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POINT PROCESS CONVERGENCE OF STOCHASTIC VOLATILITY PROCESSES WITH APPLICATION TO SAMPLE AUTOCORRELATION

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Abstract

The paper considers one of the standard processes for modeling returns in finance, the stochastic volatility process with regularly varying innovations. The aim of the paper is to show how point process techniques can be used to derive the asymptotic behavior of the sample autocorrelation function of this process with heavy-tailed marginal distributions. Unlike other non-linear models used in finance, such as GARCH and bilinear models, sample autocorrelations of a stochastic volatility process have attractive asymptotic properties. Specifically, in the infinite variance case, the sample autocorrelation function converges to zero in probability at a rate that is faster the heavier the tails of the marginal distribution. This behavior is analogous to the asymptotic behavior of the sample autocorrelations of independent identically distributed random variables.

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1. Introduction

This paper can be considered as a continuation of the efforts made by the authors and their colleagues [1, 8, 10, 11, 12, 13, 15, 21, 23] to understand the asymptotic behavior of the sample autocorrelation function (ACF) of stationary ergodic processes with heavy-tailed marginal distributions. Time series with heavy tails *and* dependence, i.e. with bursty behavior, strong oscillations and clustering of extremes, are typical in areas such as finance (see [17]) and telecommunications (see [27]). Not surprisingly, they have attracted a lot of attention over the last few years, and various models have been proposed to capture both heavy tails and dependence. Among them are the familiar linear processes (such as ARMA, FARIMA processes) with regularly varying or subexponential innovations (see [5], Section 13.3 or [17], Chapter 7 and Appendix

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A3.3) and some classes of non-linear processes, including the ARCH, GARCH and stochastic volatility processes (see [18, 25]) which are particularly popular in finance.

The time series model that has been used most frequently in financial applications is the GARCH (generalized autoregressive conditionally heteroscedastic) process. A GARCH process $\{X_t\}$ of order (p, q) (GARCH(p, q)) satisfies the recurrence equations

$$X_t = \sigma_t\, Z_t \qquad \text{and} \qquad \sigma_t^2 = \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 \qquad (t \in \mathbb{Z}), \tag{1.1}$$

where $\{Z_t\}$ is an independent identically distributed (i.i.d.) sequence of random variables, the α_is and β_js are non-negative parameters and p, q are given integers. Estimation of the parameters in the GARCH model is rather straightforward (see [19]) and this is certainly a major reason for its enormous popularity. However, the probabilistic properties of the GARCH model are difficult to derive and far from being understood. This includes the problem of the existence of a stationary solution to (1.1) given the parameters α_i, β_j and the distribution of the innovations Z_t. The tails of the finite-dimensional distributions of $\{X_t\}$ are regularly varying under general conditions on the distribution of Z_t but with the exception of the GARCH(1,1) model analytic expressions for the index of regular variation or its spectral measure are unavailable; see [17, 8, 10, 21]. In the latter three references the limit theory of the sample autocorrelation function (ACF) of GARCH processes is treated. It turns out that the limits of the sample autocorrelations are complicated functions of dependent infinite variance stable random variables (see Section 2.1 below for a definition) and that the rates of convergence in these limit theorems can be extremely slow, depending on the index of regular variation. Moreover, the sample ACFs of GARCH processes, their absolute values and squares converge in distribution to non-degenerate limits, if their variances are infinite.

We note at this point that it is common practice in the financial time series literature (see e.g. [26]) to consider significant values of the sample ACFs of the absolute and squared returns of price series (such as stock indices, share prices, foreign exchange rates) as evidence of stochastic volatility.

Interestingly, the behavior of the sample ACFs for the GARCH model (see [8, 10, 21]) is totally different from the sample ACF behavior of linear processes with heavy-tailed innovations (see [5], Section 13.3). In the latter case the sample autocorrelations of the process converge in probability even when the marginal distribution has infinite variance to a quantity which depends only on the coefficients of the linear filter. This quantity can be interpreted as the population autocorrelation even though it does not exist. Moreover, the heavier the tail of the marginal distribution of the linear process, the faster the rate of convergence of the sample ACF to the *population* ACF. These properties are illustrated in a simulation study in [9] that compares the sampling distributions of the ACF for GARCH and stochastic volatility models.

In this paper we consider another type of non-linear model popular in applications to financial time series: the *stochastic volatility processes* given by

$$X_t = \sigma_t\, Z_t \qquad (t \in \mathbb{Z}), \tag{1.2}$$

where $\{Z_t\}$ is a sequence of i.i.d. random variables, independent of another strictly stationary volatility sequence $\{\sigma_t\}$ of non-negative random variables. Because of the independence of the volatility process and the noise, the probabilistic structure of these processes is much easier to understand than that of the GARCH processes. Indeed, the dependence in $\{X_t\}$ is essentially inherited from that in $\{\sigma_t\}$, and heavy tails, e.g. regularly varying tails of X_t, can be easily modeled by innovations Z_t with regularly varying tails and 'light tailed' volatilities σ_t. Due to the difficulty of parameter estimation in stochastic volatility models there are, however, some arguments in the literature which support a preference for GARCH-type models; see e.g. [25] for discussion.

In this paper we give the limit theory for the sample ACF of stochastic volatility processes whose marginal distributions have regularly varying tails and infinite variance. After dispensing with some preliminary results in Section 2 we show in Section 3 that the point processes constructed from the rescaled sequence $\{X_t\}$ converge in distribution to a Poisson process. Then a continuous mapping argument is applied to derive the asymptotic behavior of the sample ACF. In Section 4 it is shown that the (normalised) sample autocorrelations converge weakly to functions of infinite variance stable limits. This implies that the sample autocorrelations converge to zero at a rate which is faster the smaller the tail parameter. These results are in the same spirit as those proved by Davis and Resnick [11, 12, 13] for the sample ACF of linear processes with regularly varying tails. Moreover, the results for the stochastic volatility model can be proved by similar methods as in those papers. At the end of Section 4 we also indicate how to derive the sample ACF behavior for the sequences $\{|X_t|^\delta\}$ for some positive δ. Also for those processes, the limit theory of the sample ACF is much easier to derive than in the GARCH case; see e.g. [8] for a comparison.

2. Preliminaries

2.1. Stable distributions

Recall that a random variable Z and its distribution are said to be *α-stable* for some $\alpha \in (0,2]$ if it has characteristic function

$$\mathrm{E}\, \mathrm{e}^{i\lambda Z} = \begin{cases} \exp\left[i\,\lambda\,\gamma - \sigma_\alpha\,|\lambda|^\alpha\,\left(1 - i\,\beta\,\mathrm{sign}(\lambda)\,\tan\tfrac{1}{2}\pi\alpha\right)\right] & \text{if } \alpha \neq 1,\\ \exp\left[i\,\lambda\,\gamma - \sigma_\alpha\,|\lambda|\,\left(1 + i\,\beta\,(2/\pi)\,\mathrm{sign}(\lambda)\,\ln|\lambda|\right)\right] & \text{if } \alpha = 1. \end{cases}$$

Here $\gamma \in \mathbb{R}$, $\beta \in [-1,1]$ and $\sigma_\alpha > 0$ represent the location, skewness, and scale parameters, respectively ([24] is a general reference on stable distributions and processes).

2.2. The tails of products of independent random variables

We will frequently make use of a result by Breiman [4] about the regular variation of products of non-negative random variables ξ and η. Assume ξ is regularly varying with index $\alpha > 0$, i.e.,

$$\lim_{x\to\infty} \frac{\Pr\{\xi > cx\}}{\Pr\{\xi > x\}} = c^{-\alpha}\,, \quad c > 0\,.$$

If $\mathrm{E}\,\eta^{\alpha+\epsilon} < \infty$ for some $\epsilon > 0$, then $\xi\eta$ is regularly varying with index α and

$$\Pr\{\xi\eta > x\} \sim \mathrm{E}\,\eta^\alpha \Pr\{\xi > x\} \qquad (x \to \infty). \tag{2.1}$$

2.3. The tails of a stochastic volatility model

Recall the definition of the stochastic volatility model from (1.2). Suppose Z is regularly varying with index α and tail balancing condition

$$\lim_{x\to\infty}\frac{\Pr\{Z>x\}}{\Pr\{|Z|>x\}}=p \qquad \text{and} \qquad \lim_{x\to\infty}\frac{\Pr\{Z\le -x\}}{\Pr\{|Z|>x\}}=q\,, \tag{2.2}$$

where $p+q=1$ for some $p\in[0,1]$. Then by virtue of Breiman's result (2.1), we know that as $x\to\infty$,

$$\Pr\{X>x\}\sim \mathrm{E}\sigma^\alpha\ \Pr\{Z>x\} \qquad \text{and} \qquad \Pr\{X\le -x\}\sim \mathrm{E}\sigma^\alpha\ \Pr\{Z\le -x\}\,, \tag{2.3}$$

provided $E\sigma^{\alpha+\epsilon}<\infty$ for some $\epsilon>0$. In what follows, we assume that (2.2) holds, and we also require

$$\mathrm{E}|Z|^\alpha=\infty\,. \tag{2.4}$$

Then Z_1Z_2 is also regularly varying with index α satisfying (see equations (3.2) and (3.3) in [13])

$$\frac{\Pr\{Z_1Z_2>x\}}{\Pr\{|Z_1Z_2|>x\}}\to \widetilde{p}\,:=\,p^2+(1-p)^2 \qquad (x\to\infty).$$

Another application of (2.1) implies that X_1X_h is regularly varying with index α:

$$\begin{aligned}\Pr\{X_1X_h>x\}\,&=\,\Pr\{Z_1Z_2\sigma_1\sigma_h>x\}\,\sim\,\mathrm{E}[\sigma_1\sigma_h]^\alpha\ \Pr\{Z_1Z_2>x\}\,,\\ \Pr\{X_1X_h\le -x\}\,&=\,\Pr\{Z_1Z_2\sigma_1\sigma_h\le -x\}\,\sim\,\mathrm{E}[\sigma_1\sigma_h]^\alpha\ \Pr\{Z_1Z_2\le -x\}\,,\end{aligned} \tag{2.5}$$

provided $\mathrm{E}[\sigma_1\sigma_h]^{\alpha+\epsilon}<\infty$ for some $\epsilon>0$.

3. Point process convergence

In this section we prove distributional convergence of the point processes constructed from the stochastic volatility process $(X_t\}$ defined in (1.2). The innovations Z_t are always assumed to be regularly varying with index $\alpha>0$, satisfying the tail balancing condition (2.2) and the moment condition (2.4). For an excellent account of point processes and their weak convergence, see the monograph by Daley and Vere-Jones [6].

In Davis and Resnick [13], convergence in distribution for a sequence of point processes with points based on cross-products of the sequence $\{Z_t\}$ was established. As we show below, the same result holds for the cross-products based on the stochastic volatility process. To describe the limit point process (and it is also germane to the study of the sample ACF), let

$$\sum_{k=1}^\infty \varepsilon_{\widetilde{P}_{k,0}}\,,\ \ \sum_{k=1}^\infty \varepsilon_{\widetilde{P}_{k,1}}\,,\ \ldots\,,\ \sum_{k=1}^\infty \varepsilon_{\widetilde{P}_{k,h}}$$

be independent Poisson processes on $\overline{\mathbb{R}}\backslash\{0\}$ (here $\overline{\mathbb{R}}=[-\infty,\infty]$) with intensity measures

$$\begin{aligned}\widetilde{\lambda}_0(\mathrm{d}x)&=\alpha\left[px^{-\alpha}\mathbf{1}_{(0,\infty)}(x)+(1-p)(-x)^{-\alpha}\mathbf{1}_{(-\infty,0)}(x)\right]\,\mathrm{d}x\,,\\ \widetilde{\lambda}_i(\mathrm{d}x)&=\alpha\left[\widetilde{p}x^{-\alpha}\mathbf{1}_{(0,\infty)}(x)+(1-\widetilde{p})(-x)^{-\alpha}\mathbf{1}_{(-\infty,0)}(x)\right]\mathrm{d}x \qquad (i=1,\ldots,h),\end{aligned}$$

respectively. (Here, $\varepsilon_x(\cdot)$ refers to the point measure with unit mass at the point $\{x\}$.) Setting

$$P_{k,0} = \|\sigma_1\|_\alpha \widetilde{P}_{k,0}\,, \quad P_{k,i} = \|\sigma_1\sigma_{1+i}\|_\alpha \widetilde{P}_{k,i} \qquad (k \geq 1,\ i = 1, \ldots, h),$$

where $\|Y\|_\alpha = (\mathrm{E}|Y|^\alpha)^{1/\alpha}$, it follows that

$$\sum_{k=1}^{\infty} \varepsilon_{P_{k,0}}\,, \ \sum_{k=1}^{\infty} \varepsilon_{P_{k,1}}\,, \ldots, \ \sum_{k=1}^{\infty} \varepsilon_{P_{k,h}}$$

are independent Poisson processes with intensity functions

$$\lambda_0(\mathrm{d}x) = \|\sigma_1\|_\alpha^\alpha\, \widetilde{\lambda}_0(\mathrm{d}x) \qquad \text{and} \qquad \lambda_i(\mathrm{d}x) = \|\sigma_1\sigma_{1+i}\|_\alpha^\alpha\, \widetilde{\lambda}_i(\mathrm{d}x) \qquad (i \geq 1).$$

Let $\{a_n\}$ and $\{b_n\}$ be the respective $(1 - n^{-1})$-quantiles of $|Z_1|$ and $|Z_1Z_2|$ defined by

$$a_n = \inf\big\{x\colon \Pr\{|Z_1| > x\} \leq n^{-1}\big\} \quad \text{and} \quad b_n = \inf\big\{x\colon \Pr\{|Z_1Z_2| > x\} \leq n^{-1}\big\}. \tag{3.1}$$

We start with a result for the case when the volatility process $(\sigma_t\}$ is m-dependent.

Theorem 3.1. *Suppose $\{X_t\}$ is the stochastic volatility process given by* (1.2)*, where the marginal distribution of Z satisfies* (2.2) *and* (2.4)*. Let $\{\sigma_t\}$ be a stationary m-dependent sequence of non-negative random variables such that, for fixed $h \geq 1$, and some $\epsilon > 0$,*

$$\mathrm{E}\sigma_1^{\alpha+\epsilon} < \infty \qquad \textit{and} \qquad \mathrm{E}[\sigma_1\sigma_{1+k}]^{\alpha+\epsilon} < \infty \quad (k = 1, \ldots, h).$$

Let $\mathbf{Y}_{n,t} = (a_n^{-1}X_t, b_n^{-1}X_tX_{t+1}, \ldots, b_n^{-1}X_tX_{t+h})$, where $\{a_n\}$ and $\{b_n\}$ are given in (3.1)*. Then*

$$N_n = \sum_{t=1}^{n} \varepsilon_{\mathbf{Y}_{n,t}} \ \xrightarrow{d}\ N = \sum_{i=0}^{h}\sum_{k=1}^{\infty} \varepsilon_{P_{k,i}\mathbf{e}_i}\,, \tag{3.2}$$

where $\mathbf{e}_i \in \mathbb{R}^{h+1}$ is the basis element with ith component equal to 1 *and the rest* 0*. Here $\xrightarrow{d}$ denotes convergence in distribution in the space $M_p(\overline{\mathbb{R}}^{h+1}\backslash\{\mathbf{0}\})$ of Radon point measures on $\overline{\mathbb{R}}^{h+1}\backslash\{\mathbf{0}\}$ equipped with the vague topology* (*cf.* [6, 20, 22])*.*

Remark 3.2. The points of the limit point process N are concentrated on the coordinate axes and distributed independently according to Poisson processes with intensity measures λ_i. Using Laplace transforms, it is easy to see that N is a Poisson point process with intensity measure $\nu(\mathrm{d}y_0, \ldots, \mathrm{d}y_h) = \sum_{i=0}^{h} \lambda_i(\mathrm{d}y_i)\prod_{j\neq i}\varepsilon_0(\mathrm{d}y_j)$.

Proof. It is clear that the stochastic volatility process $\{X_t\}$ inherits the m-dependence of the underlying volatility process $\{\sigma_t\}$. It then follows that $(Y_{n,t})_{t\geq 1}$ is $(m+h+1)$-dependent for every $n \geq 1$ and satisfies the mixing condition D^* in Davis and Resnick [14]. By Theorem 2.1 of the same paper, it suffices to show the following two conditions:

$$n\ \Pr\{\mathbf{Y}_{n,1} \in \cdot\} \xrightarrow{v} \nu(\cdot)\,, \tag{3.3}$$

where $\xrightarrow{v}$ denotes vague convergence in $\overline{\mathbb{R}}^{h+1}\backslash\{\mathbf{0}\}$ and ν is the intensity measure of N as specified in Remark 3.2, and for any non-negative continuous function $g \leq 1$ with compact support on $\overline{\mathbb{R}}^{h+1}\backslash\{\mathbf{0}\}$,

$$\lim_{k\to\infty}\limsup_{n\to\infty} n\sum_{i=2}^{[n/k]} \mathrm{E}[g(\mathbf{Y}_{n,1})g(\mathbf{Y}_{n,i})] = 0\,. \tag{3.4}$$

We obtain from (2.5) and (2.3) that for $k\ge 1$,

$$n\ \Pr\{b_n^{-1}X_1X_{1+k}>x\}\sim\lambda_k(x,\infty)\qquad\text{and}\qquad n\ \Pr\{a_n^{-1}X_1>x\}\sim\lambda_0(x,\infty)\,. \tag{3.5}$$

For $j>i>1$ fixed and $M>0$ set $X^*=|X_1|\max(\sigma_i,\sigma_j)$ and $A_n=\{a_n^{-1}X^*>M\}$. Then, for any $x>0$ and $j>i>1$,

$$\begin{aligned}&\Pr\{b_n^{-1}|X_1X_i|>x,\,b_n^{-1}|X_1X_j|>x\}\\ &\le \Pr\{b_n^{-1}|Z_i|X^*>x,\,b_n^{-1}|Z_j|X^*>x,A_n\}+\Pr\{b_n^{-1}|Z_i|X^*>x,\,b_n^{-1}|Z_j|X^*>x,A_n^c\}\\ &\le \Pr\{A_n\}+\Pr\{a_n\,b_n^{-1}\,|Z|>xM^{-1}\}\,.\end{aligned}$$

An application of (2.1) shows that $\limsup_{n\to\infty} n\ \Pr\{A_n\}\le \mathrm{E}[\sigma_1\max(\sigma_i,\sigma_j)]^\alpha\,M^{-\alpha}$. Since $b_n/a_n\to\infty$ (see (3.5) in [13]), $\limsup_{n\to\infty} n\ \Pr\{a_n\,b_n^{-1}\,|Z|>xM^{-1}\}=0$. Consequently,

$$n\ \Pr\{b_n^{-1}|X_1X_i|>x,\,b_n^{-1}|X_1X_j|>x\}\to 0\,, \tag{3.6}$$

and using a similar argument we also have

$$n\ \Pr\{a_n^{-1}|X_1|>x,\,b_n^{-1}|X_1X_j|>x\}\to 0\,. \tag{3.7}$$

We are now in a position to prove (3.3) and (3.4). As in Davis and Resnick [13], we introduce the class of sets $\mathcal{S}$ consisting of rectangles B of the form

$$B=(b_0,c_0]\times(b_1,c_1]\times\cdots\times(b_h,c_h]$$

which are bounded away from $\mathbf{0}$ and $b_i<c_i$, $b_i\ne 0$, $c_i\ne 0$, $i=0,\dots,h$. Since $B\in\mathcal{S}$ is bounded away from zero, either

$$B\cap\{y\mathbf{e}_i\colon y\in\overline{\mathbb{R}}\}=\emptyset\qquad\text{for } i=0,\dots,h, \tag{3.8}$$

or

$$B\cap\{y\mathbf{e}_i\,:\,y\in\overline{\mathbb{R}}\}=\begin{cases}\{0\}^{j-1}\times(b_j,c_j]\times\{0\}^{h-j+1} & (i\ne j),\\ \emptyset, & (i=j).\end{cases} \tag{3.9}$$

That is, B has either empty intersection with all of the coordinate axes or intersects exactly one in an interval. Note that in (3.9), $b_i<0<c_i$ for $i\ne j$ and $0\in(b_j,c_j]$. The class $\mathcal{S}$ forms a DC-semi-ring (cf. e.g. [20]). It easily follows from (3.5) that

$$n\ \Pr\{\mathbf{Y}_{n,1}\in B\}\to\nu(B):=\begin{cases}0 & \text{if } B\in\mathcal{S}\text{ satisfies (3.8)},\\ \lambda_j(b_j,c_j] & \text{if } B\in\mathcal{S}\text{ satisfies (3.9)},\end{cases}$$

which proves (3.3) (here, $\lambda_j(\cdot)$ is defined by analogy with (3.5)).

As for (3.4), let $g\le 1$ be a non-negative continuous function with compact support contained in the set

$$S=\bigcup_{i=0}^{h}\{(y_0,\dots,y_h)\in\overline{\mathbb{R}}^{h+1}\,:\,|y_i|>\epsilon\}$$

for some $\epsilon > 0$. Following the arguments given for (3.5)–(3.7), one can show that for $i > 1$, $n \Pr\{\mathbf{Y}_{n,1} \in S, \mathbf{Y}_{n,i} \in S\} \to 0$. Using the m-dependence of the sequence $\{X_t\}$, we have for every fixed $k \geq 1$,

$$\begin{aligned}\limsup_{n\to\infty} n \sum_{i=2}^{[n/k]} \mathrm{E}[g(\mathbf{Y}_{n,1})g(\mathbf{Y}_{n,i})] &\leq \limsup_{n\to\infty} n \sum_{i=2}^{m+h+1} \Pr\{\mathbf{Y}_{n,1} \in S, \mathbf{Y}_{n,i} \in S\} \\ &\quad + \limsup_{n\to\infty} n \sum_{i=m+h+2}^{[n/k]} \mathrm{E}[g(\mathbf{Y}_{n,1})\, \mathrm{E}g(\mathbf{Y}_{n,i})] \\ &= k^{-1} \left(\int g\, d\nu \right)^2 ,\end{aligned}$$

and the right-hand expression converges to 0 as $k \to \infty$. This completes the proof of (3.4) and the conclusion of the theorem now follows. □

We now consider an extension of the above results to the case when the logarithm of the volatility process $\{\sigma_t\}$ is a linear process. Specifically, suppose $\{\ln \sigma_t\}$ is the linear process given by

$$\ln \sigma_t = \sum_{j=-\infty}^{\infty} \psi_j \epsilon_{t-j} \qquad (t \in \mathbb{Z}), \tag{3.10}$$

where $\{\epsilon_t\}$ is a sequence of i.i.d. mean-zero Gaussian random variables and $\{\psi_j\}$ satisfies

$$\sum_{j=-\infty}^{\infty} \psi_j^2 < \infty\, . \tag{3.11}$$

Theorem 3.3. *Let $\{X_t\}$ denote the stochastic volatility process given by* (1.2), *where the marginal distribution of Z satisfies* (2.2) *and* (2.4), *and $\{\sigma_t\}$ has the representation in* (3.10). *Then* (3.2) *holds.*

Proof. For m a fixed positive integer let $\{X_{t,m}\}$ be the stochastic volatility process based on a $(2m+1)$-dependent volatility process given by

$$\ln \sigma_{t,m} = \sum_{j=-m}^{m} \psi_j \epsilon_{t-j} \qquad (t \in \mathbb{Z}). \tag{3.12}$$

If $\{N_n^{(m)}\}$ is the sequence of point processes corresponding to the sequence $\{X_{t,m}\}$, then by Theorem 3.1,

$$N_n^{(m)} \xrightarrow{d} N^{(m)} = \sum_{i=0}^{h} \sum_{k=1}^{\infty} \varepsilon_{P_{k,i}^{(m)} \mathbf{e}_i}\, ,$$

where the points $P_{k,i}^{(m)}$ are defined the same way as the $P_{k,i}$ but with σ_t replaced by $\sigma_{t,m}$. To complete the proof, it suffices to show, by Theorem 4.2 in Billingsley [2], that

$$N^{(m)} \xrightarrow{d} N \tag{3.13}$$

and for any $\eta > 0$

$$\lim_{m\to\infty} \limsup_{n\to\infty} \Pr\{\rho(N_n, N_n^{(m)}) > \eta\} = 0\, , \tag{3.14}$$

where ρ is the metric inducing the vague topology.

Since $\sigma_{1,m}$ and σ_1 are log-normal and (3.11) holds, the α th powers of $\sigma_{i,m}$ and of their cross products are uniformly integrable. It follows that for $m \to \infty$,

$$\mathrm{E}\sigma_{1,m}^{\alpha} \to \mathrm{E}\sigma_1^{\alpha} \qquad \text{and} \qquad \mathrm{E}[\sigma_{1,m}\sigma_{1+i,m}]^{\alpha} \to \mathrm{E}[\sigma_1\sigma_{1+i}]^{\alpha}\,, \tag{3.15}$$

and hence the intensity measures $\lambda_i^{(m)}$ for the Poisson points $\{P_{k,i}^{(m)},\, k \ge 1\}$ converge vaguely to λ_i. This in turn implies that the intensity measure $\nu^{(m)}$ for $N^{(m)}$ converges vaguely to the intensity measure ν of N as $m \to \infty$, which in turn implies (3.13).

Now turning to (3.14), we have for any $\gamma > 0$ and $k \ge 1$,

$$\begin{aligned}
&\Pr\Big\{b_n^{-1}\bigvee_{t=1}^{n}|X_{t,m}X_{t+k,m} - X_tX_{t+k}| > \gamma\Big\}\\
&= \Pr\Big\{b_n^{-1}\bigvee_{t=1}^{n}|Z_tZ_{t+k}|\,|\sigma_{t,m}\sigma_{t+k,m} - \sigma_t\sigma_{t+k}| > \gamma\Big\}\\
&\le n\ \Pr\{b_n^{-1}|Z_1Z_{1+k}|\,|\sigma_{1,m}\sigma_{1+k,m} - \sigma_1\sigma_{1+k}| > \gamma\}\,.
\end{aligned}$$

An application of (2.1) shows that the limit of the last expression is asymptotically (as $\gamma \to \infty$) of the order

$$\gamma^{-\alpha}\mathrm{E}|\sigma_{1,m}\sigma_{1+k,m} - \sigma_1\sigma_{1+k}|^{\alpha}\,,$$

which, by (3.15), converges to zero as $m \to \infty$.

A similar result holds for the case when $k = 0$ and b_n is replaced by a_n^2.

The same argument provided in the proof of Theorem 2.4 of [11] can now be used to establish (3.14). □

Remark 3.4. The assumption that ϵ_t is Gaussian can be relaxed to any marginal distribution such that (3.15) and the moment conditions in Theorem 3.1 hold.

4. Limit theory for the sample ACF

In this section we study the asymptotic behavior of the sample ACF and the sample autocovariance function (ACVF) of the stochastic volatility process $\{X_t\}$ defined in (1.2). Define the sample ACVF $(\widetilde{\gamma}_{n,X}(h))$ and the sample ACF $(\widetilde{\rho}_{n,X}(h))$ of $\{X_t\}$ by

$$\widetilde{\gamma}_{n,X}(h) = \frac{1}{n}\sum_{t=1}^{n-h}X_tX_{t+h} \qquad \text{and} \qquad \widetilde{\rho}_{n,X}(h) = \frac{\sum_{t=1}^{n-h}X_tX_{t+h}}{\sum_{t=1}^{n}X_t^2} \qquad (h = 0,1,2,\ldots).$$

Theorem 4.1. *Assume $\{X_t\}$ satisfies the conditions of Theorem* 3.1. *Let $\{a_n\}$, $\{b_n\}$ be the normalizing sequences defined in* (3.1).

(1°) *If $\alpha \in (0,1)$, then*

$$n\left(a_n^{-2}\widetilde{\gamma}_{n,X}(0), b_n^{-1}\widetilde{\gamma}_{n,X}(1), \ldots, b_n^{-1}\widetilde{\gamma}_{n,X}(r)\right) \xrightarrow{d} \{V_h\}_{h=0,\ldots,r}\,,$$

where $V_0 = \sum_{k=1}^{\infty}P_{k,0}^2$ and $V_h = \sum_{k=1}^{\infty}P_{k,h}$ for $h \ge 1$.

(2°) *If $\alpha = 1$ and Z_1 has a symmetric distribution, then*

$$n\ \left(a_n^{-2}\widetilde{\gamma}_{n,X}(0), b_n^{-1}\widetilde{\gamma}_{n,X}(1), \ldots, b_n^{-1}\widetilde{\gamma}_{n,X}(r)\right) \xrightarrow{d} \{V_h\}_{h=0,\ldots,r}\,,$$

where $V_0 = \sum_{k=1}^{\infty}P_{k,0}^2$ and $\{V_h\}_{h=1,\ldots,r}$ is the distributional limit of

$$\left\{\sum_{k=1}^{\infty}P_{k,h}\mathbf{1}_{\{|P_{k,h}|>\epsilon\}}\right\}_{h=1,\ldots,r}$$

as $\epsilon \to 0$. *(See* [7] *for the existence of this limit.)*
(3°) *If* $\alpha \in (1,2)$ *and* Z_1 *has mean* 0*, then*

$$n\left(a_n^{-2}\widetilde{\gamma}_{n,X}(0), b_n^{-1}\widetilde{\gamma}_{n,X}(1), \dots, b_n^{-1}\widetilde{\gamma}_{n,X}(r)\right) \xrightarrow{d} \{V_h\}_{h=0,\dots,r} ,$$

where $V_0 = \sum_{k=1}^{\infty} P_{k,0}^2$ *and* $\{V_h\}_{h=1,\dots,r}$ *is the distributional limit of*

$$\left\{ \sum_{k=1}^{\infty} \left[P_{k,h} 1_{\{|P_{k,h}|>\epsilon\}} - \int_{\{\epsilon<|x|<\infty\}} x\lambda_k(\mathrm{d}x) \right] \right\}_{h=1,\dots,r}$$

as $\epsilon \to 0$. *(See* [7] *for the existence of this limit.)*

In all three cases, the random variables $V_0, V_1, \dots, V_r$ *are independent,* V_0 *is positive* $\alpha/2$*-stable and* $V_1, \dots, V_r$ *are* α*-stable with index* α *(see Section* 2.1*). In addition,*

$$\left(a_n^2 b_n^{-1}\widetilde{\rho}_{n,X}(h)\right)_{h=1,\dots,r} \xrightarrow{d} (V_h/V_0)_{h=1,\dots,r} .$$

The proof of the theorem is omitted since it is nearly identical to that given for Theorem 3.3 in [13] where the case of a linear process with regularly varying innovations was treated.

The conclusion of (3°) of the theorem remains valid if $\widetilde{\rho}_{n,X}(h)$ is replaced by the mean-corrected version of the ACF given by

$$\widehat{\rho}_{n,X}(h) = \frac{\sum_{t=1}^{n-h}\left(X_t - \overline{X}_n\right)\left(X_{t+h} - \overline{X}_n\right)}{\sum_{t=1}^{n}\left(X_t - \overline{X}_n\right)^2} ,$$

where $\overline{X}_n$ denotes the sample mean; see Corollary 1 in [13], p. 547.

Remark 4.2. By choosing the volatility process $\{\sigma_t\}$ to be identically 1, we can recover the limiting results obtained in Davis and Resnick [13] for the autocovariances and autocorrelations of the process $\{Z_t\}$. If $(S_0, S_1, \dots, S_r)$ denotes the limit random vector of the sample autocovariances based on $\{Z_t\}$, then there is an interesting connection between S_k and V_k, namely,

$$(V_0, V_1, \dots, V_r) \stackrel{d}{=} (\|\sigma_1\|_\alpha^2 S_0,\ \|\sigma_1\sigma_2\|_\alpha S_1, \dots,\ \|\sigma_1\sigma_{1+r}\|_\alpha S_r).$$

It follows that

$$\left(a_n^2 b_n^{-1}\widetilde{\rho}_{n,X}(h)\right)_{h=1,\dots,r} \xrightarrow{d} \left(\frac{\|\sigma_1\sigma_{h+1}\|_\alpha}{\|\sigma_1\|_\alpha^2}\,\frac{S_h}{S_0}\right)_{h=1,\dots,r} .$$

The following result shows that Theorem 4.1 remains valid for much wider classes of stochastic volatility processes.

Theorem 4.3. *The conclusions of Theorem* 4.1 *remain valid if* $\{X_t\}$ *satisfies the conditions of Theorem* 3.3.

When the random variables X_t have finite variance and $\{\sigma_t\}$ is strongly mixing with a sufficiently fast rate, then standard central limit theory with normal limits and $\sqrt{n}$-rates applies to the sample autocovariances and autocorrelations; see [16]. Indeed, the sequence $\{X_t\}$ then inherits the same rate of convergence from the $\{\sigma_t\}$ sequence.

Proof. As in the proof of Theorem 3.3, let $\{X_{t,m}\}$ be the stochastic volatility process based on the volatility process $\{\sigma_{t,m}\}$ satisfying (3.12). Since Theorem 4.1 applies to the $\{X_{t,m}\}$ process it suffices, by Theorem 4.2 in [2] to show that for all $\eta > 0$

$$\lim_{m\to\infty} \limsup_{n\to\infty} \Pr\left\{a_n^{-2}\left|\sum_{t=1}^n (X_t^2 - X_{t,m}^2)\right| > \eta\right\} = 0, \tag{4.1}$$

$$\lim_{m\to\infty} \limsup_{n\to\infty} \Pr\left\{b_n^{-1}\left|\sum_{t=1}^n (X_t X_{t+k} - X_{t,m}X_{t+k,m})\right| > \eta\right\} = 0 \qquad (k = 1,\dots,m), \tag{4.2}$$

and

$$V_h^{(m)} \xrightarrow{d} V_h \qquad (h = 0,\dots,r). \tag{4.3}$$

First we prove (4.2). Write $Y_{t,m} = \sigma_t\sigma_{t+k} - \sigma_{t,m}\sigma_{t+k,m}$. Then for $k \ge 1$ and $x > 0$,

$$\begin{aligned}
&b_n^{-1}\sum_{t=1}^n (X_t X_{t+k} - X_{t,m}X_{t+k,m})\\
&\quad = b_n^{-1}\sum_{t=1}^n \left[Z_t Z_{t+k}\mathbf{1}_{\{|Z_t Z_{t+k}|\le b_n x\}} - \mathrm{E}(Z_1 Z_1 \mathbf{1}_{\{|Z_1 Z_2|\le b_n x\}})\right] Y_{t,m}\\
&\qquad + b_n^{-1}\sum_{t=1}^n Z_t Z_{t+k}\mathbf{1}_{\{|Z_t Z_{t+k}|> b_n x\}} Y_{t,m} \;+\; b_n^{-1}\mathrm{E}(Z_1 Z_1 \mathbf{1}_{\{|Z_1 Z_2|\le b_n x\}})\sum_{t=1}^n Y_{t,m}\\
&\quad = I_1 + I_2 + I_3\,.
\end{aligned}$$

For any $\eta > 0$, we have

$$\Pr\{|I_2| > \eta\} \le n\,\Pr\{|Z_1 Z_{1+k}| > b_n x\} \to x^{-\alpha} \qquad (n\to\infty), \tag{4.4}$$

and the right-hand side converges to zero as $x \to \infty$. Now if $\alpha \in (0,1)$, we have by Karamata's theorem (see [3]),

$$\begin{aligned}
\limsup_{n\to\infty} \mathrm{E}|I_1| &\le \limsup_{n\to\infty} n\, b_n^{-1}\, \mathrm{E}\big(|Z_1 Z_{1+k}|\mathbf{1}_{\{|Z_1 Z_{1+k}|\le b_n x\}}\big)\mathrm{E}|Y_{1,m}|\\
&= x^{1-\alpha}\,\mathrm{E}|Y_{1,m}|\,\frac{\alpha}{1-\alpha} \;\to\; 0 \qquad (m\to\infty),
\end{aligned} \tag{4.5}$$

and similarly for I_3. On the other hand, if $\alpha \in [1,2)$ then by the independence of $\{Y_{t,m}\}$ and $\{Z_t\}$, we have that $\mathrm{var}(I_1)$ is bounded above by

$$\begin{aligned}
&n\, b_n^{-2}\,\mathrm{var}(Z_1 Z_{1+k}\mathbf{1}_{\{|Z_1 Z_{1+k}|\le b_n x\}})\,\mathrm{E}Y_{1,m}^2\\
&\quad + 2(n-1)\, b_n^{-2}\,|\mathrm{cov}\left(Z_1 Z_{1+k}\mathbf{1}_{\{|Z_1 Z_{1+k}|\le b_n x\}}, Z_{1+k}Z_{1+2k}\mathbf{1}_{\{|Z_{1+k}Z_{1+2k}|\le b_n x\}}\right)|\,\mathrm{E}Y_{1,m}^2\\
&\qquad \le 3n\, b_n^{-2}\,\mathrm{E}\big(Z_1^2 Z_{1+k}^2\mathbf{1}_{\{|Z_1 Z_{1+k}|\le b_n x\}}\big)\mathrm{E}Y_{1,m}^2\\
&\qquad \to 3x^{2-\alpha}\,\mathrm{E}Y_{1,m}^2\,\frac{\alpha}{2-\alpha} \qquad (n\to\infty),\\
&\qquad \to 0 \qquad (m\to\infty).
\end{aligned}$$

If $\alpha = 1$, then the symmetry of Z_1 implies that $\mathrm{E}I_3 = 0$. If $\alpha > 1$, then

$$
\begin{aligned}
|\mathrm{E}I_3| &\le n\, b_n^{-1}\, \mathrm{E}\big(|Z_1 Z_{1+k}|\mathbf{1}_{\{|Z_1 Z_{1+k}|>b_n x\}}\big)\mathrm{E}|Y_{1,m}| \\
&\to x^{1-\alpha}\, \mathrm{E}|Y_{1,m}|\frac{\alpha}{\alpha-1} \qquad (n\to\infty), \\
&\to 0 \qquad (m\to\infty).
\end{aligned}
$$

The limit in (4.2) now follows easily for all $\alpha \in (0,2)$.

The argument for (4.1) is nearly identical so we omit it. Using the representation described in Remark 4.2, we have

$$V_0^{(m)} = \|\sigma_{1,m}\|_\alpha^2\, S_0 \quad \text{and} \quad V_h^{(m)} = \|\sigma_{1,m}\sigma_{1+h,m}\|_\alpha\, S_h\,.$$

By (3.15), the limits in (4.3) are immediate. □

4.1. Other Powers

It is also possible to investigate the sample ACVF and ACF of the processes $\{|X_t|^\delta\}$ for any power $\delta > 0$. This is common practice in financial time series analysis in order to detect non-linearities. We illustrate the method in the case $\delta = 1$.

Notice that $|X_t| = |Z_t|\sigma_t$, $t = 1,2,\ldots$, has a structure similar to the original process $\{X_t\}$. In particular, in the proofs of the point process convergence in Section 3 we assumed only the conditions (2.2) and (2.4) on Z, which are also satisfied for $|Z|$, and if $\mathrm{E}|Z|$ exists, also for $|Z| - \mathrm{E}|Z|$, both for $p = 1$. Hence the results in Sections 3 and 4 with $\alpha < 1$ immediately apply, with the limiting point process modified such that $\{Z_t\}$ is replaced by $\{|Z_t|\}$.

For $\alpha \in (1,2)$, one can use the following decomposition for $h \ge 1$ with $\gamma_{|X|} = \mathrm{E}|X_0 X_k|$, $\widetilde{Z}_t = |Z_t| - \mathrm{E}|Z|$ and $\widetilde{X}_t = \widetilde{Z}_t\sigma_t$:

$$
\begin{aligned}
n\,(\widetilde{\gamma}_{n,|X|}(h) - \widetilde{\gamma}_{|X|}(h)) &= \sum_{t=1}^{n-h} \widetilde{Z}_t\widetilde{Z}_{t+h}\sigma_t\sigma_{t+h} + \mathrm{E}|Z|\sum_{t=1}^{n-h}\widetilde{Z}_t\sigma_t\sigma_{t+h} \\
&\quad + \mathrm{E}|Z|\sum_{t=1}^{n-h}\widetilde{Z}_{t+h}\sigma_t\sigma_{t+h} - (\mathrm{E}|Z|)^2\sum_{t=1}^{n-h}(\sigma_t\sigma_{t+h} - \mathrm{E}\sigma_0\sigma_h) \\
&= I_1 + I_2 + I_3 + I_4\,, \quad \text{say.}
\end{aligned}
$$

Since $n^{-1}I_1 = \widetilde{\gamma}_{n,\widetilde{X}}(h)$ and $\mathrm{E}\widetilde{Z} = 0$, the results of Sections 3 and 4 are applicable to $\{\widetilde{X}_t\}$. Also notice that $\{a_n^{-2}\widetilde{\gamma}_{n,|X|}(0)\}$ converges weakly to an $\alpha/2$-stable random variable, for the same reasons as given for $\{X_t\}$. It remains to show that

$$b_n^{-1} I_j \xrightarrow{P} 0 \qquad (j = 2,3,4). \tag{4.6}$$

The same arguments as for $\widetilde{\gamma}_{n,X}(h)$ show that $a_n^{-1}I_j$, $j = 2,3$, converge to an α-stable distribution, and so, since $a_n/b_n \to 0$, (4.6) holds for $j = 2,3$. If the Gaussian process $\{\ln\sigma_t\}$ has an absolutely summable ACVF, then so does the process $\{\sigma_t\sigma_{t+h}\}$ and hence $\mathrm{var}(b_n^{-1}I_4) \to 0$ as required.

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