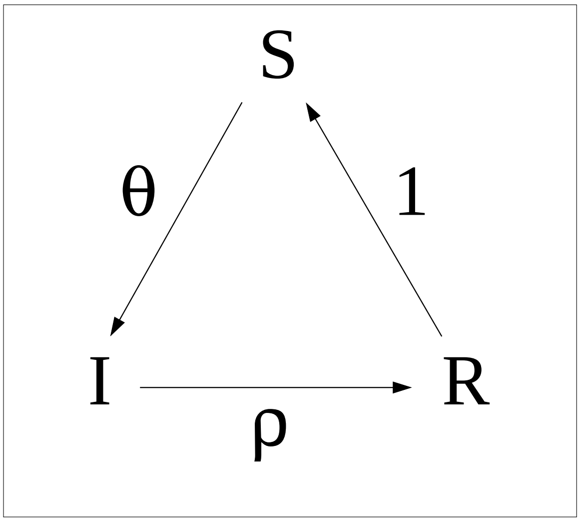
Endemic Infections

SIRS Epidemics

- The infection does not lead to permanent immunity

- A recovered individual looses immunity at some rate and re-enters the susceptible state.



Endemic Infections

SIRS Epidemics

- Again, eventually epidemics becomes extinct. However, in some cases the number of infected can stay large for long time $\Rightarrow$ infection reaches *endemic level*.

- Infection is *endemic* in a population when it is maintained without the need for external inputs.

- **e.g.** Chickenpox is endemic in the UK, but malaria is not – every year, there are a few cases of malaria acquired in the UK, but these do not lead to sustained transmission in the population.

Endemic Infections

SIRS Epidemics

As before

$$S_t = \# \text{ susceptible at time } t$$
$$I_t = \# \text{ infected at time } t$$
$$R_t = \# \text{ recovered (immune) at time } t$$

$$N \equiv S_t + I_t + R_t = \text{ population size}$$

$$\gamma_t = (S_t/N, I_t/N) \equiv (s_t, i_t)$$

$$r_t = R_t/N = 1 - s_t - i_t$$

Endemic Infections

SIRS Epidemics

MCs indexed by N with transition rates:

$$\rho\,(s \to i) = S \cdot \theta I/N = N\theta si$$
$$\rho\,(i \to r) = \rho I = N\rho i$$
$$\rho\,(r \to s) = R = Nr$$

- Will infection spread?
- If yes, how does it develop with time?
- When does it start to decline?

Endemic Infections

SIRS Epidemics

Fix N, $h > 0$

$$\mathbf{E}(s_{t+h}) = s_t + r_t h - \theta i_t s_t h + \mathbf{o}(h)$$
$$\mathbf{E}(i_{t+h}) = i_t + \theta i_t s_t h - \rho i_t h + \mathbf{o}(h)$$
$$\mathbf{E}(r_{t+h}) = r_t + \rho i_t h - r_t h + \mathbf{o}(h)$$

Get "mean field approximation" as $h \to 0$

$$\left\{ \begin{array}{rcl} \dfrac{ds_t}{dt} & = & r_t - \theta i_t s_t \\ \dfrac{di_t}{dt} & = & \theta i_t s_t - \rho i_t \end{array} \right. \equiv F(\gamma_t)$$

Endemic Infections

SIRS mean path

- Epidemiology: infection is in an endemic steady state if

$$R_0 \times S = 1$$

- $R_0 = \theta/N \times 1/\rho \Longrightarrow s_\infty = \rho/\theta$

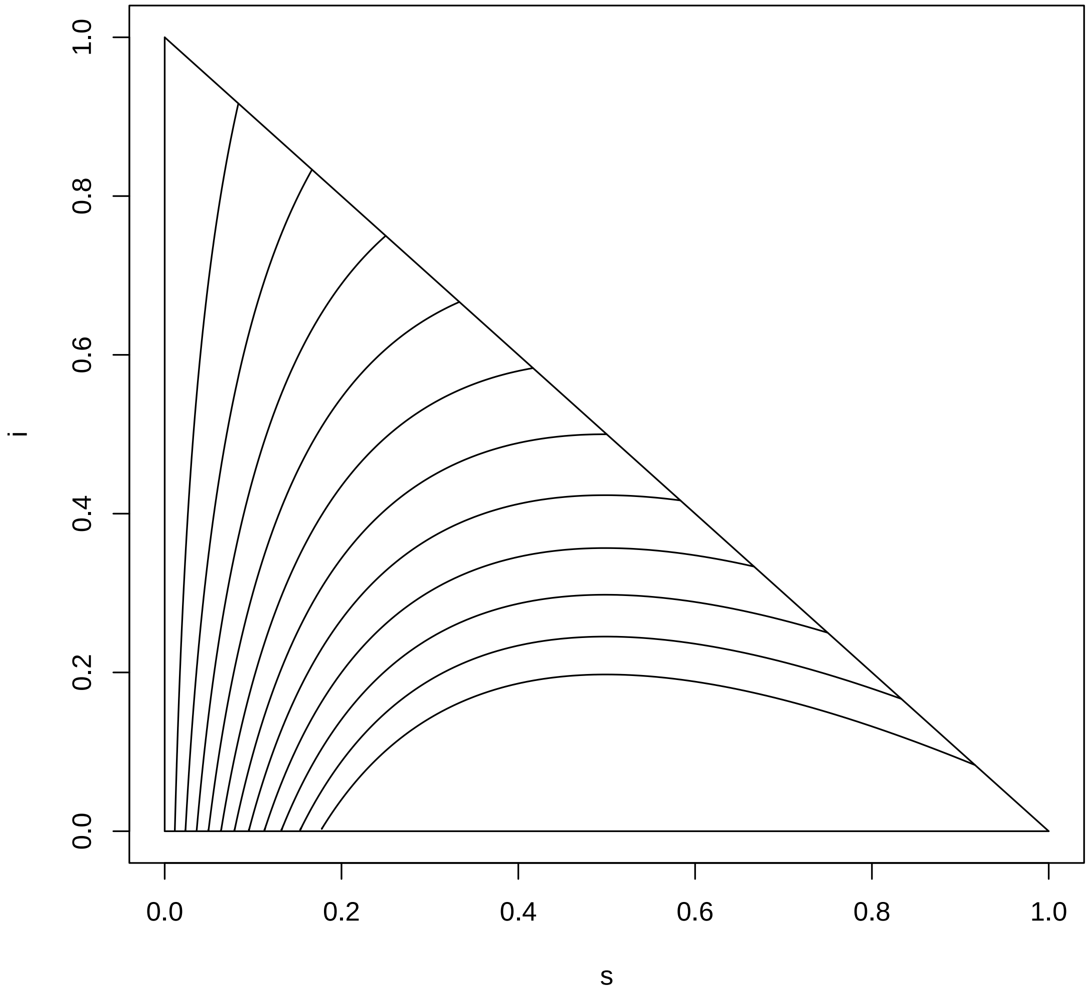
- Alternatively,

$$\begin{cases} \dfrac{ds_t}{dt} = r_t - \theta i_t s_t = 0 \\ \dfrac{di_t}{dt} = \theta i_t s_t - \rho i_t = 0, \end{cases}$$

$\Longrightarrow s_\infty = \rho/\theta, i_\infty = \frac{1-s_\infty}{1+\theta s_\infty} = \frac{1-\rho/\theta}{1+\rho}$

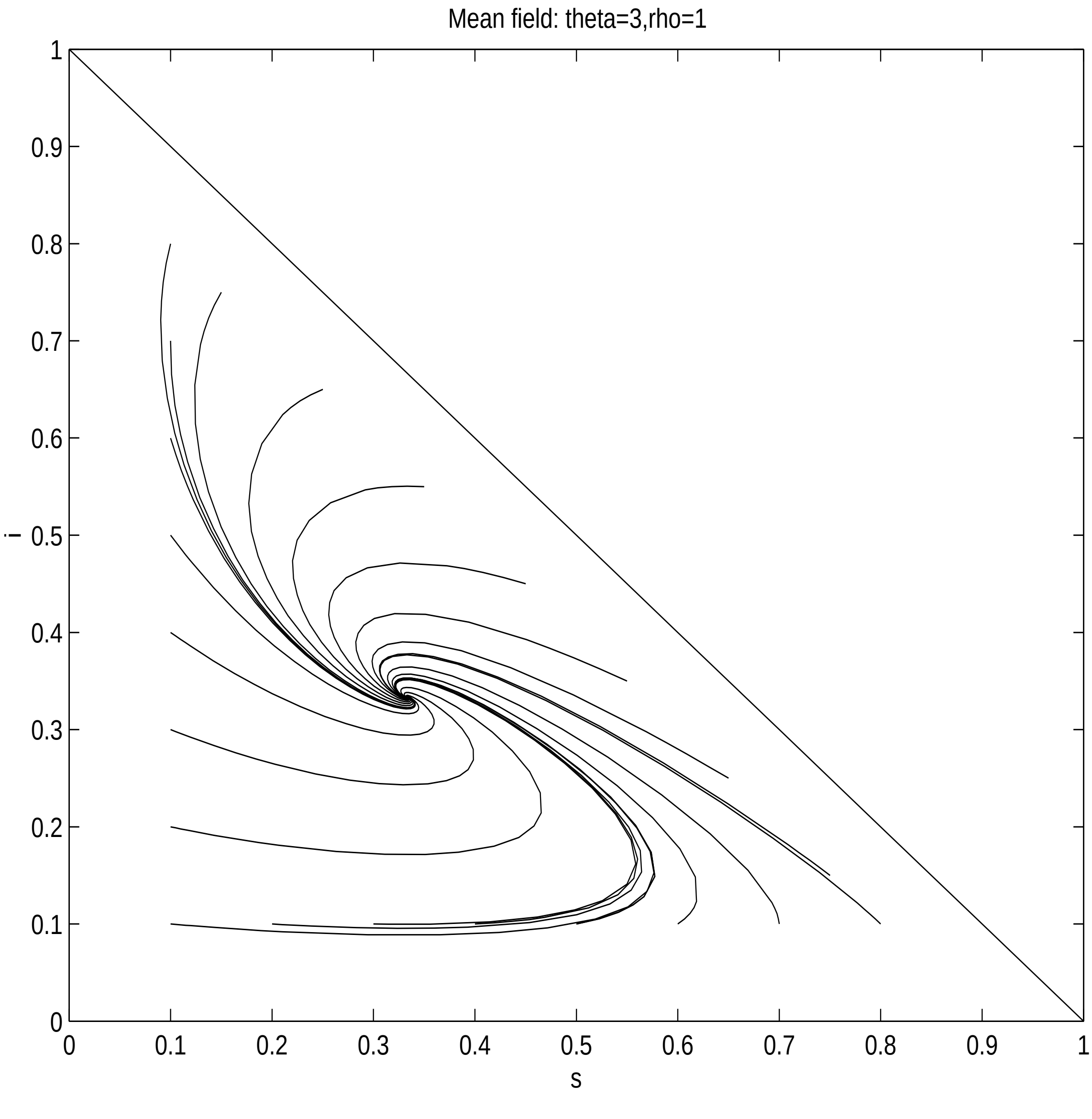
Endemic Infections

SIR mean path



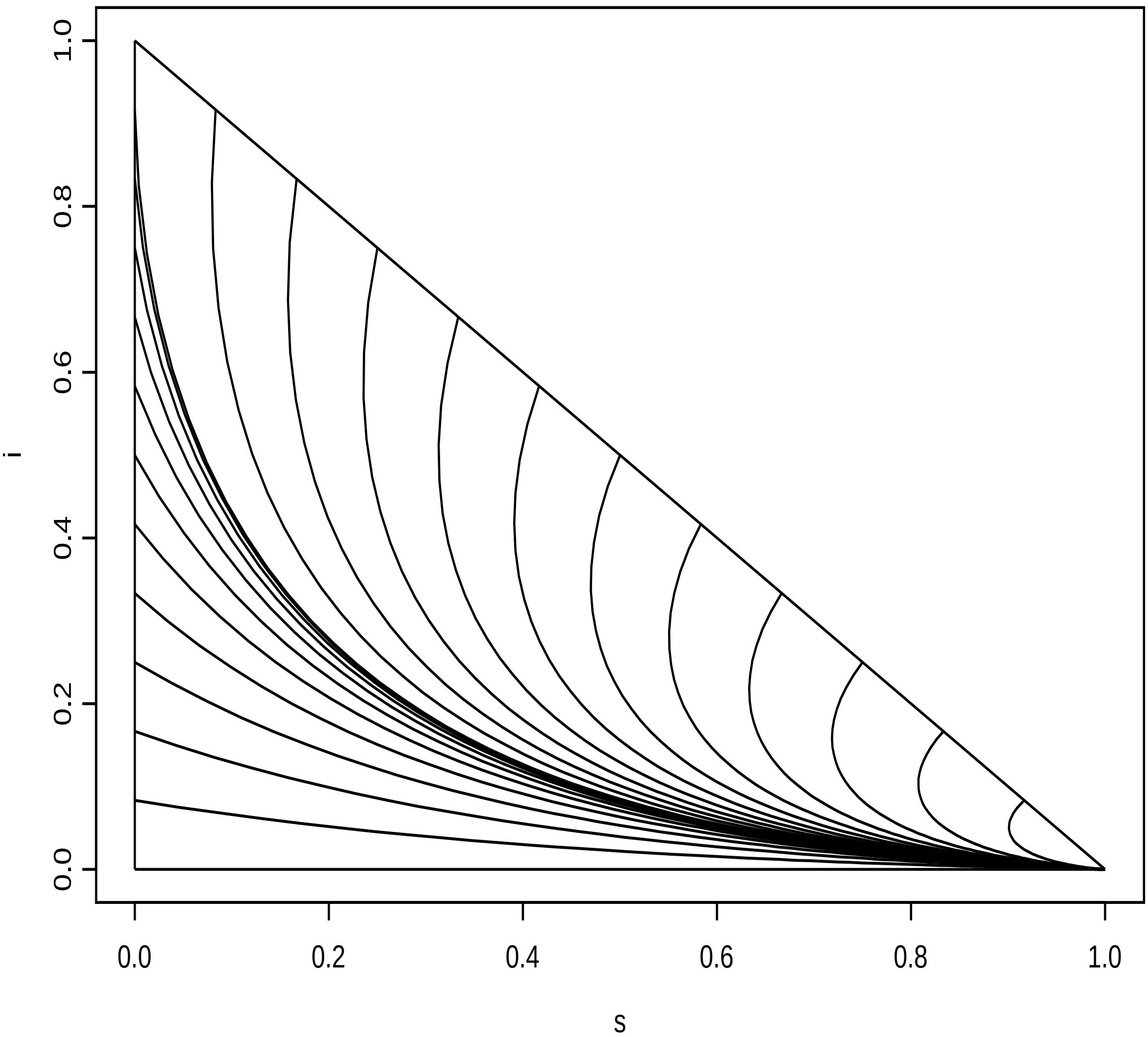
Endemic Infections

SIRS mean path: $\theta = 3$, $\rho = 1$



Endemic Infections

SIRS mean path: $\theta = 1$, $\rho = 2$



Endemic Infections

SIRS Epidemics

$(\bar{\gamma}_t)_{t\geq 0}$ - solution of mean path ODE,
i.e. $\dot{\gamma} = F(\gamma)$

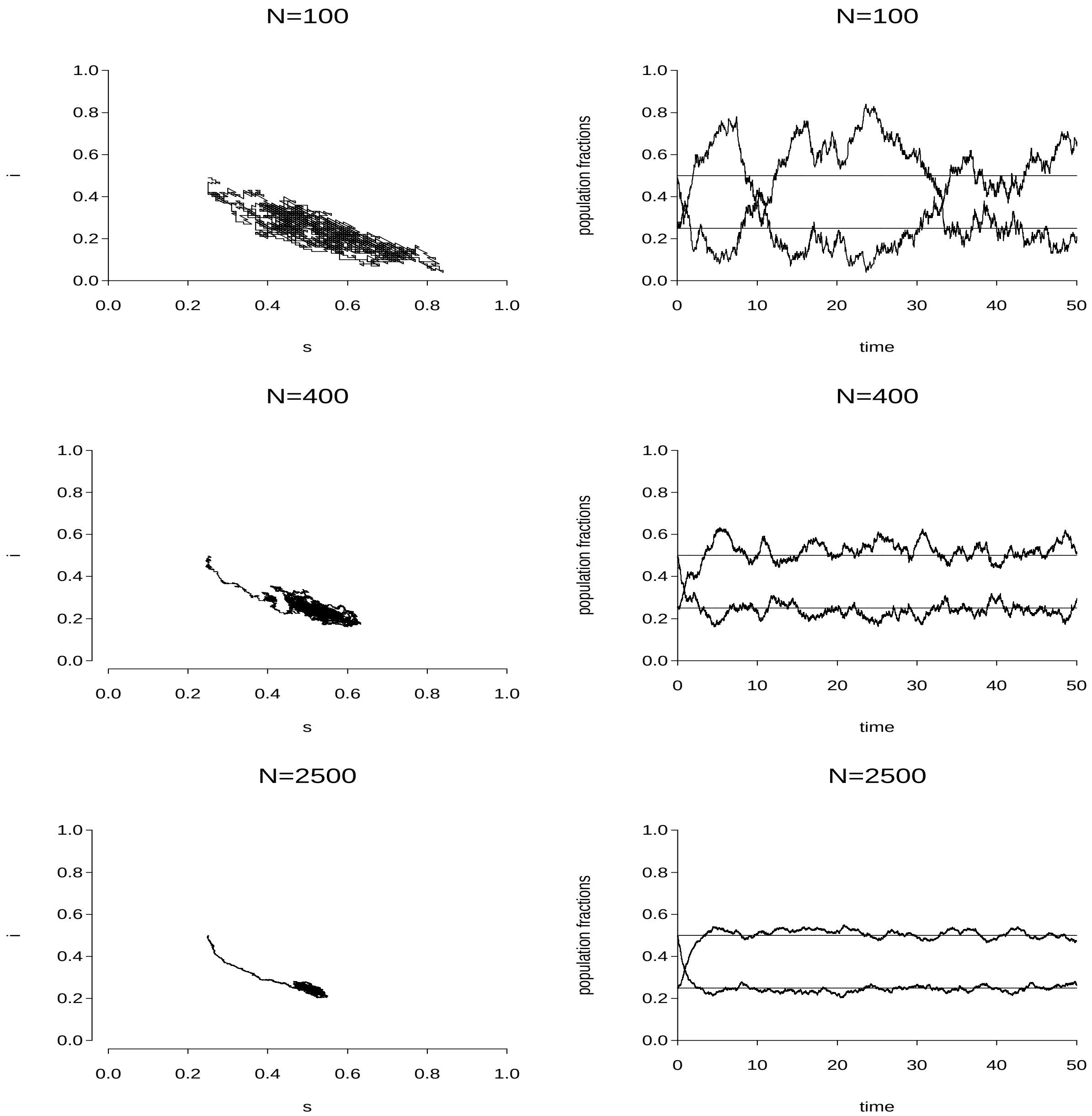
$\left(\gamma_t^N\right)_{t\geq 0}$ - random path

Theorem 1. *If* $\gamma_0^N \to \bar{\gamma}_0$ *as* $N \to \infty$ *then for all* $T > 0$, $\epsilon > 0$

$$\lim_{N\to\infty} \mathbf{P}\left(\sup_{t\leq T} |\gamma_t^N - \bar{\gamma}_t| > \epsilon\right) = 0.$$

Endemic Infections

SIRS Epidemics: $\theta = 2$, $\rho = 1$



Endemic Infections

Time to Extinction

- For all N, infection dies out with prob.1, why?

- How long until this happens?

- If $Y \sim$ Geometric(q) then $\mathbf{E}(Y) = \frac{1}{q}$

- Connection to "most likely" path

Endemic Infections

Exit path LDP

- Large Deviations for exit paths (LDP).

- Standard methods

  – Contraction Principle

    Cont. $f : \mathcal{X} \to \mathcal{Y}$ & LDP for μ^N on $\mathcal{X}$
    $\Rightarrow$ LDP for $\mu^N \circ f^{-1}$ on $\mathcal{Y}$.

  – Wentzell and Freidlin

- Dangers of diffusion approximations

Endemic Infections

Exit path LDP

Fix $\gamma = (s_t, i_t)_{t \geq 0} \in \mathcal{AC}[0, \infty)$

Let $\lambda, \mu, \nu \geq 0$ s.t.

$$\begin{cases} \dfrac{ds_t}{dt} = \nu_t - \lambda_t \\ \dfrac{di_t}{dt} = \lambda_t - \mu_t \end{cases}$$

and let

$\psi_t = \lambda_t + \mu_t + \nu_t$ and $\varphi_t = \theta s_t i_t + \rho i_t + r_t$

Define measure $Q \sim \lambda_t, \mu_t, \nu_t$

$$\mathbf{P}^N (||\bar{\gamma} - \gamma||_T < \delta) = \mathbf{E}_Q^N \left(I\{||\bar{\gamma} - \gamma||_T < \delta\} \cdot \frac{dP}{dQ} \right)$$

Endemic Infections

Rate function $I(\gamma)$

$$I(\gamma) = \inf_{\lambda,\mu,\nu} \int_0^T \left(\lambda_t \ln \left(\frac{\lambda_t}{\theta s_t i_t} \right) + \mu_t \ln \left(\frac{\mu_t}{\rho i_t} \right) \right.$$
$$\left. + \nu_t \ln \left(\frac{\nu_t}{r_t} \right) - (\psi_t - \varphi_t) \right) dt, \;\; \gamma \in \mathcal{AC}[0,\infty)$$

$$I(\gamma) = \infty, \;\; \gamma \notin \mathcal{AC}[0,\infty)$$

Theorem 2. *For all $T > 0$*

$$\lim_{\delta \to 0} \underline{\lim}_{N \to \infty} \frac{1}{N} \ln \mathbf{P}^N \left(||\bar{\gamma} - \gamma||_T < \delta \right)$$
$$= \lim_{\delta \to 0} \overline{\lim}_{N \to \infty} \frac{1}{N} \ln \mathbf{P}^N \left(||\bar{\gamma} - \gamma||_T < \delta \right) = -I(\gamma),$$

i.e.

$$\mathbf{P}^N \left(||\bar{\gamma} - \gamma||_T < \delta \right) \approx e^{-NI(\gamma)}.$$

Time until extinction

$$\tau^N = \inf\{t : i_t = 0\} = \text{time to extinction}$$

$$\bar{I} = \inf_\gamma I_\tau(\gamma) = \text{“minimal cost” of exit}$$

Theorem 3. . *Then for any* $\epsilon > 0$

$$\lim_{N\to\infty} \mathbb{P}\left(e^{N(\bar{I}-\epsilon)} \le \tau^N \le e^{N(\bar{I}+\epsilon)}\right) = 1.$$