Two levels of mixing

*Threshold parameter, $R_*$*

- Recall, $N = mn$, where $m$ is number of groups and $n$ is the size of the group.

$$N^{-1} \lambda_G = \text{rate of global infectious contacts},$$
$$\lambda_L = \text{rate of local infectious contacts},$$
$$\gamma = \text{rate of recovery}.$$

- The threshold parameter $R_* = \mu R_G$, where $R_G = \lambda_G / \gamma$ is the basic reproductive ratio for the global epidemic, $\mu$ is the average size of a clump = the average size of a local epidemic.
Two levels of mixing

Deterministic approximation

Let

\[ x_i(t) = \text{number of susceptibles at time } t \]
\[ y_i(t) = \text{number of infecteds at time } t \]

in group \( i \) at time \( t \), \( i = 1, \ldots, m \).

Then

\[
\frac{dx_i}{dt} = - \left( \lambda_L y_i + N^{-1} \lambda_G \sum_{j \neq i} y_j \right) x_i
\]
\[
\frac{dy_i}{dt} = \left( \lambda_L y_i + N^{-1} \lambda_G \sum_{j \neq i} y_j \right) x_i - \gamma y_i
\]
Two levels of mixing

Deterministic approximation

• Usual definition:

  reproductive ratio = expected number of infectious contacts made by a single infected in a wholly susceptible population

• So here, \( R_0 = \left( (n - 1)\lambda_L + \lambda_G \right) / \gamma \)

• In deterministic system, major epidemic occurs when \( R_0 > 1 \).
Two levels of mixing

Deterministic approximation

• Re-parameterize

\[
\lambda_L = (n - 1)^{-1}\lambda_L \text{ and } \gamma = 1
\]

so that \( R_0 = \lambda_L + \lambda_G \).

• \( R_* = \mu R_G = \mu N^{-1}\lambda_G N = \mu \lambda_G \), where \( \mu \) can be obtained numerically.

• For various group sizes \( n \), can plot \((\lambda_G, \lambda_L)\)
so that \( R_* = 1 \).

• As \( n \to \infty \), \( R_* \to R_0 \) but \( R_0 \) is not the right threshold for fixed \( n \)!
Brief intro to large deviations

Simple example

$X_1, X_2, \ldots, X_n$ are i.i.d. $N(0, 1)$, $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$

- Chebychev gives WLLN
  \[ P(|\bar{X}_n| > \delta) \leq \text{Var}(\bar{X}_n)/\delta^2 \to 0. \]

- Moreover, $\sqrt{n}\bar{X}_n \sim N(0, 1)$, so typical fluctuations of $\bar{X}_n$ are $O(1/\sqrt{n})$.

- Also,
  \[ P(|\bar{X}_n| > \delta) = 2(1 - \Phi(\delta\sqrt{n})) \quad (1) \]
Brief intro to large deviations

Simple example

\[ \frac{x}{x^2+1} \phi(x) \leq 1 - \Phi(x) \leq \frac{1}{x} \phi(x) \] and (1) imply

\[ P(|\bar{X}_n| > \delta) \leq 2 \frac{1}{\delta \sqrt{n}} \times \frac{1}{\sqrt{2\pi}} e^{-\delta^2 n/2} \]

\[ P(|\bar{X}_n| > \delta) \geq 2 \frac{\delta \sqrt{n}}{1 + \delta^2 n} \times \frac{1}{\sqrt{2\pi}} e^{-\delta^2 n/2} \]

so that

\[ \frac{1}{n} \log P(|\bar{X}_n| > \delta) \to -\delta^2 / 2. \]
Brief intro to large deviations

Simple example

• Above is a typical LD statement: “usual” fluctuations of $\bar{X}_n$ are on the order of $O(1/\sqrt{n})$ but with exponentially small probabilities $\bar{X}_n$ takes values on the order of $O(1)$.

• Assumption of normality can be relaxed - Cramer’s theorem.

• Independence can be relaxed - Gartner-Ellis’s theorem.

• The notion can be extended to “atypical” paths of stochastic processes - Mogulskii’s theorem and Schilder’s theorem.
Brief intro to large deviations

Definition

• Rate function $I : \Omega \rightarrow [0, \infty]$ if level sets
  $\{\omega \in \Omega : I(\omega) \leq \alpha\}$ are compact for $\alpha \geq 0$.

• $\{\mu_\epsilon\}$ satisfies LDP with a rate fn $I$, if

  $- \inf_{\omega \in \Gamma^0} I(\omega) \leq \liminf_{\epsilon \rightarrow 0} \epsilon \log \mu_\epsilon(\Gamma) \leq \limsup_{\epsilon \rightarrow 0} \epsilon \log \mu_\epsilon(\Gamma) \leq - \inf_{\omega \in \bar{\Gamma}} I(\omega)$

  for any $\Gamma \subset \Omega$.

• Note, $\inf_{\omega \in \Omega} I(\omega) = 0$, so there exists $\omega$ s.t. $I(\omega) = 0$. 
Brief intro to large deviations

Cramer’s Theorem

• $X_1, X_2, \ldots$ i.i.d. $\sim \mu$ in $\mathbb{R}$.

• Cumulant generating fn

$$\Lambda(t) = \log M(t) = \log \left( \mathbb{E} e^{tX_1} \right).$$

• Fenchel-Legendre transform of $\Lambda$

$$\Lambda^*(x) = \sup_{t \in \mathbb{R}} \{ tx - \Lambda(t) \}$$

• Cramer’s theorem:

If $\bar{X}_n \sim \mu_n$ then $\mu_n$ satisfies LDP with $\Lambda^*$. 
Brief intro to large deviations

*Cramer’s Theorem: normal case*

- $X_1, X_2, \ldots$ i.i.d. $\mathcal{N}(\mu, \sigma^2)$.

- $\Lambda(t) = \mu t + \sigma^2 t^2$, 
  $\Lambda^*(x) = \sup_{t \in \mathbb{R}} (tx - t\mu - \sigma^2 t^2 / 2)$.

- $\frac{\partial}{\partial t} = x - \mu - \sigma t$, and $t^* = \frac{x - \mu}{\sigma}$, so that 
  $\Lambda^*(x) = \frac{(x - \mu)^2}{2\sigma^2}$.

- $\mu = 0, \sigma = 1$ then $\Lambda^*(x) = x^2 / 2$
  but $\{|\bar{X}_n| \geq \delta\} = \{X_n \leq -\delta\} \cap \{X_n \geq \delta\}$ and 
  $\inf_{|\bar{X}_n| \geq \delta} \Lambda^*(x) = \delta^2 / 2$.
Brief intro to large deviations

*Mogulskii’s theorem for RWs*

• Let $X_1, X_2, \ldots$ i.i.d. with $\Lambda(t)$ and $\Lambda^*(x)$.

$$
Z_n(t) = \frac{1}{n} \sum_{i=1}^{[nt]} X_i, \quad 0 \leq t \leq 1
$$

\[ \sim \mu_n \text{ on } L_\infty([0, 1]) \]

• Mogulskii’s theorem:

$\mu_n$ satisfy LDP with

$$
I(\varphi) = \begin{cases} 
\int_0^1 \Lambda^*(\varphi'(t))dt, & \text{if } \varphi(t) \in AC[0, 1] \\
\infty, & \text{otherwise}.
\end{cases}
$$
Brief intro to large deviations

Danger of approximations

• Sometimes in disease modeling a normal approximation (diffusion) to the count process is used.

• This leads to erroneous results. Indeed, the whole cumulant function is approximated by the first two moments!

• If increments are Poisson(\(\lambda\)) then

\[ \Lambda^*(x) = x \log x/\lambda - x + \lambda \]

would be approximated by

\[ (x - \lambda)^2/2\lambda. \]