Criticality in Epidemic Models

Epidemic models often exhibit the so-called threshold phenomena, or quantitatively different behavior above and below criticality. In general, below criticality the major epidemic is impossible or unlikely, whereas when the reproductive number is above one, a major epidemic is possible. We will describe the final outcome of the infection spread for several epidemic models in both subcritical and supercritical cases.

Reed-Frost Model

$$S \longrightarrow I \longrightarrow R$$

- (S_t, I_t) = state at time $t = 0, 1, 2, \ldots$

  S_t = # susceptible at time t
  I_t = # infected at time t

- 1 unit time ≈ incubation period
  ≈ recovery time

Reed-Frost Model

- $p = 1 - q =$ probability of encounter
  for any (S, I) pair

  $\Rightarrow$ S-person at time t remains S at $t+1$
  w/prob. q^{I_t}

  $\Rightarrow$ $S_{t+1} \sim \mathbf{Bin}(S_t, q^{I_t})$
  $I_{t+1} = S_t - S_{t+1}$

- $(S_0, I_0) = (n, m)$

  epidemic continues until

  $$\tau = \text{inf}\{t : I_t = 0\}$$

- $S_\tau =$# susceptibles that avoid infection
  $K = n + m - S_\tau$
  $= I_0 + I_1 + I_2 + \ldots =$ total damage

Reed-Frost Model

Distribution of S_τ? ($\Rightarrow K$?)

- Let $m = I_0 = \mathbf{o}(n)$, and

  $q = e^{-\lambda/n} \Rightarrow p = \mathbf{O}(n^{-1})$

  $\Rightarrow \mathbf{E}(\#\text{contacts}) \approx \text{fixed}$

$\Longrightarrow$ in the beginning

$$\begin{aligned} I_{t+1} &\approx \mathbf{Bin}\left(S_t, 1 - e^{-\lambda I_t/n}\right) \\ &\approx \mathbf{Bin}(n, \lambda I_t/n) \\ &\approx \mathbf{Poisson}(\lambda I_t), \end{aligned}$$

so that

$I_{t+1} \approx$ sum of I_t indep. $\mathbf{Poisson}(\lambda)$ r.v.'s

Reed-Frost Model

- Initially, $I_0, I_1, I_2, \ldots$ is $\approx$ G-W process w/offspring $\sim \mathbf{Poisson}(\lambda)$

- Such G-W process is

  $\lambda \leq 1 \Rightarrow$ subcritical

  $\Rightarrow$ progeny is finite (known Lap. tnsf.)

  $\lambda > 1 \Rightarrow$ supercritical

  $\Rightarrow$ w/prob. $\sigma^m < 1$

  progeny is finite (known Lap. tnsf.)

  w/prob. $1 - \sigma^m$

  I_t gets large, G-W approx. breaks

Reed-Frost Model

When $\lambda > 1$ w/prob. $1-\sigma^m$ both S_t and I_t are large, LLN gives

$$\begin{cases} S_{t+1} = S_t q^{I_t} \\ I_{t+1} = S_t(1-q^{I_t}) \end{cases}$$

$$s_t = S_t/n,\ i_t = I_t/n \Rightarrow s_0 = 1, i_0 = \frac{m}{n} = \mu$$

$$\begin{cases} s_{t+1} = s_t e^{-\lambda i_t} \\ i_{t+1} = s_t(1-e^{-\lambda i_t}) \end{cases}$$

and

$$s_\infty = 1 \cdot e^{-\lambda(i_0+i_1+\dots)} = e^{\lambda(1+\mu-s_\infty)}$$

$$I_\infty \approx n i_\infty = n(1-s_\infty)$$

Reed-Frost Model

$K = I_0 + I_1 + I_2 + \ldots =$ total epidemics

- If $\frac{m}{n} \to \mu > 0 \Longrightarrow \frac{K - n(1-s_\infty)}{\sqrt{n(1-s_\infty)}} \to N(0,1)$

- If $\frac{m}{n} \to \mu = 0 \Longrightarrow$
  - $\lambda \leq 1 \Rightarrow K \sim$ progeny of G-W process
  - $\lambda > 1 \Rightarrow$
    - <u>w/prob. σ^m</u> $K \sim$ progeny of G-W
    - <u>w/prob. $1 - \sigma^m$</u> $\frac{K - n(1-s_\infty)}{\sqrt{n(1-s_\infty)}} \to N(0,1)$

General epidemic process

- Continuous time, $S_0 = n$, $I_0 = m$

- Again $S \to I \to R$ w/transition probabilities conditional on $(S_t, I_t) = (S, I)$

$$\mathbf{P}_t\Big((S_{t+h}, I_{t+h}) = (S-, I+1)\Big) = \theta S I h + \mathbf{o}(h)$$
$$\mathbf{P}_t\Big((S_{t+h}, I_{t+h}) = (S, I-1)\Big) = \rho I h + \mathbf{o}(h)$$
$$\mathbf{P}_t\Big((S_{t+h}, I_{t+h}) = (S, I)\Big) = \\ = 1 - \theta S I h - \rho I h + \mathbf{o}(h),$$

all other transitions w/prob. $\mathbf{o}(h)$

General epidemic process

Sellke's beautiful coupling argument

- wlog take $\theta = 1$
- Index S-indiv. by i and I-indiv. by j, so $1 \leq i \leq n$ and $1 \leq j \leq m$.
- $\left\{ l_i \right\}_{i=1}^{n}$ - "resistance to infection"

  $l_{(1)} < l_{(2)} < \ldots < l_{(n)}$ - order statistics

  i.i.d. $\mathbf{Exp}(1)$.
- $\left\{ \widehat{r}_j \right\}_{j=1}^{m}$ - "infectious period" for I-indiv.

  $\left\{ r_i \right\}_{i=1}^{n}$ - "infectious period" for S-indiv.

  i.i.d. $\mathbf{Exp}(\rho)$.

  $r^{(k)} = r_i$ if $l_{(k)} = l_i$

General epidemic process

- Originally infected remain for $\widehat{r}_j$ time units

- S-indiv. indexed i accumulates "exposure" at rate I_t at time t

- When "exposure" $> l_i \Rightarrow$ $S \longrightarrow I$ then stays infected for r_i time units

General epidemic process

- ν = number of new infections

- If $l_{(1)} > \sum_{j=1}^{m} \widehat{r}_j$ then $\nu = 0$

  else S-indiv. with $l_{(1)}$ is infected and $\nu \geq 1$

- Induction

$$\nu + 1 = \min\left\{k : l_{(k)} > \sum_{j=1}^{m} \widehat{r}_j + \sum_{i=1}^{k-1} \widehat{r}^{(i)}\right\}$$

  if inequality doesn't hold for $\forall k$ then $\nu = n$.

General epidemic process

Processes indexed by k

Theorem. If $n_k \to \infty$, $\rho_k \to \infty$, and

$$n_k \exp\left\{-\frac{n_k + m_k}{\rho_k}\right\} \to b, \qquad 0 < b < \infty,$$

then $S_k(\infty) = n_k - \nu_k \to_{\mathcal{D}} \mathbf{Poisson}(b)$.

Let $R = \sum_{j=1}^{m} \hat{r}_j + \sum_{i=1}^{\nu} \hat{r}^{(i)}$ total exposure.

General epidemic process

Outline of the proof

- $\rho = \mathbf{o}(n+m)$

- Almost birth-and-death process

- Death-rate to birth-rate is
$$\frac{\rho I_t}{S_t I_t} = \frac{\rho}{S_t} < \frac{\rho}{\epsilon(n+m)} \equiv q$$
on $\Big\{S_t > \epsilon(n+m)\Big\}$.

- Prob. of extinction with $S(\infty) > \epsilon(n+m)$

  $<$ Prob. of extinction with D-B ratio q

  $=$ $q^m \to 0$.

General epidemic process

Outline of the proof (cont'd)

- Most people become infected, LLN gives

$$R \approx \sum_{j=1}^{m} \widehat{r}_j + \sum_{i=1}^{n} \widehat{r}^{(i)} \approx \frac{n+m}{\rho}$$

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$$\begin{aligned} S(\infty) &= \#\text{of } l_i\text{'s} > R \approx \frac{n+m}{\rho} \\ &\approx \mathbf{Bin}\left(n, e^{-\frac{n+m}{\rho}}\right) \\ &\approx \mathbf{Poisson}(b). \end{aligned}$$

General sampling procedure

- $(S_0, I_0) = (n, m)$.

- I-person draws size of contacts $\sim p$
  $p = (p_0, p_1, p_2, \ldots)$, moments = (λ, σ^2)

- I_t contacts <u>all</u> at time t: $\{S_t\} \cup \{I_t\}$

- $U_k = \#$ of susceptibles after k samplings.

$$U_0 = S_0 = n, I_0 = m$$
$$K_t = I_0 + I_1 + \ldots + I_t = n + m - S_t$$

  can recover

$$S_{t+1} = U_{K_t}, I_t = S_t - S_{t+1}$$

General sampling procedure

- The process runs until $I_t = 0$

  $\Longrightarrow S_t = S_{t+1}$, and therefore

  $$\begin{aligned} K_t &= n + m - S_t = n + m - S_{t+1} \\ &= n + m - U_{K_t} \end{aligned}$$

  for the first time.

- Let $Y_k \equiv U_k - nw^k$, $(w \equiv 1 - \lambda/n)$

  2 moments + tightness show that

  $$\{n^{-1/3} Y_{tn^{2/3}}\} \Rightarrow \sigma W_t.$$

General sampling procedure

The outcome of an almost critical epidemic
Martin Löf (1998)

- Total epidemic $K = \sum_t I_t$ = first time when U_k hits $n + m - k$, or equivalently when $n - U_k - k$ hits $-m$

- Let $\lambda = 1 + a/n^{1/3}$ and $m = bn^{1/3}$, same as

$$n^{1/3}\Big(at - t^2/2 + \mathbf{O}(n^{-1/3})\Big) - Y_{tn^{2/3}}$$

hits $-bn^{1/3}$, simplifies to

$$W_t \text{ hits } b + at - t^2/2 \rightsquigarrow T.$$

- Finally, recall $K = n^{2/3}T$.