ST 430
Homework #6: Solutions

6.4.6  a) In order for \( \alpha \) to be 0.07, \( P(60 - \bar{Y}^* \leq \bar{Y} \leq 60 + \bar{Y}^* \mid \mu = 60) = 0.07 \). Equivalently,
\[
P\left(\frac{60 - \bar{Y}^* - 60}{8.0/\sqrt{36}} \leq \frac{-\bar{Y} - 60}{8.0/\sqrt{36}} \leq \frac{60 + \bar{Y}^* - 60}{8.0/\sqrt{36}}\right) = P(-0.75\bar{Y}^* \leq Z \leq 0.75\bar{Y}^*) = 0.07.
\]
But
\[
P(-0.09 \leq Z \leq 0.09) = 0.07, \text{ so } 0.75\bar{Y}^* = 0.09, \text{ which implies that } \bar{Y}^* = 0.12.
\]

b) \( 1 - \beta = P(\text{reject } H_0 \mid H_1 \text{ is true}) = P(59.88 \leq \bar{Y} \leq 60.12 \mid \mu - 62) = P\left(\frac{59.88 - 62}{8.0/\sqrt{36}} \leq Z \leq \frac{60.12 - 62}{8.0/\sqrt{36}}\right) = P(-1.59 \leq Z \leq -1.41) = 0.0793 - 0.0559 = 0.0234.
\]

c) For \( \alpha = 0.07, \pm z_{\alpha/2} = \pm 1.81 \) and \( H_0 \) should be rejected if \( \bar{Y} \) is either
\[
1) \leq 60 - 1.81 \cdot \frac{8.0}{\sqrt{36}} = 57.50 \text{ or } 2) \geq 60 + 1.81 \cdot \frac{8.0}{\sqrt{36}} = 62.41. \text{ Suppose } \mu = 62. \text{ Then}
\]
\[
1 - \beta = P(\bar{Y} \leq 57.59 \mid \mu = 62) + P(\bar{Y} \geq 62.41 \mid \mu = 62) = P(Z \leq -3.31) + P(Z \geq 3.31) = 0.0005 + 0.3783 = 0.3788.
\]

6.4.8  If \( n = 45 \), \( H_0 \) will be rejected when \( \bar{Y} \) is either 1) \( \leq 10 - 1.96 \cdot \frac{4}{\sqrt{45}} = 8.83 \) or 2) \( \geq 10 + 1.96 \cdot \frac{4}{\sqrt{45}} = 11.17 \). When \( \mu = 12, \beta = P(\text{accept } H_0 \mid H_1 \text{ is true}) = P(\bar{Y} \leq 11.17 \mid \mu = 12)
\]
\[
= P\left(\frac{11.17 - 12}{4/\sqrt{45}} \leq Z \leq \frac{11.17 - 12}{4/\sqrt{45}}\right) = P(-1.39 \leq Z \leq -0.32) = 0.0023. \text{ It follows that a}
\]
\[
\text{sample of size } n = 45 \text{ is sufficient to keep } \beta \text{ smaller than } 0.20 \text{ when } \mu = 12.
\]

6.4.10 a) \( P(\text{Type I error}) = P(\text{reject } H_0 \mid H_0 \text{ is true}) = P(\bar{Y} \geq 3.20 \mid \lambda = 1) = \int_{3.20}^{\infty} e^{-\lambda}d\lambda = 0.04.
\]

b) \( P(\text{Type II error}) = P(\text{accept } H_0 \mid H_1 \text{ is true}) = P(\bar{Y} < 3.20 \mid \lambda = 4/3) = \int_{0}^{3.20} \frac{3}{4} e^{-3y/4}dy + \int_{3.20}^{\infty} e^{-3u}du = 0.91.
\]

6.4.16  If \( H_0 \) is true, \( X = X_1 + X_2 \) has a binomial distribution with \( n = 6 \) and \( p = \frac{1}{2} \). Therefore,
\[
\alpha = P(\text{reject } H_0 \mid H_0 \text{ is true}) = P(\bar{X} \geq 3 \mid p = \frac{1}{2}) = \sum_{k=4}^{6} \binom{6}{k} \left(\frac{1}{2}\right)^k \left(1 - \frac{1}{2}\right)^{6-k} = 7/2^6 = 0.11.
\]
6.4.18 a) \[ \alpha = P(\text{reject } H_0 \mid H_0 \text{ is true}) = P(Y \leq 2 \mid \lambda = 6) = \sum_{k=0}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} = 0.062. \]

b) \[ \beta = P(\text{accept } H_0 \mid H_1 \text{ is true}) = P(Y \geq 3 \mid \lambda = 4) = 1 - P(Y \leq 2 \mid \lambda = 4) = 1 - \sum_{k=0}^{2} \frac{e^{-4} 4^k}{k!} = 1 - 0.238 = 0.762. \]

6.4.20 \[ \beta = P(\text{accept } H_0 \mid H_1 \text{ is true}) = P(Y < \ln 10 \mid \lambda) = \int_0^{\ln 10} \lambda e^{-\lambda} d\lambda = 1 - e^{-\ln 10} = 1 - 10^{-1}. \]

6.4.21 \[ \alpha = P(\text{reject } H_0 \mid H_0 \text{ is true}) = P(Y_1 + Y_2 \leq k \mid \theta = 2). \] When \( H_0 \) is true, \( Y_1 \) and \( Y_2 \) are uniformly distributed over the square defined by \( 0 \leq Y_1 \leq 2 \) and \( 0 \leq Y_2 \leq 2 \), so the joint pdf of \( Y_1 \) and \( Y_2 \) is a plane parallel to the \( Y_1Y_2 \)-axis at height \( \frac{1}{4} \left( f_{Y_1}(Y_1) f_{Y_2}(Y_2) = \frac{1}{2} \cdot \frac{1}{2} \right) \). By geometry, \( \alpha \) is the volume of the triangular wedge in the lower left-hand corner of the square over which \( Y_1 \) and \( Y_2 \) are defined. The hypotenuse of the triangle in the \( Y_1Y_2 \)-plane has the equation \( y_1 + y_2 = k \). Therefore, \( \alpha = \text{area of triangle} \times \text{height of wedge} = \frac{1}{2} k \cdot k \cdot \frac{1}{4} = k^2 / 8. \) For \( \alpha \) to be 0.05, \( k = \sqrt{0.04} = 0.63 \).

6.4.22 \[ \alpha = P(\text{reject } H_0 \mid H_0 \text{ is true}) = P(Y_1Y_2 \leq k^* \mid \theta = 2). \] If \( \theta = 2 \), the joint pdf of \( Y_1 \) and \( Y_2 \) is the horizontal plane \( f_{Y_1,Y_2}(Y_1,Y_2) = \frac{1}{4} \). \( 0 \leq y_1 \leq 2, 0 \leq y_2 \leq 2 \). Therefore, \( \alpha = P(Y_1Y_2 \leq k^* \mid \theta = 2) = \frac{k^*}{4} \left( \int_{0}^{2} \int_{0}^{k^* / 2} \frac{1}{4} y_1 y_2 dy_1 dy_2 = \frac{k^*}{4} \left( \int_{0}^{2} \frac{k^*}{4} y_1 dy_1 \right) \right) = \frac{k^*}{4} \left( \int_{0}^{k^* / 2} k^* \ln \frac{k^*}{2} \right). \) By trial and error, \( k^* = 0.087 \) makes \( \alpha = 0.05 \).

Note: The \( k^* \) value in 6.4.22 is incorrect. The correct value is approximately 0.0349.