ST430 Spring 2005 **Midterm I** February 14, 2005, 11:00am–11:50am

There are three questions, with a total of 40 points. **Please start a new page for each question and do not use both sides.**

1. [Total 12 points]

   Let $Y_1, \ldots, Y_n$ denote a random sample from the probability density function

   $$f_Y(y;\theta) = (\theta+1)y^{\theta}, \quad 0 < y < 1, \quad \theta > -1.$$

   (a) [4] Use either the Fisher–Neyman or the Factorization Theorem, determine which of the following is sufficient for θ: $\hat{\theta}_1 = \sum Y_i$, $\hat{\theta}_2 = \sum Y_i^2$ or $\hat{\theta}_3 = \sum \ln Y_i$? Note: you need to show your work to receive full credit, and you may need to use the identity $x = e^{\ln x}$.

   (b) [4] Find an estimator for θ using the method of moments.

   (c) [4] Find an estimator for θ using the method of maximum likelihood.

2. [Total 16 points]

   Let $Y_1, \ldots, Y_n$ be a random sample from

   $$f_Y(y;\theta) = \begin{cases} \frac{3y^2}{\theta^3} & \text{if } 0 < y < \theta, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) [4] Use the method of maximum likelihood to find an estimator for θ. Hint: differentiation does not work here.

   (b) [4] Show that $\hat{\theta}_1 = \frac{4}{3}\bar{Y}$ is an unbiased estimator of θ, where $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$.

   (c) [4] First calculate $E(Y_i)$ and $E(Y_i^2)$, and then find $\text{Var}(\hat{\theta}_1)$.

   (d) [4] Find the Cramer–Rao lower bound for $f_Y(y;\theta)$. Compare your answer with $\text{Var}(\hat{\theta}_1)$. Any comments?

3. [Total 12 points]

   Assume that $X_1, \ldots, X_9$ are iid $N(2\mu, 3\mu^2)$ and $Y_1, \ldots, Y_8$ are iid $N(\frac{\mu}{2}, 2\mu^2)$. Also assume $X_1, \ldots, X_9$ are independent of $Y_1, \ldots, Y_8$. Let $\bar{X} = \frac{1}{9}\sum_{i=1}^{9} X_i$ and $\bar{Y} = \frac{1}{8}\sum_{i=1}^{8} Y_i$. We estimate μ with $\hat{\mu} = c(\bar{X} + \bar{Y})$, where c is a constant.

   (a) [4] Find c so that $\hat{\mu}$ is unbiased for μ.

   (b) [4] Express $\text{MSE}(\hat{\mu})$, the mean-squared-error of $\hat{\mu}$, in terms of μ and c.

   (c) [4] Find the best value of c so that $\text{MSE}(\hat{\mu})$ is minimized.

—— End of Midterm I ——