Bayesian Inference for Epidemic Models

Features of Epidemic Data

- High dependance is inherently present.

- Never observe the entire process: cannot observe actual infection times.

- Sometimes observe symptoms onset but most often just the final number of infected individuals.

- True epidemic chain - who infects whom - is almost never observed.
Bayesian Inference for Epidemic Models

*Generalized Stochastic Epidemic*

- Recall, $I_0 = m$, $S_0 = n$.

- Infectious period has an arbitrary but specified distribution $\sim T_I$.

- Person-to-person infectious contacts at rate $\lambda/n$.

- All contacts at points of independent Poisson processes.
Bayesian Inference for Epidemic Models

Generalized Stochastic Epidemic

- Let $\varphi(\theta) = E\left(e^{-\theta T_1}\right)$ be the mgf of $T_1$ and let $p^n_k = \text{probability that the final size is } k$.

- We have derived before

$$
\sum_{k=0}^{l} \frac{\binom{n-k}{l-k}p^n_k}{\varphi\left(\frac{\lambda(n-l)}{n}\right)^{k+m}} = \binom{n}{l}, \quad 0 \leq l \leq n.
$$

- Linear system is unstable under numerical solutions: very small probabilities for many final sizes. However, need all intermediate probabilities $\Rightarrow$ multiple precision arithmetics.
Bayesian Inference for Epidemic Models

Generalized Stochastic Epidemic

- Based on solutions of triangular system above, obtain posterior distribution of $\lambda$:

$$\pi(\lambda|x) \propto f(x|\lambda)\pi(\lambda),$$

where $f(x|\lambda)$ is the likelihood of the final outcome $x$ and $\pi(\lambda)$ is the prior for $\lambda$.

- Can similarly obtain posterior for $R_0$.

- Note, essentially estimating parameters from one data point. Rely on the structure imposed in the process.
Bayesian Inference for Epidemic Models

*Generalized Stochastic Epidemic*

- Employ standard MCMC method: simple Gaussian Random Walk type Metropolis algorithm.

- Recall, Metropolis algorithm provides a method of constructing a Markov Chain whose stationary distribution is the desired distribution known up to a constant.

- Proposal density $g(u, v)$ is Gaussian centered at the current value.
Bayesian Inference for Epidemic Models

Brief Refresher on Metropolis Algorithm

• Draw $v \sim g(\cdot, u)$, where $u = U^t$ is the current state of the Markov Chain.

• Compute $r = \pi(v)/\pi(u)$, where $\pi$ is the desired posterior – need only know up to a constant.

• If $r \geq 1$, set $U^{t+1} = v$, else if $r < 1$, set

$$U^{t+1} = \begin{cases} v, & \text{w/prob. } r \\ u, & \text{w/prob. } 1 - r. \end{cases}$$
Bayesian Inference for Epidemic Models

Brief Refresher on Metropolis Algorithm

- Then \( U^t \xrightarrow{\mathcal{D}} U \sim \pi \) as \( t \to \infty \).

- Let \( P = \{p_{ij}\} \) be the transition matrix of the MC defined above. Need to show that \( \pi^t = (\pi_1, \pi_2, \ldots, \pi_k) \) solves
  \[
  \pi^t P = \pi^t. \tag{1}
  \]

- Symmetry implies that MC is reversible
  \[
  \pi_i P_{ij} = \pi_i \left[ \min \left( 1, \frac{\pi_j}{\pi_i} \right) Q_{ij} \right] \\
  = \min(\pi_i, \pi_j) Q_{ij} = \min(\pi_i, \pi_j) Q_{ji} \\
  = \pi_j P_{ji}.
  \]
Bayesian Inference for Epidemic Models

*Brief Refresher on Metropolis Algorithm*

- Then
  \[
  (\pi^t P)_j = \sum_{i=1}^{k} \pi_i P_{ij} = \sum_{i=1}^{k} \pi_j P_{ji}
  \]
  \[
  = \pi_j \sum_{i=1}^{k} P_{ji} = \pi_j,
  \]
  so (1) holds and $\pi$ is the stationary distribution of the MC.

- Prior specification: the rate $\lambda \sim \text{Gamma}$ with mean 1 and variance 10,000. Some sensitivity analysis later.
Bayesian Inference for Epidemic Models

Results for Generalized Stochastic Epidemic

• Three choices of infectious period distribution, $T_I$:
  
  − Constant $\equiv 4.1$;

  − Exponential w/ mean $= 4.1$  
    $\Rightarrow$ var $= 4.1^2$;

  − Gamma w/ mean $= 4.1$, var $= 2 \cdot 2.05^2$. 
Bayesian Inference for Epidemic Models

Results for Generalized Stochastic Epidemic

Posterior estimates for $R_0$ when $x = 30$

<table>
<thead>
<tr>
<th>Infectious Period</th>
<th>Constant</th>
<th>Gamma</th>
<th>Exponential</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>1.177</td>
<td>1.219</td>
<td>1.251</td>
</tr>
<tr>
<td>Median</td>
<td>1.165</td>
<td>1.193</td>
<td>1.210</td>
</tr>
<tr>
<td>St.Dev.</td>
<td>0.21</td>
<td>0.27</td>
<td>0.34</td>
</tr>
<tr>
<td>95% CT</td>
<td>(0.80, 1.62)</td>
<td>(0.76, 1.83)</td>
<td>(0.73, 2.05)</td>
</tr>
</tbody>
</table>

Data from closed community in Nigeria. Figure p11 shows posterior density of $R_0$ for three infectious periods.
Bayesian Inference for Epidemic Models

Results for Generalized Stochastic Epidemic

Posterior estimates for $R_0$ when $x = 60$

<table>
<thead>
<tr>
<th></th>
<th>Infectious Period</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Constant</td>
</tr>
<tr>
<td>Mean</td>
<td>1.424</td>
</tr>
<tr>
<td>Median</td>
<td>1.413</td>
</tr>
<tr>
<td>St.Dev.</td>
<td>0.18</td>
</tr>
<tr>
<td>95% CT</td>
<td>(1.09,1.81)</td>
</tr>
</tbody>
</table>

Figure p13 shows posterior density of $R_0$ for three infectious periods.
Bayesian Inference for Epidemic Models

Results for Generalized Stochastic Epidemic

- When \( x = 30 \), the MLE of \( R_0 \) (Rida, 1991) is 1.108, and the posterior mode is about 1.15. Martingale methods (Becker, 1989) give an estimate of 1.10.

- The posterior variance of \( R_0 \) increases with the variance of the infectious period. Also, the posterior distribution becomes skewed to the right.

- Similar findings for \( x = 60 \).

- Figure p15 shows posterior density for two priors on \( R_0 \): Uniform and Gamma.
Bayesian Inference for Epidemic Models

*Epidemics with two levels of mixing*

- Recall, individuals are grouped into social cliques (or neighborhoods).

- Global contacts at rate $\lambda_G$ plus additional contacts at rate $\lambda_L$ within these groups.

- Let $m_j$ be number of groups of size $j$, $m = \sum_j m_j$ be number of groups, and $N = \sum_j j m_j$ total number of individuals.
Bayesian Inference for Epidemic Models

Epidemics with two levels of mixing

• Exact likelihood is numerically intractable.

• Have to resort to asymptotic results for final outcome as number of groups $m \to \infty$.

• Instead of modelling global contacts explicitly, a given individual avoids global infection with probability $p = \exp(-\lambda_G z \mathbb{E}(T_I))$, where $z$ is the mean proportion of susceptibles that ultimately become infected.
Bayesian Inference for Epidemic Models

Epidemics with two levels of mixing

- A little more subtle approach is to replace $z \mathbb{E}(T_I)$ with $A/N$, where $A = \sum_k T_I(k)$ is the total sum of infectious periods.

- Augment the parameter space with $A$, so

$$
\pi(\lambda_L, \lambda_G, A | \tilde{n}) \propto f(\tilde{n} | \lambda_L, \lambda_G, A) f(A | \lambda_G, \lambda_L) \\
\times \pi(\lambda_L)\pi(\lambda_G),
$$

where $\tilde{n} = (n_{ij}) = \text{number of household in which i out of j susceptibles ultimately become infected}$.

- Recall also, that the threshold parameter is $R_* = \lambda_G \mathbb{E}(T_I) \nu$, where $\nu$ is the mean size of an outbreak in a group.
Bayesian Inference for Epidemic Models

*Epidemics with two levels of mixing*

- Influenza outbreak in 1977-1978 in Tecumseh, Michigan. Data for $\alpha = .1$ of the entire population was collected.

- Figure p17 shows the posterior for the threshold parameter $R_*$ for Uniform and Gamma priors.

- Figures p18, p19, and p20 show posteriors of $\lambda_G$, $\lambda_L$, and $R_*$ as a function of $\alpha$ respectively.