Introduction to Coupling

Definition. Coupling is a method of analysis, which involves defining several stochastic processes on the same probability space.

Realizations are related; goal is to use the dependence to gain insight.
Simple Example of Coupling

Gambler’s ruin problem

• Initial wealth: $a$ units of money.

• Wish to raise $N$ units.

• Play independent games:
  
  win 1 with prob. $p$,
  loose 1 with prob. $1 - p$.

• Play until reach $N$ or 0 (bankrupt).
Simple Example of Coupling

Gambler’s ruin problem

\[ P(p) = P(\text{raise money}) = P(\text{reach } N \text{ before } 0). \]

- Want to prove \( P(p) \) is increasing with \( p \).

- Suggestions?
  (see Karlin and Taylor, 1975, pp 92-94)
Simple Example of Coupling

Gambler’s ruin problem

• Let $U_1, U_2, \ldots$ be i.i.d. Uniform$[0, 1]$.

• For $p \in (0, 1)$ let

$$X_i(p) = \begin{cases} +1, & \text{if } U_i \leq p \\ -1, & \text{if } U_i > p \end{cases}$$

and

$$S_n(p) = a + \sum_{i=1}^{n} X_i(p), \quad n = 1, 2, \ldots$$

stop when reach $N$ or $0$.

• This is a realization of G-R problem.
Simple Example of Coupling

Gambler’s ruin problem

- Let $p > p'$; construct $S_n(p)$ and $S_n(p')$ from the same $U_1, U_2, \ldots$'s

\[
X_i(p) = \begin{cases}
+1, & \text{if } U_i \leq p' \\
+1, & \text{if } U_i \leq p \\
-1, & \text{if } U_i > p
\end{cases}
\]

\[
\geq \begin{cases}
+1, & \text{if } U_i \leq p' \\
-1, & \text{if } U_i \leq p = X_i(p') \\
-1, & \text{if } U_i > p
\end{cases}
\]

$\Rightarrow S_n(p) \geq S_n(p')$
Simple Example of Coupling

Gambler’s ruin problem

• If $S_n(p')$ reaches $N$ then so does $S_n(p) \Rightarrow P(p) \geq P(p')$.

• In fact, $P(p) > P(p')$:

  if $U_i \in (p', p)$ for $i = 1, 2, \ldots \max(a, N - a)$

  then $X_i(p) = +1 \leadsto S_n(p)$ absorbed at $N$

  $X_i(p') = -1 \leadsto S_n(p')$ absorbed at 0.

  probability of this $(p - p')^{\max(a, N-a)} > 0$. 

Introduction to Coupling

*General Epidemic Process*

- Relax assumption: infectious period $\sim T_I$ (arbitrary but specified distribution).

- (the new and improved indexing system!)

  $m$ initial infectives: $-(m-1), -(m-2), \ldots, 0$

  $n$ initial susceptibles: $1, 2, \ldots, n$

- $r_{-(m-1)}, r_{-(m-2)}, \ldots, r_0, \ldots, r_n$ i.i.d. $\sim T_I$

  $l_1, l_2, \ldots, l_n$ i.i.d. $\sim \text{Exp}(1)$. 

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*General Epidemic Process*

Conditional on \( S_t = S, \ I_t = I \)

\[ P_t \left( \text{susc. } j \text{ is infected in } [t, t+h] \right) = \]

\[ = P_t \left( l_j < \sum_{k=1}^{I} \min(\xi_k - t, h) \right) \leq P(l_j < Ih) = \]

\[ = 1 - e^{-Ih} = Ih + o(h), \text{ but also} \]

\[ = P_t \left( l_j < \sum_{k=1}^{I} \min(\xi_k - t, h) \right) \]

\[ \geq P_t \left( \{l_j < Ih\} \cap \left\{ \bigcap_{k=1}^{I} \{\xi_k - t > h\} \right\} \right), \]

where \( \xi_k \) is time of recovery of \( k^{th} \) infected at time \( t \).
Introduction to Coupling

*General Epidemic Process*

Conditional on $S_t = S$, $I_t = I$ (cont.)

\[
\geq P \left( l_j < Ih \right) \times \inf_{s \leq t} P \left( r_k > s + h \bigg| r_k > s \right)
\]

\[
= \left( Ih + o(h) \right) \times \inf_{s \leq t} \left( 1 + \frac{F_{T_I}(s) - F_{T_I}(s + h)}{1 - F_{T_I}(s)} \right)
\]

\[
= \left( Ih + o(h) \right)
\]

as long as $F_{T_I}$ is uniformly continuous.
Introduction to Coupling

*General Epidemic Process*

As before

\[ \nu = \# \text{ new infections}; \]
\[ R = \text{total "exposure to infection"} \]
\[ \nu = \sum_{k=-(m-1)}^{\nu} r_k, \quad \text{and} \]
\[ \nu + 1 = \inf \left\{ i : l_{(i)} > \sum_{k=-(m-1)}^{i-1} r_k \right\} \]
Introduction to Coupling

*General Epidemic Process*

• Let $\phi(\theta) = \mathbb{E}\left(e^{-\theta T_I}\right)$, be the mgf of $T_I$.

• Assume the Wald identity

$$\mathbb{E}\left(e^{-\theta R/\phi(\theta)^{\nu+m}}\right) = 1 \quad (1)$$

(proved by F. Ball, 1986.)

• Notation: $\{N\} = \{1, 2, \ldots, N\}$. 
Introduction to Coupling

General Epidemic Process

• More notation:

\[ P^n_\omega = P^n(\nu = \omega), \quad w = 0, 1, \ldots n, \]

and let \( P^n_{\{\omega\}} = \text{prob. that precisely initial susceptibles } 1, 2, \ldots, \omega \text{ are infected, so that} \]

\[ P^n_\omega = \binom{n}{\omega} P^n_{\{\omega\}}. \]

• Fix \( 0 \leq \omega \leq j \leq n. \)

We construct coupled epidemics, starting with \( n \) and \( j \) susceptibles using same \( r \)'s and \( l \)'s.
Introduction to Coupling

General Epidemic Process

- Epidemic with \( n \) susc. infects precisely \( \{\omega\} \)
  iff
  Epidemic with \( j \) susc. infects precisely \( \{\omega\} \)
  and \( l_k > R_j, k = j + 1, j + 2, \ldots, n \)

\[
P^n_{\{\omega\}} = P^j_{\{\omega\}} \times E^j \left( E \left( \prod_{k=j+1}^{n} I\{l_k > R_j\} \bigg| R_j \right) \bigg| \{\omega\} \right)
= P^j_{\{\omega\}} \times E^j \left( \exp\left(- (n - j) R_j \right) \bigg| \{\omega\} \right)
= P^j_{\{\omega\}} \times E^j \left( \exp\left(- (n - j) R_j \right) \bigg| \omega \right)
\]

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General Epidemic Process

• For $\theta = n - j$ identity (1) gives

$$\sum_{\omega=0}^{j} P_{\omega}^j \mathbb{E}^j \left( \exp(- (n-j)R_j) / \phi(n-j)^{a+\omega} \right) = 1$$

• Now $P_{\omega}^j = \binom{j}{\omega} P_{\{\omega\}}^j$ and $P_{\omega}^n = \binom{n}{\omega} P_{\{\omega\}}^n$, so

$$\sum_{\omega=0}^{j} \binom{n-\omega}{j-\omega} P_{\omega}^n / \phi(n-j)^{a+\omega} = \binom{n}{j}$$

$j = 0, 1, \ldots, n.$