E[r_i] = r_0 \exp(-k_r t_i) + u_r(1 - \exp(-k_r t_i))$$

$$Var(r_i) = (1 - \exp(-2k_r t_i))\frac{\sigma_r^2}{2k_r}$$

The equation for the Monte Carlo simulation of the $\omega(k)$−th path is given by

$$\begin{aligned} r_i^{\omega(k)} &= r_{i-1}^{\omega(k)} \cdot \exp(-k_r \cdot \Delta t) + u_r \cdot (1 - \exp(-k_r \cdot \Delta t)) \\ &\quad + \sigma_r \sqrt{\frac{1 - \exp(-2 \cdot k_r \cdot \Delta t)}{2 \cdot k_r}} \varepsilon_r^{\omega(k)} \end{aligned} \tag{2.10}$$

with $\varepsilon_r \sim N(0, 1)$ and Δt the discrete time unit.

3. Mechanism of Callable or Putable Performance-Linked PBS

Project-backed securities are types of bonds backed by a pool of projects that are capable of generating positive revenues. Revenue performance-linked PBSs [11] are bonds whose floating periodical coupon payments is proportional to the amount of revenues generated by the pool of projects. The indenture of the performance-linked PBSs requires the issuer to pay the holder a known amount, the principal at the maturity of the contract. Furthermore, the issuer is required to periodically pay investors floating coupons linked to the generated project revenues.

The tenor structure considered is $(t_0 = 0) < (t_1 = 1) < \cdots < (t_{n-1} = n - 1) < (t_n = n)$, where t_0 is the initial time and t_n is the maturity date of the PBS. Accordingly, the coupon payments are denoted by $c_1 < \cdots < c_n$, respectively. Under a risk neutral evaluation, the contractual value of the performance-linked PBS without embedded options a time t_0 is given by

$$\bar{V}_0 = \left[\sum_{j=1}^{n}(E[c_j] \cdot \overline{D}_{j,0})\right] + P_n\overline{D}_{n,0} \tag{3.1}$$

where

c_j, the coupon payment at time $t = j$, is the $\alpha\%$ share of the project revenues R_j, i.e. $c_j = \alpha\% R_j$;

R_j is the revenue at time j, which can be modeled as presented in Sec. 2.1;

$\overline{D}_{j,0} = \frac{1}{(1+E[r_j])^j}$ is the expected discount factor from time j to time 0;

r_j is the interest rate at time j, which can be modeled as presented in Sec. 2.2;

P_n is the principal the issuers are supposed to pay the holders at maturity t_n, and it is calculated as $\sum_{j=1}^{n}[((1-\alpha\%)E[R_j]) \cdot (1+E[r_{j,n}])^{n-j}]$, where $(1-\alpha\%)E[R_j]$ is the unit time step revenue share that is not included in the floating coupon, i, and $r_{j,n}$ is computed trough a bootstrapping procedure by solving the following equation

$$(1 + r_j)^j (1 + r_{j,n})^{(n-j)} = (1 + r_n)^n$$

Furthermore, the "residual value" of the PBS without embedded options at the generic time t_i is defined as

$$\bar{V}_i = \sum_{j=i+1}^{n} (E[c_j] \cdot \overline{D}_{j,i}) + P_n \overline{D}_{n,i} \tag{3.2}$$

with $\overline{D}_{j,i} = \frac{1}{(1+E[r_{i,j}])^{j-i}}$ the discount factor from time j to time i, and $r_{j,i}$ is computed by solving

$$(1 + r_i)^i (1 + r_{i,j})^{(j-i)} = (1 + r_j)^j$$

Both issuers (developers) and investors of performance-linked PBSs can be negatively affected by the volatility of the future revenue cash flows and interest rates. From an investment point of view, investors' returns will decrease and issuers' returns will increase when low revenues and high interest rates occur. Conversely, high revenues and low interest rates will increase investors' returns and will diminish issuers' returns. In order to augment the marketability of performance-linked PBSs, issuers and investors would need to hedge their risk exposures. Under normal circumstances, issuers would be hesitant to offer third parties guarantees because, beside the additional fees associated with these guarantees, third parties usually demand a tight control of the project operations. A more viable alternative solution is to integrate performance-link PBSs with risk hedging contracts such as a call option for the issuers and a put option for the investors. Both put and call are American-type options (Bermudan) that allow the option's holders an early exercise. The Bermudan call option embedded in the performance-linked PBS gives issuers the right to purchase back their debts for a known amount, the call strike price, at the specified times before maturity. Conversely, the embedded Bermudan put option gives the holders the right to have issuers pay the debt back for a different known amount, the put strike price. Pricing PBSs within this option framework require considering the options as an integral part of the bond instead of simply adding separately the options' values to the bond's value. This occurs because the value of this hybrid bond is affected by the equal right to terminate the contract offered to both issuers and buyers. In fact, the put and call options are used as protection tools when the state variables move to their negative potential of uncertainties. As soon as the issuers or buyers think the present value of their expected future payoffs is less than their immediate payoffs, they will terminate the contract. At any possible coupon payment date $\{t_i\}_{i=1,\ldots,n}$, if issuers want to terminate the PBS contract, they can exit the contract by paying, in addition to the due coupon payment c_i, the pre-established residual PBS value, Eq. (3.2), plus a penalty fee, $\psi_i^{\uparrow}$, as compensation for early termination. On the other hand, if buyers want to terminate the PBS contract, they would receive, in addition to the due coupon payment c_i, the pre-established PBS residual value, Eq. (3.2), minus a penalty fee, $\psi_i^{\downarrow}$, as compensation for early termination. The contract termination rights of the issuers and buyers are modeled as a call option with strike price $K_i^{\uparrow}$ and a put option with strike price $K_i^{\downarrow}$. Accordingly, the two option strike prices have two time-dependent components: the pre-established residual value $\bar{V}_i$ and the penalty fee ψ_i. More specifically

$$\begin{cases} K_i^{\uparrow} = \bar{V}_i + \psi_i^{\uparrow} = \left[\displaystyle\sum_{j=i+1}^{n} (E[c_j] \cdot \overline{D}_{j,i}) + P_n \overline{D}_{n,i} \right] + \psi_i^{\uparrow} \\ K_i^{\downarrow} = \bar{V}_i - \psi_i^{\downarrow} = \left[\displaystyle\sum_{j=i+1}^{n} (E[c_j] \cdot \overline{D}_{j,i}) + P_n \overline{D}_{n,i} \right] - \psi_i^{\downarrow} \end{cases} \tag{3.3}$$

where $\bar{V}_i$ is the residual value given by Eq. (3.2), and $\psi_i^{\uparrow}$ and $\psi_i^{\downarrow}$ are the penalty fee the issuers and investors are obliged to pay the counterpart if an option is exercised at t_i; two common penalty fee functions

$$\psi_i = \begin{cases} (t_n - t_i) \cdot \text{Constant} \\ \text{Constant}^{(t_n - t_i)} \end{cases} \tag{3.4}$$

are shown in Fig. 1.

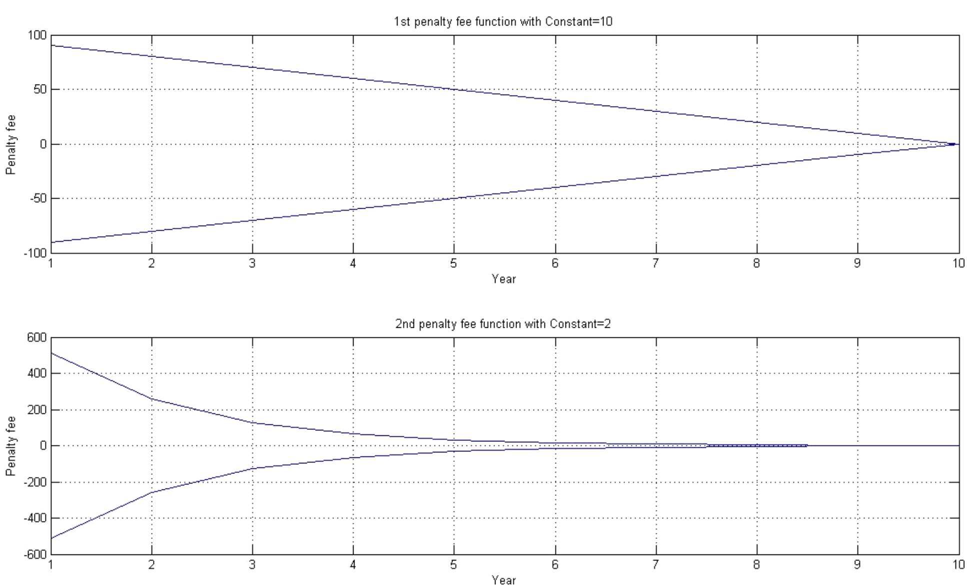

Fig. 1. Illustration of two penalty fee functions.

4. Pricing via Dynamic Programming

We first provide a precise formulation of the model considered for pricing the performance-linked PBSs with their embedded call and put options. It is assumed that the bond vest period covers the time interval $[0, 1]$ and the tenor structure is: $(t_0 = 0) < (t_1 = 1) < \cdots < (t_{n-1} = n - 1) < (t_n = n)$, where t_0 is the initial time and t_n is the maturity date of the PBS. Accordingly, the corresponding coupon payments are denoted by $c_1 < \cdots < c_n$. For $i = 1, \ldots, n-1$, the unit time steps, Δt_i^{i+1}, are equally spaced, while the vest period, Δt_0^1, can have a different time length. The possible option exercise dates are any of the coupon payment dates $t_1 <, \ldots, < t_n$. The non-constant call and put strike prices at time i are $K_i^{\uparrow}$ and $K_i^{\downarrow}$, as defined in Eq. (3.3), respectively. If investors "put" the bond at time i, they will receive $K_i^{\downarrow} + c_i$ from the issuers. On the other hand, if issuers "call" the bond at time i, they will pay $K_i^{\uparrow} + c_i$ to the investors. It is worth noting that the strike price is composed by two time-functions, the pre-established residual value $\overline{V}_i$ and the penalty fees ψ_i, that are both monotonically decreasing with time. As a result, $K_1^{\uparrow} \geq \cdots \geq K_{n-1}^{\uparrow} \geq K_n^{\uparrow}$ and $K_1^{\downarrow} \leq \cdots \leq K_{n-1}^{\downarrow} \leq K_n^{\downarrow}$, that is the earlier the issuers call the PBS, the higher is the monetary amount the issuers will pay to the investors; and the earlier the investors put the PBS, the less they will receive from the issuers (see Fig. 2).

At time t_i, the value of callable and put performance-linked PBS, V_i, is derived by considering investors and issuers' decision making strategies that maximize the profits. Issuers should call the PBS at time i if the present value of the expectation of

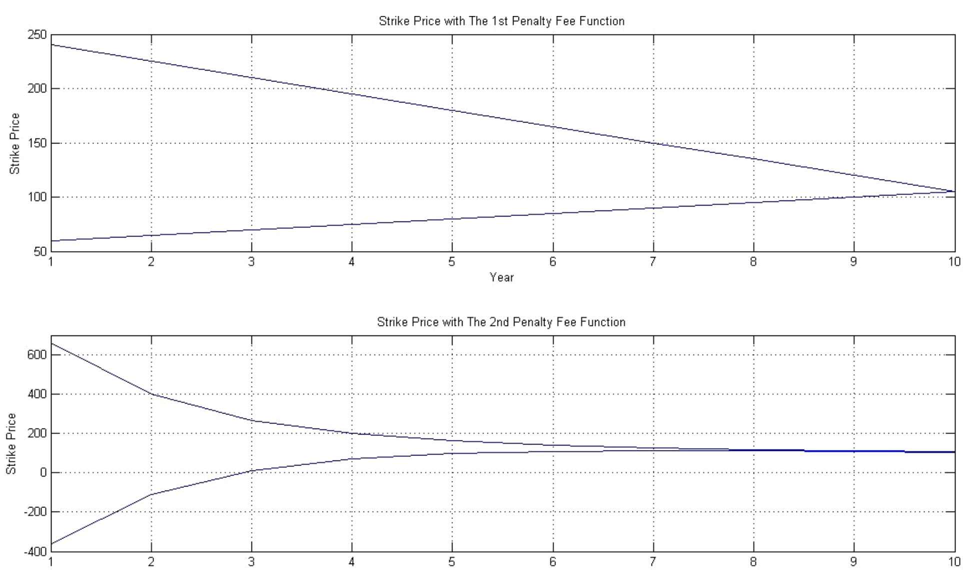

Fig. 2. Illustration of two strike prices.

the future PBS payoffs, $E[D_{i+1,i}V_{i+1}|\mathrm{F}_i]$, conditional on the information F_i known at time i, is greater than the call strike price $K_i^{\uparrow}$, where $D_{j,i} = \frac{1}{(1+r_{i,j})^{j-i}}$ the discount factor from time j to time i, and $r_{j,i}$ is computed as shown in section Sec. 3. Conversely, investors should put the bond at time i if the put strike price $K_i^{\downarrow}$ is greater than the expected present value of the future PBS payoffs, $E[D_{i+1,i}V_{i+1}|\mathrm{F}_i]$ conditional on the information F_i known at time i. Accordingly, the decision making strategies of the issuers and investors are $\min\{K_i^{\uparrow}, E[D_{i+1,i}V_{i+1}|\mathrm{F}_i]\}$ and $\max\{K_i^{\downarrow}, E[D_{i+1,i}V_{i+1}|\mathrm{F}_i]\}$, respectively. It is worth noting that it is unnecessary to include an exercise priority rule between the parties because they will never exercise their rights simultaneously as it occurs that $K_i^{\uparrow} > K_i^{\downarrow}$ for any i.

The recurrent backward Bellman equations [14] that define the value of the PBS are:

— At time n, when the issuers are contractually obliged to liquidate the debt, the value of the PBS is given by

$$V_n = P_n + c_n \tag{4.1}$$

— At a generic time $1 \le i \le n-1$, the value of the PBS is

$$V_i = \begin{cases} \max\{\min\{K_i^{\uparrow},\, E[D_{i+1,i}\,V_{i+1}|\mathrm{F}_i]\},\, K_i^{\downarrow}\} + c_i \\ \qquad \text{or equivalently} \\ \min\{\max\{K_i^{\downarrow},\, E[D_{i+1,i}V_{i+1}|\mathrm{F}_i]\},\, K_i^{\uparrow}\} + c_i \end{cases} \tag{4.2}$$

— At time 0, the PBS value is calculated by adding the present value of $V_{i=1}$ to the present value of any vest period coupon payments c_j, that is

$$V_0 = V_1 \cdot D_{1,0} + \sum^{0<j<1} c_j D_{j,0} \tag{4.3}$$

The PBS value and the optimal option exercise time are computed by solving Eqs. (4.1), (4.2) and (4.3) backward from time t_n to t_0 using an approximate dynamic programming approach combined with Monte Carlo simulation that extends the Longstaff and Schwartz's approach [5, 12]. In Eq. (4.2), the exact value of $K_i^{\uparrow}$ and $K_i^{\downarrow}$ are easily calculated from Eq. (3.3), while the approximate value of $E[D_{i+1,i}V_{i+1}\,|\,\mathrm{F}_i]$ can be calculated using the multi-linear regression Monte Carlo method (MRMC), which is a two factor model extension of the least-squares method proposed by Longstaff and Schwartz [12]. MLRM least-squares regresses the two simulated state variables $\{R_1^{\omega(k)}, \ldots, R_n^{\omega(k)}\}_{k=1,\ldots,N}$ and $\{r_1^{\omega(k)}, \ldots, r_n^{\omega(k)}\}_{k=1,\ldots,N}$ to estimate the conditional expectation $E[D_{i+1,i}V_{i+1}|\mathrm{F}_i]$. Similarly to the Longstaff Schwartz's approach [12], we define the conditional continuation value at time i as $Cont_i = E[D_{i+1,i}V_{i+1}\,|\,\mathrm{F}_i]$. The continuation value at time i is the expected present value of the stream of cash flows the investors should receive if neither the investors nor the issuers have terminated the contract at time i or earlier. The continuation value at time t_i is calculated by estimating the conditional expectation $\overline{Cont_i} = E[D_{i+1,i}V_{i+1}|R_i, r_i]$ by regressing the N-simulated present values of the future PBS payoffs $\{(D^{\omega(1)}_{i,i+1} \cdot V^{\omega(1)}_{i+1}), \ldots, (D^{\omega(n)}_{i,i+1} \cdot V^{\omega(N)}_{i+1})\}$ with the two current state variables at time t_i, the N-simulated risk-free interest rates $\{r^{\omega(1)}_i, \ldots, r^{\omega(N)}_i\}$ and revenues $\{R^{\omega(1)}_i, \ldots, R^{\omega(N)}_i\}$. The continuation value $\overline{Cont}_i$ can be represented as a liner function of the elements of orthonormal countable basis,

$$\overline{Cont}_i = \sum_{k=1}^{\infty} \left(a_k \cdot p^r_k(r_i) + b_k \cdot p^R_k(R_i)\right) \tag{4.4}$$

where p^r_k and p^R_k are the kth elements of the basis; while a_k and b_k are the associated kth constant coefficients. The simplest and most efficient approximate form of Eq. (4.4) can be derived through linear multiple regression [17], that is

$$\overline{Cont}_i \approx (a_i r_i + b_i R_i) \tag{4.5}$$

where a_i and b_i are the associated constant coefficients for r_i and R_i, respectively. The set of coefficients a_i and b_i can be estimated by least-squares regression onto the basis

$$\{\hat{a}_i, \hat{b}_i\} = \arg\min \parallel (a_i r^{\omega(k)}_i + b_i R^{\omega(k)}_i) - D^{\omega(k)}_{i+1,i} \cdot V^{\omega(k)}_{i+1,} \parallel \tag{4.6}$$

The basic equations of the algorithm of the backward dynamic programming approach for each of the kth simulated path $\omega(k)$, with $k = 1, \ldots, N$, can be derived adapting Eqs. (4.1), (4.2) and (4.3) as follows

$$\text{At}(t = n) \rightarrow V^{\omega(k)}_n = P_n + \alpha\%\, R^{\omega(k)}_n \tag{4.7}$$

At $(t = i)_{i=1,\ldots,n-1}$

$$\rightarrow V^{\omega(k)}_i = \begin{cases} K^{\uparrow}_i + \alpha\% R^{\omega(k)}_i & \text{if } \overline{Cont}^{\omega(k)}_i > K^{\uparrow}_i \\ K^{\downarrow}_i + \alpha\%\, R^{\omega(k)}_i & \text{if } \overline{Cont}^{\omega(k)}_i < K^{\downarrow}_i \\ D^{\omega(k)}_{i+1,i} \cdot V^{\omega(k)}_{i+1,} + \alpha\%\, R^{\omega(k)}_i & \text{if } K^{\downarrow}_i \leqq \overline{Cont}^{\omega(k)}_i \leqq K^{\uparrow}_i \end{cases} \tag{4.8}$$

$$\text{At}(t = 0) \rightarrow V^{\omega(k)}_0 = V^{\omega(k)}_1 \cdot D^{\omega(k)}_{1,0} + \sum^{0<j<1} R^{\omega(k)}_j\, D^{\omega(k)}_{j,0} \tag{4.9}$$

Then the expected value of the callable and putable PBS can be calculated as

$$V_0 = \frac{1}{N} \sum_{k=1}^{N} V^{\omega(k)}_0 \tag{4.10}$$

It is worth noting from Eq. (4.8) that if the $K^{\downarrow}_i \leq \overline{Cont}_i \leq K^{\uparrow}_i$, no parties will exercise their option rights. Therefore, the putable and callable PBS creates a protection for both parties outside the range$[\psi^{\downarrow}_i, \psi^{\uparrow}_i]$ around the residual value Eq. (4.3). Furthermore, because at initialization of the contract the simple performance–linked PBS is worth $\bar{V}_0$, Eq. (3.1), and the callable and putable performance-linked PBS is worth V_0, Eq. (4.10), the actual value of the embedded put and call options θ at time t_0 is the difference between the two values, $\theta = (V_0 - \bar{V}_0)$.

If $\theta > 0$, the hedging protection of the investors is more valuable than the hedging protection of the issuers. In this case, issuers are entitled to receive as much as $(V_0 - \bar{V}_0)$to make the deal fair. If $\theta < 0$, the protection values are reverse and the investors are entitled to receive the compensation $(\bar{V}_0 - V_0)$from the issuers. Finally, protection equilibrium occurs when $\theta = 0$, that is when the issuers' protection value counterbalances the investors' protection value, and no compensation is required.

5. Numerical Example

The assumed PBS indenture specifies that the term period equals to 10 years, coupon payments are due every 6 months, and no vest period. The tenor structure is $(t_0 = 0) < (t_1 = 1) <, \ldots, < (t_{n-1} = n-1) < (t_n = n)$, where t_0 is the initial time, and there are $n = 20$ payment dates. Both contractual parties, issuer and investor, can terminate the transaction every six months, i.e. at each coupon payment date. The other parameters are all listed in Table 1.

It is assumed that each semiannual coupon payment is 40% of the corresponding semiannual revenues generated by the pool of infrastructure projects. The two types of strike functions from Eq. (3.3) are shown in Fig. 3 along with the initial and residual value of the PBS, which is calculated using Eqs. (3.3) and (3.2).

The approximate dynamic programming multi-linear regression method is carried out by first simulating the revenue and risk-free rate paths, $\{R_1^{\omega(k)}, \ldots, R_n^{\omega(k)}\}_{k=1,\ldots,N}$ and $\{r_1^{\omega(k)}, \ldots, r_n^{\omega(k)}\}_{k=1,\ldots,N}$, and then calculating recursively Eq. (4.4), Eq. (4.7) and Eq. (4.8) for each simulated path $\omega(k)$. Convergence studies of the callable and putable performance-linked PBS were performed by ranging the number of simulated paths N from 10,000 to 100,000, with $N_i = 10,000 + 1000 \cdot i$ and $i = 1, \ldots, 99$. The convergence results for the two types of strike functions are shown in the left side of Fig. 4.

If the relative error, $RE^{N_i} = \frac{(\theta^{N_{i+1}} - \theta^{N_i})}{\theta^{N_i}}$, is set equal to a tolerated error of 0.5%, the number of paths that makes the error converge, i.e. burn-in path N_i^*, is $93,000$ and $14,000$ for the 1st strike price function and 2nd strike price function, respectively. The error convergence results are presented in the right side of Fig. 4.

Table 1. Input parameters for monte carlo numerical analysis.

Risk-free Interest Rate		Revenue	
Mean Reverting Speed	$k_r = 0.05$	Mean Reverting Speed	$k_R = 0.05$
Long-run Equilibrium Level	$u_r = 0.05$	Long-run Equilibrium Level	$u_R = 100$
Volatility	$\sigma_r = 0.004$	Volatility	$\sigma_R = 4$
Correlation coefficient b/w r and R			$rho = 0.5$
Maturity			$T = 10$
Interval			$\Delta t = 0.5$
risk-adjusted discount rate			$u = 0.055$
Revenue α% Share			40%
Constant Parameter in ψ_1			15
Constant Parameter in ψ_2			3

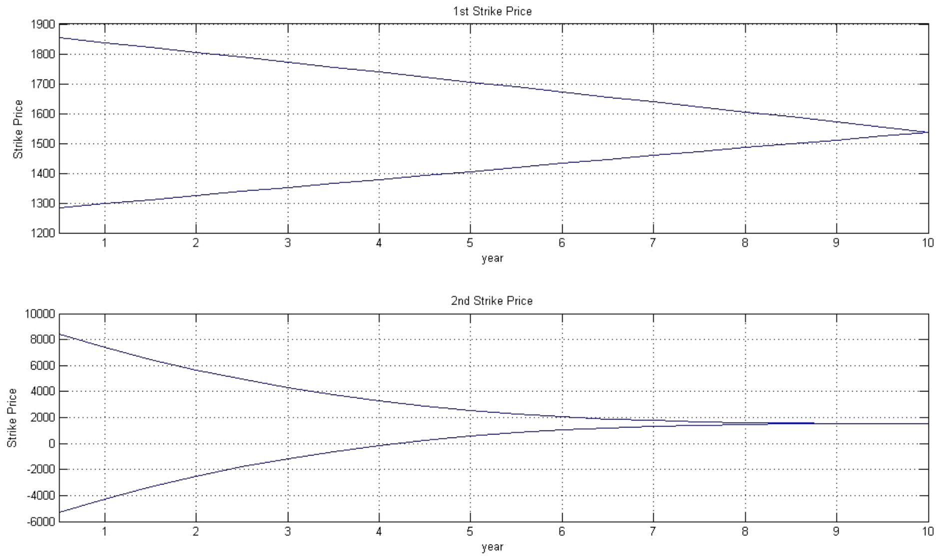

Fig. 3. Residual value and strike prices relative to ψ_1 and ψ_2.

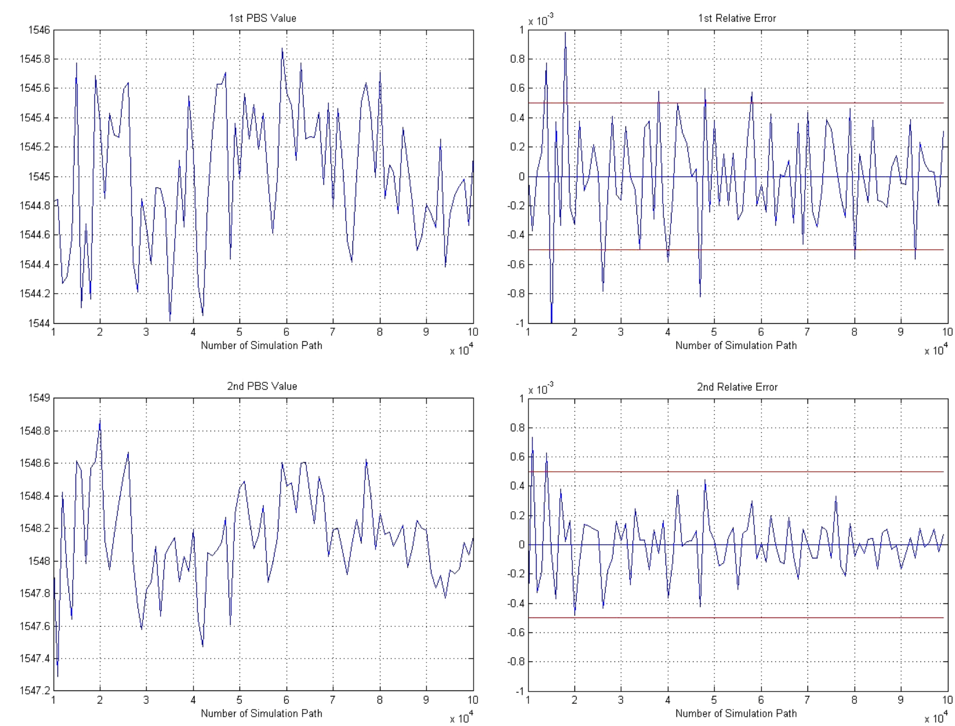

Fig. 4. Value and error convergence results for specific N path numbers.

6. Numerical Example Extension

The numerical example can be extended to the case where the strike prices $K_i^{\uparrow}$ and $K_i^{\downarrow}$ are not anymore deterministic but random processes depending on the revenue R_i as in Eq. (6.1).

$$\begin{cases} K_i^{\uparrow} = \left[\sum_{j=i+1}^{n} (E[c_j|R_i] \cdot \overline{D}_{j,i}) + P_n \overline{D}_{n,i} \right] + \psi_i^{\uparrow} \\ K_i^{\downarrow} = \left[\sum_{j=i+1}^{n} (E[c_j|R_i] \cdot \overline{D}_{j,i}) + P_n \overline{D}_{n,i} \right] - \psi_i^{\downarrow} \end{cases} \tag{6.1}$$

Unlike deterministic residual PBS value in Eq. (3.3), the first component of Eq. (6.1) gives the expected payout, given the revenue and the interest rate at time i, if both the issuer and investor do not exercise their options until maturity. This new formulation of the strike prices would be more fair to both issuer and the investor than the definition of Eq. (3.3). This is because when R_i is high, the investor expects the residual PBS payoff to be high. Therefore it seems unfair to the investor if the issuer exercises her call option using the strike price $K_i^{\uparrow}$ defined in Eq. (3.3), which might be lower than the investor's expected payoff. On the other hand, when R_i is low, the issuer expects the residual PBS payoff to be low. If $K_i^{\downarrow}$ is defined as in Eq. (3.3), the issuer would pay more than she expects when the investor exercise her option. It is worth noting that, due to the mean reverting feature of the revenue process (2.1), the values of the random strike prices of Eq. (6.1) closely float around the deterministic strike prices defined in Eq. (3.3). For the computational analysis, the conditional expected payoff in Eq. (6.1) is given by

$$\begin{aligned} E[c_{i+1}^{\omega(k)} \mid R_i^{\omega(k)}] &= \alpha\% R_i^{\omega(k)} \exp(-k_R \Delta t) \\ &\quad + \left(u_R - \frac{u - r_i^{\omega(k)}}{k_R} \right) (1 - \exp(-k_R \Delta t)) \end{aligned} \tag{6.2}$$

For each simulated path $\omega(k)$, the corresponding pairs of strike price sequences is

$$\begin{cases} K_i^{\uparrow,\omega(k)} = \left[\sum_{j=i+1}^{n} (E[c_j^{\omega(k)} \mid R_i^{\omega(k)}] \cdot \overline{D}_{j,i}) + P_n \overline{D}_{n,i} \right] + \psi_i^{\uparrow} \\ K_i^{\downarrow,\omega(k)} = \left[\sum_{j=i+1}^{n} (E[c_j^{\omega(k)} \mid R_i^{\omega(k)}] \cdot \overline{D}_{j,i}) + P_n \overline{D}_{n,i} \right] - \psi_i^{\downarrow} \end{cases} \tag{6.3}$$

The convergence results for the two types of strike functions are shown in the left side of Fig. 5. If the relative error and tolerated error is defined the same as section §5, the number of paths that makes the error converge, i.e. burn-in path N_i^*, is 47,000 and 18,000 for the 1st strike price function and 2nd strike price function, respectively. The error convergence results are presented in the right side of Fig. 5.

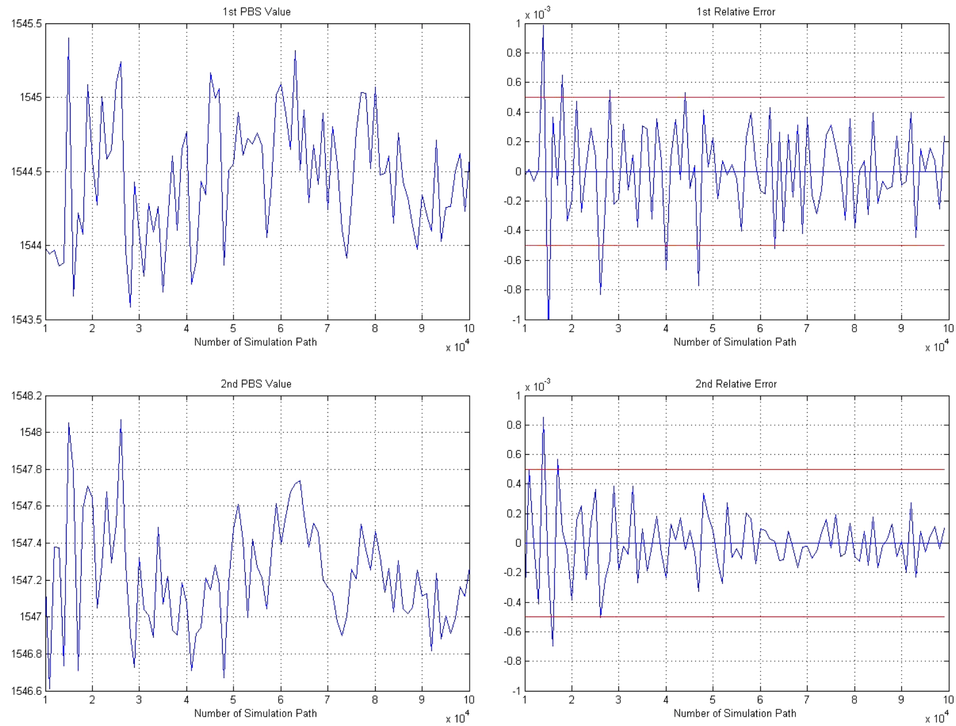

Fig. 5. Value and error convergence results for specific N path numbers under stochastic strike prices.

7. Conclusions

We have shown how to implement the pricing of callable and putable revenue performance-linked PBS with two random factors, the revenue generated by a pool of projects and the risk-free interest rate. This debt instrument allows both issuers and investors to hedge the risks against the detrimental fluctuation of project revenues and interest rates. Under the reasonable assumptions of non-arbitrage and no default, the pricing of the performance-linked PBS, along with the embedded options, is performed with a new dynamic programming procedure, the multi-linear regression Monte Carlo method, which is both fast and precise.

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