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EXTREME VALUE THEORY FOR SPACE-TIME PROCESSES WITH HEAVY-TAILED DISTRIBUTIONS

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ABSTRACT. Many real-life time series often exhibit clusters of outlying observations that cannot be adequately modeled by a Gaussian distribution. Heavy-tailed distributions such as the Pareto distribution have proved useful in modeling a wide range of bursty phenomena that occur in areas as diverse as finance, insurance, telecommunications, meteorology, and hydrology. Regular variation provides a convenient and unified background for studying multivariate extremes when heavy tails are present. In this paper, we study the extreme value behavior of the space-time process given by

$$X_t(\mathbf{s}) = \sum_{i=0}^{\infty} \psi_i(\mathbf{s}) Z_{t-i}(\mathbf{s}), \quad \mathbf{s} \in [0,1]^d,$$

where $(Z_t)_{t \in \mathbb{Z}}$ is an iid sequence of random fields on $[0,1]^d$ with values in the Skorokhod space $\mathbb{D}([0,1]^d)$ of càdlàg functions on $[0,1]^d$ equipped with the J_1-topology. The coefficients ψ_i are deterministic real-valued fields on $\mathbb{D}([0,1]^d)$. The indices $\mathbf{s}$ and t refer to the observation of the process at location $\mathbf{s}$ at time t. For example, $X_t(\mathbf{s}), t = 1, 2, \ldots$, could represent the time series of annual maxima of ozone levels at location $\mathbf{s}$. The problem of interest is determining the probability that the maximum ozone level over the entire region $[0,1]^2$ does not exceed a given standard level $f \in \mathbb{D}([0,1]^2)$ in n years. By establishing a limit theory for point processes based on $(X_t(\mathbf{s}))$, $t = 1 \ldots, n$, we are able to provide approximations for probabilities of extremal events. This theory builds on earlier results of de Haan and Lin [11] and Hult and Lindskog [13] for regular variation on $\mathbb{D}([0,1]^d)$ and Davis and Resnick [7] for extremes of linear processes with heavy-tailed noise.

1. INTRODUCTION

Building on the recent theory developed by de Haan and Lin [11] and Hult and Lindskog [13] for random functions with values in the space of càdlàg functions, we study the asymptotic theory for point processes and extremes of filtered processes of the form

$$(1.1) \qquad X_t(\mathbf{s}) = \sum_{i=0}^{\infty} \psi_i(\mathbf{s}) \, Z_{t-i}(\mathbf{s}), \quad \mathbf{s} \in [0,1]^d.$$

Here $(Z_t)_{t \in \mathbb{Z}}$ is an iid sequence of random fields on $[0,1]^d$ with values in the Skorokhod space $\mathbb{D} = \mathbb{D}([0,1]^d)$ of càdlàg functions equipped with the J_1-topology; see Bickel and Wichura [2] for definitions and properties related to this topology. The ψ_i's are deterministic real-valued fields on $\mathbb{D}$.

The indices $\mathbf{s}$ and t refer to a measurement taken at location $\mathbf{s}$ at time t. For example, $X_t(\mathbf{s})$, $t = 1, 2, \ldots$, could represent the time series of annual maxima of ozone levels at location $\mathbf{s}$. One of the problems of interest is determining the probability that the maximum ozone level over the entire region $[0,1]^2$ does not exceed a given standard level $f \in \mathbb{D}([0,1]^2)$ in n years. Another example, mentioned in de Haan and Lin [11], concerns the probability that the water level $X_t(s)$ on day t

at location $s \in [0,1]$ along the Dutch coast will not breach the dykes. Here $f(s)$ is a function that represents the height of the dykes at location s. Then the probability of interest is

$$P\left(\max_{t=1,\ldots,n} X_t(s) \leq f(s) \text{ for all } s \in [0,1]\right).$$

A third example is the windspeed $X_t(\mathbf{s})$ along a building at time t and location $\mathbf{s}$ on the face of a building.

Figure 1.1. *An autoregressive random field $X_t = 0.9 X_{t-1} + Z_t$, $t = 0, 1, 2, 3$, (top left to bottom right) with a regularly varying Lévy random field with index $\alpha = 1$, see Section 4.2.*

Serial dependence enters in the model (X_t) through the linear filter with weights ψ_j, $j = 1, 2, \ldots$. For example, at each fixed location $\mathbf{s}$ we have a linear time series model $(X_t(\mathbf{s}))_{t \in \mathbb{Z}}$. If Z_t is a second order stationary random field with mean 0 and covariance function $\gamma_Z(\mathbf{s})$, then the serial
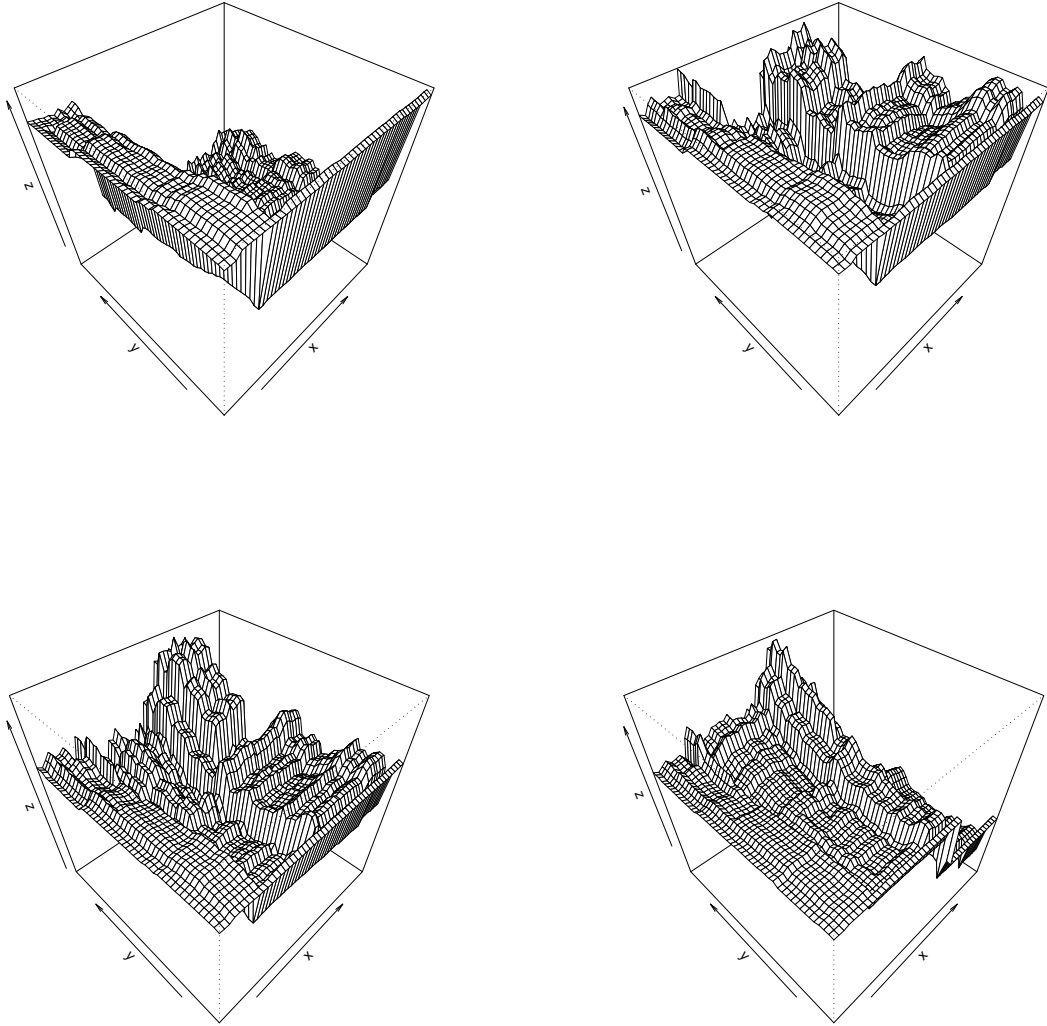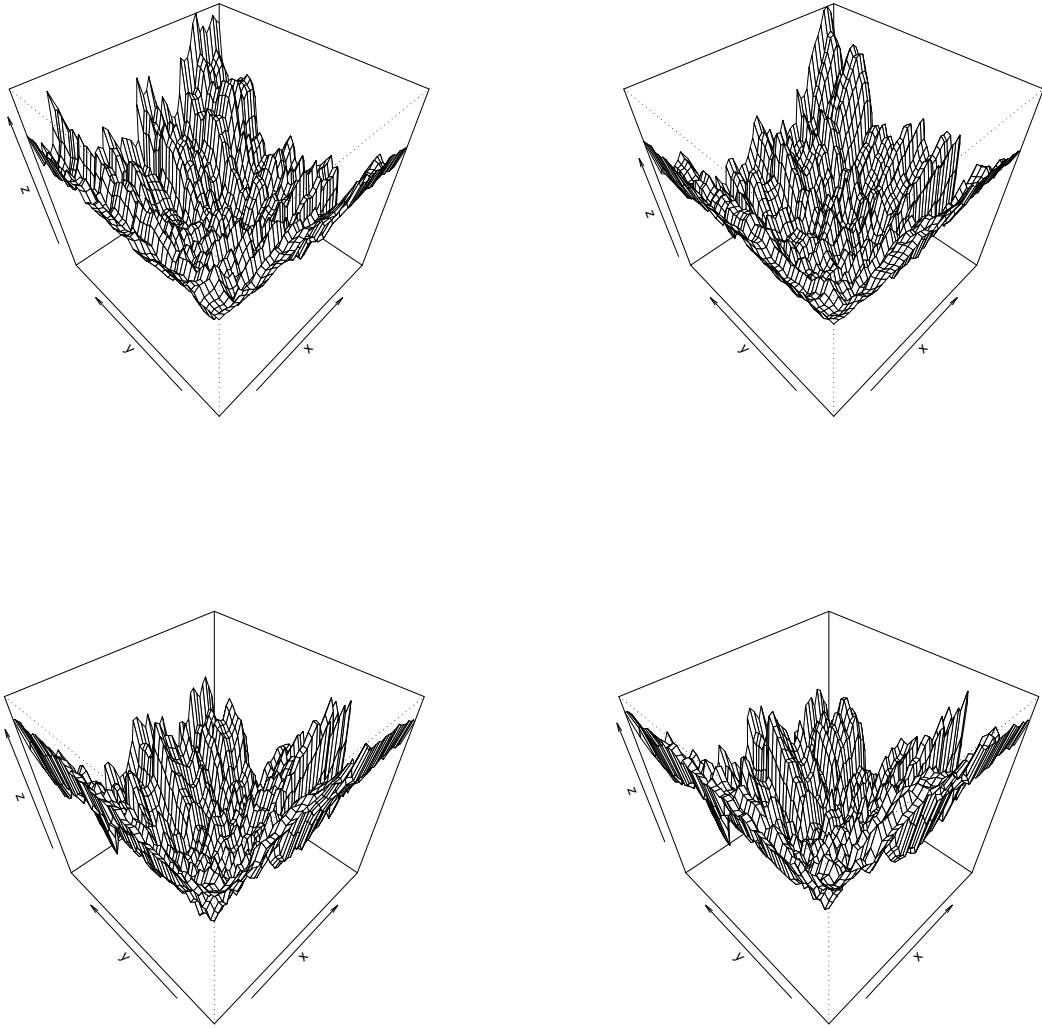
Figure 1.2. *An autoregressive random field $X_t = -0.8X_{t-1} + Z_t$, $t = 0, 1, 2, 3$, (top left to bottom right) with a regularly varying Lévy random field with index $\alpha = 4$, see Section 4.2.*

autocorrelation of X_t at location $\mathbf{s}$ is given by

$$\mathrm{Cor}(X_t(\mathbf{s}), X_{t+h}(\mathbf{s})) = \sum_{i=0}^{\infty} \psi_i(\mathbf{s})\psi_{i+h}(\mathbf{s}) / \sum_{i=0}^{\infty} \psi_i^2(\mathbf{s}),$$

provided $\sum_j \psi_j^2(\mathbf{s}) < \infty$. The spatial dependence among these linear time series is governed by the spatial dependence in the noise (Z_t). In particular, the spatial covariance function at fixed time t is given by

$$\mathrm{Cov}(X_t(\mathbf{s}_1), X_t(\mathbf{s}_2)) = \left(\sum_{i=0}^{\infty} \psi_i(\mathbf{s}_1)\psi_i(\mathbf{s}_2) \right) \gamma_Z(\mathbf{s}_2 - \mathbf{s}_1).$$

If the linear filter weights $\psi_j(\mathbf{s})$ are space-invariant, i.e., $\psi_j(\mathbf{s}) = \psi_j$ for all j and $\mathbf{s}$, then $(X_t(\mathbf{s})$ is stationary in both space and time with a multiplicative covariance function given by

$$\mathrm{Cov}(X_t(\mathbf{s}_1), X_{t+h}(\mathbf{s}_2)) = \left(\sum_{i=0}^{\infty} \psi_i \psi_{i+h}\right) \gamma_Z(\mathbf{s}_2 - \mathbf{s}_1).$$

Realizations from two autoregressive (AR) spatial processes are displayed in Figures 1.1 and 1.2. The AR(1) process is given by $X_t = \phi X_{t-1} + Z_t$ which corresponds to the linear process in (1.1) with coefficients $\psi_j(\mathbf{s}) = \phi^j$. The realizations in the figures correspond to $t = 0, 1, 2, 3$ with $\phi = .9$ (Figure 1.1), $\phi = -.8$ (Figure 1.2) and noise (Z_t) which is a regularly varying Lévy random field with $\alpha = 4$ (see Section 4.2). The process defined by (1.1) allows for modeling of both the dependence in time and space in a flexible way. While one can introduce serial dependence of random fields in more complicated ways, we will restrict attention to the linear case in this paper.

Many real-life time series often exhibit clusters of outlying observations that cannot be adequately modeled by a Gaussian distribution. Heavy-tailed distributions such as the Pareto distribution have proved useful in modeling a wide range of bursty phenomena that occur in finance, insurance, telecommunications, meteorology, hydrology; see Embrechts et al. [8] and the collection of papers [9] for specific examples and references. The theory of regular variation provides a convenient and unified background for studying multivariate extremes when heavy tails are present; see Resnick [22] for the basic theory and Basrak et al. [1], de Haan and Lin [11], Hult and Lindskog [13] for some recent developments. The novelty of the papers by de Haan and Lin [11] and Hult and Lindskog [13] is the precise formulation of regular variation for random functions with values in $\mathbb{C}[0,1]$ and $\mathbb{D}[0,1]$. This serves as a starting point for what we consider in this paper.

This paper is organized as follows. In Section 2, we introduce the notion of a regularly varying random field with values in $\mathbb{D}$ and we quote some preliminary results that will be frequently used in the sequel. In Section 3, we apply the notion of regular variation on $\mathbb{D}$ to max-stable fields. The section culminates with a representation of a max-stable random field in terms of a homogeneous Poisson process and iid random fields. This representation was proposed by Schlather [25] as one possibility for simulating max-stable random fields. De Haan and Pereira [12] formulate one and two dimensional families of parametric models for spatial extremes that are based on a similar representation for stationary random fields. These parameters can be used to describe a form of dependence between the random field at any two locations. In Section 4, we continue with some examples of regularly varying random fields. These include regularly varying Lévy and sαs random fields. In Section 5 we deal with the regular variation on $\mathbb{D}$ of the linear process X_t and study some of its consequences. In Section 5.2 we establish convergence for the sequence of point processes based on the points X_t, properly normalized, towards a compound Poisson process. This may be viewed as an extension of the seminal result by Davis and Resnick [7] for linear processes. Applications of these results to problems in extreme value theory, including the calculation of the probability of exceedances of high thresholds by the X_t's and the extremal index of the sequence $(|X_t|_\infty)$, are given in Section 5.3.

2. Preliminaries on regular variation on $\mathbb{D}$

2.1. **Definition and properties of regularly varying random fields.** In this section we introduce the essential ingredients about regular variation on $\mathbb{D}$ that will be required for the results in Section 5. We closely follow the discussion in de Haan and Lin [11], Hult and Lindskog [13] and Hult et al. [15]. Denote by $\mathbb{D} = \mathbb{D}([0,1]^d, \mathbb{R})$ the space of càdlàg functions $x : [0,1]^d \to \mathbb{R}$ equipped with a metric d_0 which is equivalent to the J_1-metric and such that it makes $\mathbb{D}$ a complete separable linear metric space; see Bickel and Wichura [2] and Billingsley [3]. We denote by $\mathbb{S}_\mathbb{D}$ the "unit sphere" $\{x \in \mathbb{D} : |x|_\infty = 1\}$ with $|x|_\infty = \sup_{\mathbf{s} \in [0,1]^d} |x(\mathbf{s})|$, equipped with the relativized topology of $\mathbb{D}$. Define $\overline{\mathbb{D}}_0 = (0, \infty] \times \mathbb{S}_\mathbb{D}$, where $(0, \infty]$ is equipped with the metric $\rho(x, y) = |1/x - 1/y|$ making it

complete and separable. For any element $x \in \overline{\mathbb{D}}_0$, we write $x = (|x|_\infty, \widetilde{x})$, where $\widetilde{x} = x/|x|_\infty$. Then $\overline{\mathbb{D}}_0$, equipped with the metric $\max\{\rho(x,y), d_0(\widetilde{x}, \widetilde{y})\}$, is a complete separable metric space. The topological spaces $\mathbb{D}\backslash\{0\}$, equipped with the relativized topology of $\mathbb{D}$, and $(0,\infty) \times \mathbb{S}_\mathbb{D}$, equipped with the relativized topology of $\overline{\mathbb{D}}_0$, are homeomorphic; the function T given by $T(x) = (|x|_\infty, \widetilde{x})$ is a homeomorphism. Hence

$$\mathcal{B}(\overline{\mathbb{D}}_0) \cap [(0,\infty) \times \mathbb{S}_\mathbb{D}] = \mathcal{B}(T(\mathbb{D}\backslash\{0\})),$$

i.e., the sets of the Borel σ-field $\mathcal{B}(\overline{\mathbb{D}}_0)$ that are of interest to us can be identified with the usual Borel sets on $\mathbb{D}$ (viewed in spherical coordinates) that do not contain the zero function. For notational convenience we will throughout the paper identify $\mathbb{D}$ with the product space $[0,\infty) \times \mathbb{S}_\mathbb{D}$ so that expressions like $\overline{\mathbb{D}}_0\backslash\mathbb{D}$ $(= \{\infty\} \times \mathbb{S}_\mathbb{D})$ make sense. We denote by $\mathcal{B}(\overline{\mathbb{D}}_0) \cap \mathbb{D}$ the Borel sets $B \in \mathcal{B}(\overline{\mathbb{D}}_0)$ such that $B \cap [\{\infty\} \times \mathbb{S}_\mathbb{D}] = \emptyset$. Notice that a bounded set B of $\overline{\mathbb{D}}_0$ is a set bounded away from zero, i.e., there exists $\delta > 0$ such that $|x|_\infty > \delta$ for all $x \in B$.

In addition, we will make use of the space $\mathbb{C} = \mathbb{C}([0,1]^d)$ of continuous functions on $[0,1]^d$ equipped with the uniform topology. Completely analogously to $\overline{\mathbb{D}}_0$, $\mathbb{S}_\mathbb{D}$, etc., we will use the notation $\overline{\mathbb{C}}_0$, $\mathbb{S}_\mathbb{C}$, etc.

Regular variation on $\mathbb{R}^d$ (for random vectors) is typically formulated in terms of vague convergence on $\mathcal{B}(\overline{\mathbb{R}}_0^d)$, where $\overline{\mathbb{R}}_0^d = \overline{\mathbb{R}}^d\backslash\{\mathbf{0}\})$ and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$; see Resnick [21, 22]. The topology on $\overline{\mathbb{R}}_0^d$ is chosen so that $\mathcal{B}(\overline{\mathbb{R}}_0^d)$ and $\mathcal{B}(\mathbb{R}^d)$ coincide on $\mathbb{R}^d\backslash\{\mathbf{0}\}$. Moreover, $B \subset \overline{\mathbb{R}}_0^d$ is relatively compact (or bounded) if and only if $B \cap \mathbb{R}^d$ is bounded away from $\mathbf{0}$ (i.e., $\mathbf{0} \notin \overline{B \cap \mathbb{R}^d}$) in $\mathbb{R}^d$.

The vector $\mathbf{X}$ with values in $\mathbb{R}^d$ is *regularly varying* with index $\alpha > 0$ and *spectral measure* σ on the Borel σ-field of the unit sphere $\mathbb{S}^{d-1} = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| = 1\}$ if there exists a sequence of constants $a_n \to \infty$ such that

$$n\, P(|\mathbf{X}| > t\, a_n, \widetilde{\mathbf{X}} \in \cdot) \overset{w}{\to} t^{-\alpha}\, \sigma(\cdot)\,, \quad t > 0\,,$$

where $\overset{w}{\to}$ denotes weak convergence and, as before, $\widetilde{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$ for $\mathbf{x} \neq \mathbf{0}$. It is always possible to choose (a_n) such that $P(|\mathbf{X}| > a_n) \sim n^{-1}$, and then σ is a probability measure. Equivalently, $\mathbf{X}$ is regularly varying if there exists a sequence $a_n \to \infty$ (which can be chosen as above) and a non-null Radon measure μ on $\mathcal{B}(\overline{\mathbb{R}}_0^d)$ such that $\mu(\overline{\mathbb{R}}^d\backslash\mathbb{R}^d) = 0$ and

$$n\, P(a_n^{-1}\mathbf{X} \in \cdot) \overset{v}{\to} \mu(\cdot)\,,$$

where $\overset{v}{\to}$ denotes vague convergence on the Borel σ-field $\mathcal{B}(\overline{\mathbb{R}}_0^d)$.

Regular variation on $\mathbb{D}$ is naturally expressed in terms of $\widehat{w}$-convergence of boundedly finite measures on $\overline{\mathbb{D}}_0$; for details on $\widehat{w}$-convergence and its relationship with vague and weak convergence we refer to Appendix A2.6 in Daley and Vere-Jones [6], cf. also Kallenberg [16] and Resnick [21, 22]. A boundedly finite measure assigns finite mass to bounded sets. A sequence of boundedly finite measures (m_n) on a complete separable metric space $\mathbb{E}$ converges to the measure m in the $\widehat{w}$-topology, $m_n \overset{\widehat{w}}{\to} m$, if $m_n(B) \to m(B)$ for every bounded Borel set B with $m(\partial B) = 0$. Equivalently, $m_n \overset{\widehat{w}}{\to} m$ refers to

$$(2.1) \qquad\qquad m_n(f) = \int_\mathbb{E} f\, dm_n \to \int_\mathbb{E} f\, dm = m(f)$$

for all bounded continuous functions f on $\mathbb{E}$ which vanish outside a bounded set. If the state space $\mathbb{E}$ is locally compact ($\overline{\mathbb{R}}_0^d$ is locally compact while $\overline{\mathbb{D}}_0$ is not), then a boundedly finite measure is called a Radon measure, and $\widehat{w}$-convergence coincides with vague convergence and we write $m_n \overset{v}{\to} m$. Finally we notice that if $m_n \overset{\widehat{w}}{\to} m$ and $m_n(\mathbb{E}) \to m(\mathbb{E}) < \infty$, then $m_n \overset{w}{\to} m$.

We say that the random field X with values in $\mathbb{D}$ (and its distribution) are *regularly varying* with index $\alpha > 0$ and *spectral measure* σ on $\mathbb{S}_\mathbb{D}$, if there exists a sequence of constants $a_n \to \infty$ such

that

$$(2.2) \qquad n\,P(|X|_\infty > t\,a_n\,, \widetilde{X} \in \cdot\,) \overset{w}{\to} t^{-\alpha}\,\sigma(\cdot)\,, \quad t > 0\,,$$

where $\overset{w}{\to}$ denotes weak convergence on the Borel σ-field $\mathcal{B}(\mathbb{S}_\mathbb{D})$. One can always choose (a_n) such that $P(|X|_\infty > a_n) \sim n^{-1}$, and then σ is a probability measure on $\mathbb{S}_\mathbb{D}$. The convergence in (2.2) is equivalent to

$$(2.3) \qquad n\,P(a_n^{-1}\,X \in \cdot\,) \overset{\widehat{w}}{\to} m(\cdot)\,,$$

where $\overset{\widehat{w}}{\to}$ denotes $\widehat{w}$-convergence on the Borel σ-field $\mathcal{B}(\overline{\mathbb{D}}_0)$ and m is a finitely bounded measure with the property that $m(\overline{\mathbb{D}}_0 \backslash \mathbb{D}) = 0$; see Hult and Lindskog [13] for a proof of the equivalence between (2.3) and (2.2).

We will often make use of the following useful result by Hult and Lindskog [13] proved for $d = 1$. The proof for $d > 1$ is analogous and therefore omitted. The result characterizes a regularly varying random field in terms of the finite-dimensional distributions and the modulus of continuity. Write, for an $x \in \mathbb{D}$, $\delta > 0$ and a set $A \subset [0,1]^d$,

$$w''(x, \delta) = \sup_{\mathbf{s}_1 \le \mathbf{s} \le \mathbf{s}_2\,, |\mathbf{s}_2 - \mathbf{s}_1| \le \delta} \min(|x(\mathbf{s}) - x(\mathbf{s}_1)|, |x(\mathbf{s}) - x(\mathbf{s}_2)|)\,,$$

$$w(x, A) = \sup_{\mathbf{s}_1, \mathbf{s}_2 \in A} |x(\mathbf{s}_1) - x(\mathbf{s}_2)|\,.$$

Lemma 2.1. *The random field X with values in $\mathbb{D}$ is regularly varying if and only if there exist a sequence (a_n) satisfying $n\,P(|X|_\infty > a_n) \to 1$ and a collection of Radon measures $m_{\mathbf{s}_1,\ldots,\mathbf{s}_k}$, $\mathbf{s}_i \in [0,1]^d$, $i = 1,\ldots,k$, $k \ge 1$, not all of them being the null measure, with $m_{\mathbf{s}_1,\ldots,\mathbf{s}_k}(\overline{\mathbb{R}}^k \backslash \mathbb{R}^k) = 0$, such that the following conditions hold:*

(1) *The following relation holds:*

$$(2.4) \qquad n\,P(a_n^{-1}(X(\mathbf{s}_1),\ldots,X(\mathbf{s}_k)) \in \cdot\,) \overset{v}{\to} m_{\mathbf{s}_1,\ldots,\mathbf{s}_k}(\cdot)\,,$$

for all $\mathbf{s}_i \in [0,1]^d$, $i = 1,\ldots,k$, $k \ge 1$, where $\overset{v}{\to}$ refers to vague convergence on the Borel σ-field $\mathcal{B}(\overline{\mathbb{R}}_0^k)$.

(2) *For any $\epsilon, \eta > 0$ there exist $\delta \in (0, 0.5)$ and n_0 such that for $n \ge n_0$,*

$$(2.5) \qquad n\,P(w''(X, \delta) > a_n \epsilon) \le \eta\,,$$

$$(2.6) \qquad n\,P(w(X, [0,1]^d \backslash [\delta, 1-\delta]^d) > a_n \epsilon) \le \eta\,.$$

The measures $m_{\mathbf{s}_1,\ldots,\mathbf{s}_k}$, $\mathbf{s}_i \in [0,1]^d$, $i = 1,\ldots,k$, $k \ge 1$, determine the limiting measure m in the definition of regular variation of X.

2.2. **Regular variation, point process convergence and convergence of maxima.** Next we connect regular variation on $\mathbb{D}$ with the weak convergence of the point processes

$$N_n = \sum_{i=1}^{n} \varepsilon_{a_n^{-1} X_i}\,, \quad n \ge 1\,,$$

and the maxima $a_n^{-1} \max_{i=1,\ldots,n} X_i$, where the X_i's are iid copies of a regularly varying random field X with values in $\mathbb{D}$ and ε_x is Dirac measure at x.

Lemma 2.2. *Let $X, X_1, X_2, \ldots$ be an iid sequence of $\mathbb{D}$-valued random fields. In items (2) and (3) we assume in addition that X has non-negative sample paths.*

(1) *The field X is regularly varying with index $\alpha > 0$ and limiting measure m as in (2.3) if and only if $N_n \overset{d}{\to} N$ in $M_p(\overline{\mathbb{D}}_0)$, the space of point measures with state space $\overline{\mathbb{D}}_0$ equipped with the $\widehat{w}$-topology, where N is a Poisson random measure with mean measure m (PRM(m)).*

(2) *If the relation*

$$(2.7) \qquad a_n^{-1} \max_{t=1,\ldots,n} X_t \overset{d}{\to} Z$$

holds in $\mathbb{D}$ for some non-degenerate random field Z then X is regularly varying on $\mathbb{D}$ for some positive α.

(3) *Conversely, assume that X is regularly varying with index $\alpha > 0$. Then (2.7) holds for some Z in the sense of the finite-dimensional distributions. Moreover, if a $\mathbb{C}$-valued version of Z exists then the convergence of the finite-dimensional distributions in (2.7) can be extended to convergence in $\mathbb{D}$.*

Proof. (1) This result follows by an adaptation of Proposition 3.21 in Resnick [22]. While this proposition applies to weak convergence of point processes with a locally compact state space, our state-space $\overline{\mathbb{D}}_0$ is not locally compact. However, the proof (which only involves Laplace functionals of the underlying point processes) remains valid if one changes from vague convergence used in [22] to $\widehat{w}$-convergence as described above; see Daley and Vere-Jones [6], Chapter 9 and Appendix A2.6. The proof of (1) in the case $d = 1$ and for non-negative càdlàg X on $[0,1]$ can also be found in Theorem 2.4 of de Haan and Lin [11]. (The proof is given under the assumption that $\alpha = 1$ which does not restrict generality.)

(2) and (3) The proof follows by an adaptation of the proof in Theorem 2.4 in [11] who consider the case of non-negative X on $[0,1]$. The extension to $d > 1$ does not provide additional difficulties. $\qquad\square$

A consequence of Lemma 2.2 (and indeed of finite-dimensional extreme value theory, see Resnick [22], Section 5.4) is that regular variation of X on $\mathbb{D}$ with index α implies that for any choice of $\mathbf{s}_i \in [0,1]^d$, $i = 1,\ldots,k$, $k \geq 1$,

$$(2.8) \qquad a_n^{-1} \left(\max_{t=1,\ldots,n} X_t(\mathbf{s}_i) \right)_{i=1,\ldots,k} \overset{d}{\to} (Z(\mathbf{s}_i))_{i=1,\ldots,k} \ .$$

The distribution of $(Z(\mathbf{s}_i))_{i=1,\ldots,k}$ is a multivariate extreme value distribution with Fréchet marginals with index α and exponent measure $m_{\mathbf{s}_1,\ldots,\mathbf{s}_k}$ which is described in part (1) of Lemma 2.1. By part (1) of the lemma it also follows that the point process convergence $N_n \overset{d}{\to} N$ in $M_p(\overline{\mathbb{D}}_0)$ for some PRM(m), N, implies (2.8). However, tightness of the sequence $(a_n^{-1} \max_{t=1,\ldots,n} X_t)$ in $\mathbb{D}$ and the tightness condition for regular variation given by (2.5) and (2.6) are in general not equivalent in $\mathbb{D}$. An assumption such as continuity of the limit Z in (2.7) is in general needed. A counterexample showing that regular variation of X on $\mathbb{D}$ does not imply (2.7) was kindly communicated to us by Yohann Gentric.

2.3. **Regular variation of products of random variables.** We will often make use of a simple result on the products of independent random variables, which we will refer to as Breiman's result. See Breiman [4], cf. Basrak et al. [1] for a proof and some multivariate extensions.

Lemma 2.3. *Assume ξ, η are non-negative random variables, η is regularly varying with index $\alpha > 0$ and one of the following conditions holds:*

(1) $0 < E\xi^{\alpha+\delta} < \infty$ *for some $\delta > 0$.*

(2) $0 < E\xi^{\alpha} < \infty$ *and $P(\eta > x) \sim c\,x^{-\alpha}$ as $x \to \infty$ for some $c > 0$.*

Then $P(\xi\eta > x) \sim E\xi^{\alpha} P(\eta > x)$ as $x \to \infty$.

The proof of the result under condition (2) is difficult to find in the literature, but it follows easily by intersecting the event $\{\xi\eta > x\}$ with the events $\{\xi > \epsilon x\}$ and $\{\xi \leq \epsilon x\}$ for $\epsilon > 0$ sufficiently small and by observing that $P(\xi > x) = o(x^{-\alpha})$.

3. Applications to max-stable random fields

3.1. **Preliminaries.** The class of max-stable random fields provides a good collection of examples of regularly varying random fields. It is common (see Resnick [22], de Haan and Lin [11]) to assume that all one-dimensional marginals of a max-stable process are Fréchet with index 1. Of course, the marginals of the process X can be transformed to obtain marginals with any other extreme value distribution. For this reason, we confine our discussion to the case $\alpha = 1$.

Following de Haan [10], a random field X on $[0,1]^d$ is called *max-stable* with unit Fréchet marginals (i.e., $P(X(\mathbf{s}) \leq x) = \mathrm{e}^{-x^{-1}}$, $x > 0$, for every $\mathbf{s} \in [0,1]^d$), if for iid copies X_i of X and every $k \geq 1$,

$$(3.9) \qquad k\,X \stackrel{d}{=} \max_{i=1,\ldots,k} X_i\,,$$

where $\stackrel{d}{=}$ denotes equality of the finite-dimensional distributions. Condition (3.9) is equivalent to the existence of a field Y with regularly varying finite-dimensional distributions with index $\alpha = 1$ such that for iid copies Y_i of Y and a suitable sequence (a_n) of positive constants,

$$(3.10) \qquad a_n^{-1} \max_{i=1,\ldots,n} Y_i \stackrel{d}{\to} X\,,$$

where $\stackrel{d}{\to}$ represents convergence of the finite-dimensional distributions. The finite-dimensional distributions of X have the following canonical form,

$$(3.11) \qquad P(X(\mathbf{s}_1) \leq y_1, \ldots, X(\mathbf{s}_k) \leq y_k) = \exp\left\{ -\int_0^1 \left(\max_{i=1,\ldots,k} \frac{f_i(x)}{y_i} \right) dx \right\}\,,$$

for some suitable choice of non-negative L^1 functions f_i which have integral 1, i.e., $\int_0^1 f_i(y)\,dy = 1$, see Resnick [22], Proposition 5.11. It follows that the finite-dimensional distributions of X are regularly varying with index $\alpha = 1$. Moreover, from (3.10), we have the relation

$$n\,P(a_n^{-1}(Y(\mathbf{s}_i))_{i=1,\ldots,k} \in ([0,y_1] \times \cdots \times [0,y_k])^c) \to \int_0^1 \left(\max_{i=1,\ldots,k} \frac{f_i(x)}{y_i} \right) dx\,,$$

which identifies the measure $m_{\mathbf{s}_1,\ldots,\mathbf{s}_k}$ in (2.4).

According to the defining property (3.9), max-stability is only a property of the finite-dimensional distributions of the field X. In what follows, this notion is strengthened to requiring that X satisfies (3.9) and assumes values in $\mathbb{D}$. Interestingly, with this additional assumption that X lives in $\mathbb{D}$, regular variation of X on $\mathbb{D}$ is automatic. This is the content of the following lemma.

Lemma 3.1. *Assume that X is a max-stable process with values in $\mathbb{D}$. Then X is regularly varying on $\mathbb{D}$ with index 1.*

Proof. The proof follows from part (2) of Lemma 2.2 by taking $Y_i = X_i$ for iid copies of the $\mathbb{D}$-valued max-stable field X. Equality of the finite-dimensional distributions of X and $n^{-1} \max_{t=1,\ldots,n} X_i$ implies equality in distribution in $\mathbb{D}$ which in turn yields weak convergence in $\mathbb{D}$. Hence X is regularly varying on $\mathbb{D}$ with index 1. $\qquad\square$

Remark 3.2. For a max-stable field X with unit Fréchet marginals there exists a unique measure m such that $n\,P(n^{-1}X \in \cdot) \stackrel{\widehat{w}}{\to} m$ and $m(tB) = t^{-1}m(B)$ for any bounded set B and $t > 0$. Moreover, for any $\mathbf{s}_i \in [0,1]^d$ and $y_i \geq 0$, $i = 1, \ldots, k$, $k \geq 1$,

$$(3.12) \qquad P(X(\mathbf{s}_1) \leq y_1, \ldots, X(\mathbf{s}_k) \leq y_k) = \mathrm{e}^{-m(A^c)}\,,$$

where

$$(3.13) \qquad A = \{ z \in \overline{\mathbb{D}}_0 : z(\mathbf{s}_i) \leq y_i\,, i = 1, \ldots, k \}\,.$$

The measure m is often referred to as the *exponent measure* of X and uniquely determines the finite-dimensional distributions of X, i.e., $\mathrm{e}^{-m(A^c)}$ coincides with the right-hand side of (3.11).

3.2. **A representation of a max-stable random field.** In this section, we construct a max-stable $\mathbb{D}$-valued random field X with unit Fréchet marginals and we show that it is regularly varying on $\mathbb{D}$ with index 1.

To start with, consider a unit rate Poisson process on $(0, \infty)$. An increasing enumeration of the points of the process is denoted by $(\Gamma_i)_{i \geq 1}$. Consider an iid sequence $Y, Y_1, Y_2, \ldots$, of random fields on $[0,1]^d$ with values in $\mathbb{D}$ independent of (Γ_i). Moreover, assume that $0 < EY^+(\mathbf{s}) < \infty$ for all $\mathbf{s} \in [0,1]^d$, where a^+ is the positive part of the real number a. Define the càdlàg random field

$$(3.14) \qquad X(\mathbf{s}) = \sup_{j \geq 1} \Gamma_j^{-1} Y_j(\mathbf{s}) = \sup_{j \geq 1} \Gamma_j^{-1} Y_j^+(\mathbf{s}), \quad \mathbf{s} \in [0,1]^d.$$

Notice that the second equality is due to the fact that $0 < EY^+(\mathbf{s}) < \infty$ and therefore $Y_j(\mathbf{s}) > 0$ infinitely often with probability 1 for every $\mathbf{s}$. In view of (3.14) we assume that $Y(\mathbf{s})$ is positive a.s. and $EY(\mathbf{s}) < \infty$. For $\mathbf{s} \in [0,1]^d$, the random variable $X(\mathbf{s})$ is well defined by virtue of the strong law of large numbers $\Gamma_j/j \overset{\text{a.s.}}{\to} 1$ and since $Y_j(\mathbf{s})/j \overset{\text{a.s.}}{\to} 0$ by the Borel-Cantelli lemma.

Representation (3.14) was introduced by Schlather [25] as a model for max-stable random fields. We show that (3.14) yields a representation of any max-stable field in $\mathbb{D}$.

Theorem 3.3. *The $\mathbb{D}$-valued random field X is max-stable if and only if X has representation (3.14) for some iid sequence (Y_i) of $\mathbb{D}$-valued random fields such that $Y > 0$ a.s. and $E|Y|_\infty < \infty$. In either case, X is regularly varying on $\mathbb{D}$ with index 1 and spectral measure σ given by*

$$(3.15) \qquad \sigma(S) = E\left(|Y|_\infty I_S(\widetilde{Y})\right) \Big/ E|Y|_\infty, \quad S \in \mathcal{B}(\mathbb{S}_\mathbb{D}).$$

Proof. We first show that X given by (3.14) with $E|Y|_\infty < \infty$ is in $\mathbb{D}$. Since $E|Y|_\infty < \infty$, it follows from the strong law of large numbers and the Borel-Cantelli lemma that $|Y_j|_\infty/\Gamma_j \overset{\text{a.s.}}{\to} 0$. Therefore $\sup_{1 \leq j \leq n} \Gamma_j^{-1} Y_j$ converges a.s. as $n \to \infty$ in the uniform topology to X given in (3.14) which is finite a.s.

Next we show that the random field (3.14) is max-stable in the sense of (3.9). Consider the point process $N = \sum_{j=1}^\infty \varepsilon_{\Gamma_j^{-1} Y_j}$ with state space $\overline{\mathbb{D}}_0$. Using standard arguments, the log-Laplace functional of N is given by

$$\log E \exp\{-N(f)\} = -E\left(\int_0^\infty (1 - \mathrm{e}^{-f(sY)}) s^{-2} \, ds\right),$$

where f is a bounded continuous function on $\overline{\mathbb{D}}_0$ with bounded support. Define a measure m on the Borel σ-field of $\overline{\mathbb{D}}_0$ by

$$m(\{x \in \overline{\mathbb{D}}_0 : |x|_\infty > t, \widetilde{x} \in S\}) = E|Y|_\infty \, t^{-1} \sigma(S), \quad t > 0, \quad S \in \mathcal{B}(\mathbb{S}_\mathbb{D}),$$

where the probability measure σ is given by (3.15). Recall that for $z \in \overline{\mathbb{D}}_0$ and $s > 0$ we have $f(sz) = f((s|z|_\infty, \widetilde{z}))$. It follows that the log-Laplace functional is equal to

$$-E\left(\int_0^\infty (1 - \mathrm{e}^{-f(s|Y|_\infty, \widetilde{Y})}) s^{-2} \, ds\right) = -E|Y|_\infty \int_{\mathbb{S}_\mathbb{D}} \int_0^\infty (1 - \mathrm{e}^{-f(t,\theta)}) t^{-2} \, dt \, \sigma(d\theta).$$

This is the log-Laplace functional of $\mathrm{PRM}(m)$ on $\overline{\mathbb{D}}_0$, hence N is $\mathrm{PRM}(m)$. For any $\mathbf{s}_i \in [0,1]^d$, $i = 1, \ldots, k$, and $\mathbf{y} \in \mathbb{R}_+^k$ consider the finite-dimensional set $A \subset \overline{\mathbb{D}}_0$ given by (3.13). Then we have

$$P(N(A^c) = 0) = P(X(\mathbf{s}_1) \leq y_1, \ldots, X(\mathbf{s}_k) \leq y_k) = \mathrm{e}^{-m(A^c)}.$$

Since by definition of m, $m(tB) = t^{-1} m(B)$ for any bounded set B and $t > 0$, we conclude from Remark 3.2, in particular equation (3.12), that m is the exponent measure of a max-stable random field. This proves max-stability of X.

The regular variation with index 1 of the max-stable random field X given in (3.14) follows from Lemma 3.1. The spectral measure (3.15) was calculated in the course of the proof above.

Now we prove the converse. Assume that X is max-stable with unit Fréchet marginals. As such it has an exponent measure $\widetilde{m}$ with corresponding spectral (probability) measure $\widetilde{\sigma}$. Let (Y_j) be an iid sequence of positive $\mathbb{S}_{\mathbb{D}}$-valued random fields with distribution $\widetilde{\sigma}$. We will show that X has the same distribution as the $\mathbb{D}$-valued random field

$$X^* = c \sup_{j \geq 1} \Gamma_j^{-1} Y_j \,,$$

where

$$c = m(\{x \in \overline{\mathbb{D}}_0 : |x|_\infty > 1\}) \,.$$

By the direct part of the proof, X^* is well defined, max-stable, regularly varying with index 1 and has exponent measure given by

$$m(\{x \in \overline{\mathbb{D}}_0 : |x|_\infty > t \,, \widetilde{x} \in S\}) = c\, t^{-1}\, \widetilde{\sigma}(S) \,, \quad t > 0 \,, \quad S \in \mathcal{B}(\mathbb{S}_{\mathbb{D}}) \,.$$

This means that the exponent measures m and $\widetilde{m}$ coincide. Hence the max-stable fields X and X^* have the same finite-dimensional distributions. This concludes the proof. $\square$

Remark 3.4. The condition $E|Y|_\infty < \infty$ can be verified in general circumstances. A prime example is a $\mathbb{C}$-valued centered Gaussian random field which we can interpret as a mean zero Gaussian random element with values in a separable Banach space. Then it is well known that the tail $P(|Y|_\infty > y)$ decays exponentially fast, in particular $E(|Y|_\infty^p) < \infty$ for all $p > 0$, see Landau and Shepp [17], Marcus and Shepp [20], cf. Ledoux and Talagrand [18]. Representation (3.14) is advantageous for simulating max-stable fields as advocated by Schlather [25]. In Figure 3.5 we show two realizations of max-stable fields on $[0,1]^2$ based on the representation (3.14), where Y is an isotropic Gaussian random field with exponential and Gaussian covariance functions, respectively. These fields were generated in the software package R, using the RandomFields package written by Schlather.
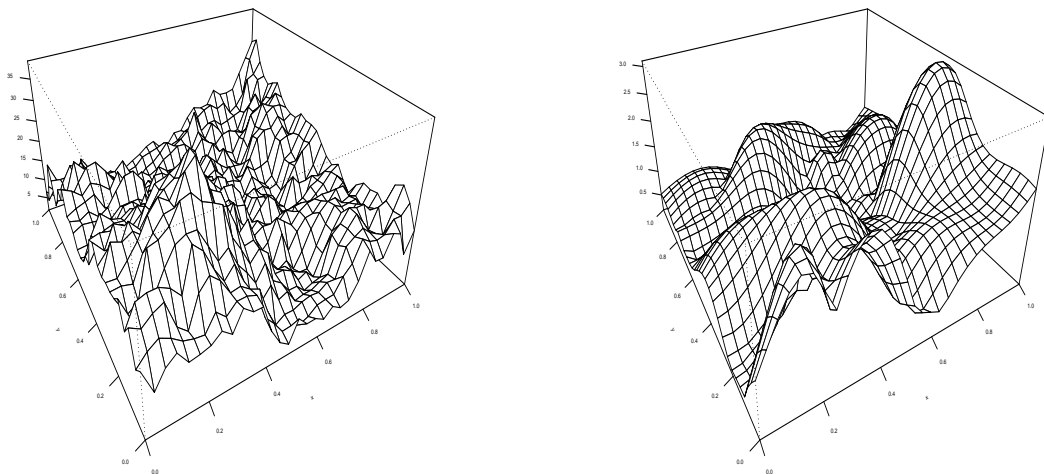

Figure 3.5. *Two realizations of a max-stable field using the representation* (3.14)*, where Y is an isotropic Gaussian random field on $[0,1]^2$ with exponential* (left) *and Gaussian* (right) *covariance functions.*

Remark 3.6. It follows by direct calculation, see Section 4.1 below, that $\Gamma_1^{-1}Y$ is regularly varying with index 1 and has the same limit measure m on $\overline{\mathbb{D}}_0$ as $X = \sup_{j\geq 1}\Gamma_j^{-1}Y_j$. This means that the extreme behavior of a max-stable random field X is determined only by the first term in the supremum. This can also be seen from the fact that for every $\epsilon > 0$,

$$(3.16) \qquad n\,P\Big(n^{-1}\Big|\sup_{j\geq 2}\Gamma_j^{-1}Y_j\Big|_\infty > \epsilon\Big) \to 0, \quad n \to \infty.$$

Indeed, we have

$$n\,P\Big(n^{-1}\Big|\sup_{j\geq 2}\Gamma_j^{-1}Y_j\Big|_\infty > \epsilon\Big) \leq n\sum_{j\geq 2}P\Big(\Gamma_j^{-1}\,|Y|_\infty > \epsilon\,n\Big)$$

$$= n\int_0^\infty\left(\sum_{j=1}^\infty P(\Gamma_j \leq (\epsilon n)^{-1}y) - P(\Gamma_1 \leq (\epsilon n)^{-1}y)\right)P(|Y|_\infty \in dy)$$

$$(3.17) \qquad = n\int_0^\infty\left(\frac{y}{\epsilon n} - (1 - \mathrm{e}^{-y/(\epsilon n)})\right)P(|Y|_\infty \in dy).$$

Observe that $f_n(y) = n[y/(\epsilon n) - (1-\mathrm{e}^{-y/(\epsilon n)})] \leq c\,y$ for some $c > 0$, all $y > 0$. Moreover, $f_n(y) \to 0$ as $n \to \infty$ for every $y > 0$. Since $E|Y|_\infty < \infty$ by assumption, a dominated convergence argument yields that for every $\epsilon > 0$ the right-hand side in (3.17) converges to zero. Combining the arguments above, we conclude that (3.16) holds.

4. Examples of regularly varying random fields

In this section we consider some more examples of regularly varying $\mathbb{D}$-valued random fields X. In Section 3 we have already studied the class of max-stable random fields which constitute an important family of $\mathbb{D}$-valued regularly varying random fields.

4.1. **A simple multiplicative field.** Let Y be a càdlàg random field and suppose that η is a non-negative regularly varying random variable with index $\alpha > 0$, independent of Y. Assume that η and $\xi = |Y|_\infty$ satisfy the conditions of Breiman's Lemma 2.3. For example, the assumptions on ξ are satisfied for Gaussian Y. Define the $\mathbb{D}$-valued random field

$$X(\mathbf{s}) = \eta\,Y(\mathbf{s}), \quad \mathbf{s} \in [0,1]^d.$$

An application of Lemma 2.3 yields that X is regularly varying on $\mathbb{D}$ with index α. Indeed, for any Borel set $S \subset \mathbb{S}_\mathbb{D}$ and $t > 0$,

$$n\,P\left(|\eta Y|_\infty > a_n\,t, \widetilde{Y} \in S\right) \to t^{-\alpha}E\left(|Y|_\infty^\alpha\,I_S(\widetilde{Y})\right)\Big/E|Y|_\infty^\alpha = t^{-\alpha}\sigma(S), \quad t > 0,$$

where (a_n) is chosen such that $P(|X|_\infty > a_n) \sim n^{-1}$. The right-hand side of this relation has the form given in (2.2). Using Breiman's result, one can easily calculate asymptotic expressions related to the finite-dimensional distributions of X. For example, for positive y_i, $i = 1, \ldots, k$,

$$n\,P(a_n^{-1}(|X(\mathbf{s}_1)|, \ldots, |X(\mathbf{s}_k)|) \in (y_1,\infty) \times \cdots \times (y_k,\infty)) = n\,P\left(a_n^{-1}\eta\min_{i=1,\ldots,k}(|Y(\mathbf{s}_i)|/y_i) > 1\right)$$

$$\to E\left(\min_{i=1,\ldots,k}(|Y(\mathbf{s}_i)|/y_i)^\alpha\right).$$

In addition, we may conclude that condition (2) of Lemma 2.1 is satisfied for X.

We also mention that, if Y has mean zero and finite second moment and η has finite second moment, then X and Y have the same correlation structure.

Despite its simplicity, the multiplicative model serves as an approximation to the large values of some important regularly varying random fields. Those include the max-stable fields (see Remark 3.6), but also the sαs random fields considered in Section 4.4.

4.2. **Regularly varying Lévy fields.** We consider a $\mathbb{D}$-valued random field X which has independent and stationary increments and for $\mathbf{s} \in [0,1]^d$ the log-characteristic function of $X(\mathbf{s})$ is given by

$$\log E\mathrm{e}^{\,i\,t\,X(\mathbf{s})} = -\|[\mathbf{0},\mathbf{s}]\| \int_{\mathbb{R}_0} \left(\mathrm{e}^{\,i\,t\,y} - 1 - i\,t\,y\,I_{[-1,1]}(y)\right)\,\nu(dy)\,,$$

where ν is a Lévy measure on $\overline{\mathbb{R}}_0$, satisfying $\int_{\overline{\mathbb{R}}_0}(1 \wedge y^2)\,\nu(dy) < \infty$ and $|A|$ is the Lebesgue measure of any set A. Following standard theory for Lévy processes (see Sato [24]), we call X a *Lévy random field*. We may and do assume that X has càdlàg sample paths. For fixed $\mathbf{s}$, it follows from Hult and Lindskog [13] that $X(\mathbf{s})$ is regularly varying with limit measure $\|[\mathbf{0},\mathbf{s}]\|\mu$ for some Radon measure μ on $\overline{\mathbb{R}}_0$ if and only if the Lévy measure ν is regularly varying in the sense that for some sequence of constants $a_n \to \infty$,

$$n\,\nu(a_n \cdot) \xrightarrow{v} \mu \quad \text{on } \overline{\mathbb{R}}_0\,.$$

Therefore we assume that $X(\mathbf{s})$ is regularly varying with index α for some $\mathbf{s} \in (0,1]^d$. Following the ideas in Hult and Lindskog [13] in the case $d = 1$, the càdlàg random field X is regularly varying on $\mathbb{D}$ and the limit measure m in (2.3) can be identified as $m = (\mathbb{LEB} \times \mu) \circ T^{-1}$, where $\mathbb{LEB}$ denotes Lebesgue measure and $T : [0,1]^d \times \overline{\mathbb{R}}_0 \to \overline{\mathbb{D}}_0$ is given by $T(\mathbf{t},x) = x\,I_{[\mathbf{t},\mathbf{1}]}(\mathbf{s})$, $\mathbf{s} \in [0,1]^d$. We conclude that the following property of m in spherical coordinates holds. Let θ have distribution on $\{1,-1\}$ given by

$$P(\theta = 1) = \mu((1,\infty))/\mu(\{x \in \overline{\mathbb{R}} : |x| > 1\} = 1 - P(\theta = -1)\,,$$

independent of $\mathbf{U}$ which has a uniform distribution on $(0,1)^d$. The spectral measure is then given by

$$\sigma(\cdot) = P\left(\theta(I_{[\mathbf{U},\mathbf{1}]}(\mathbf{s}))_{\mathbf{s}\in[0,1]^d} \in \cdot\right)\,.$$

Hence, for $y > 0$,

$$\frac{m(\{x \in \overline{\mathbb{D}}_0 : |x|_\infty > y\,, \widetilde{x} \in \cdot\,\})}{m(\{x \in \overline{\mathbb{D}}_0 : |x|_\infty > 1\})} = y^{-\alpha}\,\sigma(\cdot)\,.$$

4.3. **Regularly varying Ornstein-Uhlenbeck processes.** Consider an Ornstein-Uhlenbeck process X on $[0,1]$ driven by a regularly varying Lévy process L, i.e., a Lévy field with $d = 1$, see Section 4.2. It has the stochastic integral representation

$$X(s) = \int_0^s \mathrm{e}^{-\lambda\,(s-y)}\,L(dy)\,, \quad s \in [0,1]\,,$$

where $\lambda > 0$ is a constant. It follows from Hult and Lindskog [13], Example 24, that if L is regularly varying with index $\alpha > 0$, then X is regularly varying on $\mathbb{D}$ with the same index and its spectral measure σ on $\mathbb{S}_\mathbb{D}$ is given by

$$\sigma(\cdot) = P(\theta(\mathrm{e}^{-\lambda(s-U)}I_{[U,1]}(s))_{s\in[0,1]} \in \cdot)\,,$$

where θ and U are as defined in Section 4.2. This example can be extended to filter functions $f(s,y)$ more general than the exponential function $f(s,y) = \mathrm{e}^{-\lambda(s-y)}$ (see Hult and Lindskog [13]) as well as to certain classes of predictable integrand processes (see Hult and Lindskog [14]). In particular, for special choices of the function f and regularly varying Lévy processes L one gets regularly varying continuous-time ARMA (CARMA) processes; see for example Brockwell [5].

4.4. **Regularly varying sαs series.** In this section we consider the random field

$$(4.1) \qquad X = \sum_{i=1}^{\infty} r_i \, \Gamma_i^{-1/\alpha} \, Y_i \,,$$

where (Γ_i) is an increasing enumeration of the points of a unit rate Poisson process on $(0, \infty)$, independent of the sequences (r_i) and (Y_i), and $\alpha \in (0, 2)$. Here (Y_i) is an iid sequence of $\mathbb{D}$-valued random fields and (r_i) is an iid Rademacher sequence, i.e., $P(r_i = \pm 1) = 0.5$. If $E(|Y(\mathbf{s}_i)|^{\alpha}) < \infty$, $\mathbf{s}_i \in [0, 1]^d$, $i = 1, \dots, k$, it follows from the theory of α-stable processes that the $\mathbb{R}^k$-valued infinite series $(X(\mathbf{s}_1), \dots, X(\mathbf{s}_k))$ converges a.s. and represents an sαs-stable random vector; see Samorodnitsky and Taqqu [26], Chapter 3. In particular, the finite-dimensional distributions are regularly varying with index α.

We will always assume that the infinite series in (4.1) converges a.s. in $\mathbb{D}$. Necessary and sufficient conditions for the a.s. convergence of (4.1) in $\mathbb{D}$ or in the space $\mathbb{C}$ in terms of distributional characteristics of Y_i are known in some special cases. We discuss some of them.

Example 4.1. Assume that Y_1 assumes values in $\mathbb{C}$ and $0 < E(|Y_1|_{\infty}^{\alpha}) < \infty$ for some $\alpha \in (0, 2)$. It follows from the reasoning in Ledoux and Talagrand [18], Chapter 5, in particular Corollary 5.5, that the infinite series (4.1) represents a symmetric α-stable (sαs) random field with values in $\mathbb{C}$ and every sαs random field with values in $\mathbb{C}$ has such a series representation. It is also shown on p. 135 in [18] that

$$(4.2) \qquad t^{\alpha} \, P \left(\Big| \sum_{i=2}^{\infty} r_i \, \Gamma_i^{-1/\alpha} \, Y_i \Big|_{\infty} > t \right) \to 0 \,, \quad t \to \infty \,,$$

and that $X = \Gamma_1^{-1/\alpha} r_1 \, Y_1 + R$ is regularly varying on $\overline{\mathbb{C}}_0$ (see pp. 134-136 in [18]) with the straightforward interpretation of $\overline{\mathbb{C}}_0$. Choosing (a_n) such that

$$P(\Gamma_1^{-1/\alpha} |Y_1|_{\infty} > a_n) \sim E|Y_1|_{\infty}^{\alpha} \, a_n^{-\alpha} \sim n^{-1} \,,$$

and following the argument in [18], for any Borel set $S \in \mathcal{B}(\mathbb{S}_{\mathbb{C}})$ which is a continuity set with respect to the limiting measure,

$$n \, P \left(|X|_{\infty} > a_n \, t \,, \widetilde{X} \in S \right) \quad \sim \quad n \, P \left(\Gamma_1^{-1/\alpha} |Y_1|_{\infty} > a_n \, t \,, \widetilde{Y}_1 \in S \right)$$

$$(4.3) \qquad\qquad\qquad\qquad \to \quad t^{-\alpha} \frac{E \left(|Y_1|_{\infty}^{\alpha} I_S(\widetilde{Y}_1) \right)}{E(|Y_1|_{\infty}^{\alpha})} \,.$$

Hence X is regularly varying with index α and spectral measure given on the right-hand side. Notice that this measure is only determined by the distribution of the first term in the series representation, and this is completely analogous to the case of max-stable random fields, see Remark 3.6. $\qquad\square$

For Y_i with values in $\mathbb{D}$, such general results about the a.s. convergence of the series (4.1) are not readily available. Indeed, the proof relies on the fact that the $\Gamma_i^{-1/\alpha} r_i Y_i$'s are random elements in a separable Banach space such as $\mathbb{C}$. However, results by Rosiński [23] in the case $d = 1$, in particular his Theorem 5.1, indicate that special cases of (4.1) with $\mathbb{D}$-valued Y_i's represent sαs Lévy motion on $[0, 1]$. An exception is the case $\alpha \in (0, 1)$:

Example 4.2. Assume $\alpha \in (0, 1)$. We have for $m, h \geq 0$,

$$\Big| \sum_{i=m}^{m+h} \Gamma_i^{-1/\alpha} \, r_i \, Y_i \Big|_{\infty} \leq \sum_{i=m}^{m+h} \Gamma_i^{-1/\alpha} \, |Y_i|_{\infty}$$

If $E(|Y|_{\infty}^{\alpha}) < \infty$ then the right-hand side converges to 0 a.s. as $m, h \to \infty$. Hence the left-hand side is a Cauchy sequence with respect to the uniform topology. Since $\mathbb{D}$ can be made a complete

separable metric space (after completing the J_1-metric) and a.s. convergence in the uniform sense implies a.s. convergence in the J_1-sense, we conclude that the infinite series (4.1) converges a.s. in $\mathbb{D}$. Hence X is an element of $\mathbb{D}$. Adapting the argument on pp. 124–127 in Ledoux and Talagrand [18], we conclude that

$$n\,P\left(a_n^{-1}\sum_{i=2}^{\infty}\Gamma_i^{-1/\alpha}|Y_i|_\infty > \epsilon\right) = o(1)\,, \quad \epsilon > 0\,.$$

This proves that X inherits its tail behavior from the first term in the series (4.1), hence it is regularly varying on $\mathbb{D}$. $\qquad\square$

5. Regular variation of linear combinations of random fields

5.1. **Regular variation.** In this section we prove regular variation of the linear processes $\sum_{i=1}^k \psi_i\, Z_i$, where (Z_i) is an iid sequence of regularly varying random fields with values in $\mathbb{D}$. We start with a result for the truncated series.

Lemma 5.1. *Assume that Z_1 is regularly varying with index α and limiting measure m_Z, ψ_i, $i = 1,\ldots,k$, are deterministic functions in $\mathbb{D}$ with $\min_{i=1,\ldots,k}|\psi_i|_\infty > 0$. Then $\sum_{i=1}^k \psi_i\, Z_i$ is regularly varying with index α and limiting measure*

$$\mu^{(k)} = \sum_{i=1}^{k} m_Z \circ \psi_i^{-1}\,,$$

where $\psi_i^{-1}(B) = \{x \in \overline{\mathbb{D}}_0 : \psi_i\, x \in B\}$.

Proof. For the sake of illustration we focus on the case $k = 2$; the general case $k > 2$ following from an inductive argument. We first note that $\psi_i Z_1$ is regularly varying. This follows by a direct application of Lemma 2.1 which yields

$$n\,P(a_n^{-1}\psi_i\, Z_1 \in \cdot) \overset{\widehat{w}}{\to} \nu_i = m_Z \circ \psi_i^{-1}\,, \quad i = 1,2\,.$$

Hence $Y_i = \psi_i Z_i$ are independent regularly varying random elements with values in $\mathbb{D}$. Next we show that $Y_1 + Y_2$ is regularly varying. Since Y_1, Y_2 are independent it follows from standard regular variation theory (see Resnick [21, 22] or Hult and Lindskog [13]) that

$$n\,P(a_n^{-1}(\mathbf{Y}_1, \mathbf{Y}_2) \in (d\mathbf{u}, d\mathbf{v})) \overset{v}{\to} \nu_{1:\mathbf{s}_1,\ldots,\mathbf{s}_k}(d\mathbf{u})\,\varepsilon_{\mathbf{0}}(d\mathbf{v}) + \nu_{2:\mathbf{s}_1,\ldots,\mathbf{s}_k}(d\mathbf{v})\,\varepsilon_{\mathbf{0}}(d\mathbf{u})\,,$$

where $\varepsilon_{\mathbf{0}}$ is Dirac measure concentrated at $\mathbf{0} \in \mathbb{R}^k$, $\nu_{i:\mathbf{s}_1,\ldots,\mathbf{s}_k}$ are the restrictions of the measures ν_i as defined in (2.4) and

$$\mathbf{Y}_i = (Y_i(\mathbf{s}_1),\ldots Y_i(\mathbf{s}_k))\,, \quad i = 1,2\,.$$

Here $\overset{v}{\to}$ denotes vague convergence on the Borel σ-field $\mathcal{B}(\overline{\mathbb{R}}_0^{2k})$. It follows from a multivariate version of Breiman's result (see Basrak et al. [1]) that linear transformations of $(\mathbf{Y}_1, \mathbf{Y}_2)$ are regularly varying. Hence $\mathbf{Y}_1 + \mathbf{Y}_2$ is regularly varying with index α and limiting measure given by $\mu^{(2)}$ defined above.

Finally, we verify the tightness conditions (2.5) and (2.6) in Lemma 2.1. Notice that

$$w''(Y_1 + Y_2, \delta) \;\leq\; w''(Y_1, \delta) + w''(Y_2, \delta)\,,$$

$$w(Y_1 + Y_2, [0,1]^d\backslash[\delta, 1-\delta]^d) \;\leq\; w(Y_1, [0,1]^d\backslash[\delta, 1-\delta]^d) + w(Y_2, [0,1]^d\backslash[\delta, 1-\delta]^d)\,,$$

implying (2.5) and (2.6) for $Y_1 + Y_2$ by the corresponding relations for Y_1 and Y_2. This proves the lemma. $\qquad\square$

Lemma 5.2. *Assume that Z_1 is regularly varying with index α and limiting measure m_Z and (ψ_i) is a sequence of deterministic functions in $\mathbb{D}$ with $\min_i |\psi_i|_\infty > 0$ and*

$$\sum_{i=1}^{\infty} |\psi_i|_\infty^{\min(1,\alpha-\epsilon)} < \infty$$

for some $\epsilon \in (0,\alpha)$. Then the infinite series $X = \sum_{i=1}^{\infty} \psi_i Z_i$ converges a.s. in $\mathbb{D}$. Moreover, X is regularly varying with index α and limiting measure

$$\mu = \sum_{i=1}^{\infty} m_Z \circ \psi_i^{-1}.$$

Proof. For fixed $m \geq 1$, write $X^{(m)} = \sum_{i=1}^{m} \psi_i Z_i$. The infinite series defining $X(\mathbf{s})$ is bounded by

$$|X(\mathbf{s})| \leq \sum_{i=1}^{\infty} |\psi_i|_\infty |Z_i|_\infty.$$

The right-hand side is a.s. convergent as a consequence of regular variation of $|Z_1|_\infty$ and the summability conditions on $(|\psi_i|_\infty)$; see Davis and Resnick [7]. Hence the infinite series $X(\mathbf{s})$ converges a.s. for every $\mathbf{s}$. Moreover, $|X^{(m)} - X|_\infty \to 0$ a.s. By virtue of this fact and since uniform convergence in $\mathbb{D}$ implies convergence in the Skorokhod metric, we conclude that the limiting random function X is an element of $\mathbb{D}$.

Now we turn to regular variation of X. By Lemma 5.1, $X^{(m)}$ is regularly varying with index α for every $m \geq 1$. By the characterization (2.1) of $\widehat{w}$-convergence, it suffices to show that for any bounded continuous f with support vanishing outside a bounded set,

$$n\, Ef(X/a_n) = \int f(x)\, [n\, P(a_n^{-1} X \in dx)] \to \int f(x)\, \mu(dx).$$

Note that for any $m \geq 1$,

$$n\, Ef(X^{(m)}/a_n) \to \sum_{i=1}^{m} \int f(\psi_i\, x)\, m_Z(dx) = \int f(x)\, \mu^{(m)}(dx).$$

Also, as $m \to \infty$,

$$(5.1) \qquad \sum_{i=1}^{m} \int f(\psi_i\, x)\, m_Z(dx) \to \sum_{i=1}^{\infty} \int f(\psi_i\, x)\, m_Z(dx) = \int f(x)\, \mu(dx).$$

This can be seen as follows. Suppose the support of f is contained in the set $\{x : |x|_\infty > c\}$ and $K = \max_{x \in \mathbb{D}} |f(x)|$. Then

$$\left| \sum_{i=m+1}^{\infty} \int f(\psi_i\, x)\, m_Z(dx) \right| \quad \leq \quad K \sum_{i=m+1}^{\infty} m_Z(\{x : |\psi_i\, x|_\infty > c\})$$

$$\leq \quad K \sum_{i=m+1}^{\infty} m_Z(\{x : |\psi_i|_\infty |x|_\infty > c\})$$

$$= \quad K\, m_Z\{x : |x|_\infty > c\}) \sum_{i=m+1}^{\infty} |\psi_i|_\infty^{\alpha} \to 0, \quad m \to \infty.$$

To complete the proof we show that

$$(5.2) \qquad \lim_{m \to \infty} \limsup_{n \to \infty} n\, E \left| f(X/a_n) - f(X^{(m)}/a_n) \right| = 0.$$

Set

$$w'(x, \epsilon) = \sup \left\{ |f(x) - f(y)| : y \in \mathbb{D}\backslash\{0\}\, , d(x, y) < \epsilon \right\},$$

where d is the metric on $\overline{\overline{\mathbb{D}}}_0$ induced by the Skorokhod topology. Note that $d(X^{(m)}/a_n, X/a_n) \le a_n^{-1}|X^{(m)} - X|_\infty$. Hence we may conclude that

$$n\, E\left| f(X/a_n) - f(X^{(m)}/a_n) \right|$$

$$\le\ n\, E\left[w'(X^{(m)}/a_n, \epsilon)\, I_{[0,\epsilon]}(|X - X^{(m)}|_\infty/a_n) \left(I_{(c,\infty)}(|X^{(m)}|_\infty/a_n) + I_{(c,\infty)}(|X|_\infty/a_n) \right) \right]$$

$$(5.3) \qquad + n\, K\, P(|X - X^{(m)}|_\infty > a_n\epsilon)\,.$$

For ϵ small, the first term may be bounded by

$$n\, E\left[w'(X^{(m)}/a_n, \epsilon) \left(I_{(c,\infty)}(|X^{(m)}|_\infty/a_n) + I_{(c-\epsilon,\infty)}(|X^{(m)}|_\infty/a_n) \right) \right]$$

$$\le\ 2\, nE\left[w'(X^{(m)}/a_n, \epsilon)\, I_{(c-\epsilon,\infty)}(|X^{(m)}|_\infty/a_n) \right]\,.$$

Using the $\widehat{w}$-convergence, the limit of this expression is

$$2 \int w'(x, \epsilon)\, I_{(c-\epsilon,\infty)}(|x|_\infty)\, \mu^{(m)}(dx) \overset{m\to\infty}{\to} 2 \int w'(x, \epsilon)\, I_{(c-\epsilon,\infty)}(|x|_\infty)\, \mu(dx) \overset{\epsilon\to 0}{\to} 0\,.$$

Finally, as to the second term in (5.3), we have

$$|X - X^{(m)}|_\infty \le \sum_{i=m+1}^{\infty} |\psi_i|_\infty\, |Z_i|_\infty\,,$$

which is regularly varying on $(0, \infty]$; see e.g. Embrechts et al. [8], Lemma A3.26. Hence

$$\lim_{m\to\infty} \limsup_{n\to\infty} n\, P(|X - X^{(m)}|_\infty > \epsilon\, a_n) \le \epsilon^{-\alpha} \lim_{m\to\infty} \sum_{i=m+1}^{\infty} |\psi_i|_\infty^\alpha = 0\,.$$

This completes the proof. $\square$

5.2. **Point process convergence.** In this section we use the results about the regular variation of the linear combinations for showing point process convergence of the scaled linear process (X_t) defined in (1.1).

Proposition 5.3. *For $m \ge 1$ fixed, consider the sequence of point processes*

$$I_n = \sum_{t=1}^{n} \varepsilon_{a_n^{-1}(Z_t, \ldots, Z_{t-m+1})}$$

defined on $(\overline{\overline{\mathbb{D}}}_0)^m$. Then $I_n \overset{d}{\to} I$ where $\overset{d}{\to}$ denotes convergence in distribution of point processes on the space $\widehat{M}((\overline{\overline{\mathbb{D}}}_0)^m)$ and

$$I = \sum_{i=1}^{\infty} \left[\varepsilon_{(P_i,0,\ldots,0)} + \varepsilon_{(0,P_i,0,\ldots,0)} + \cdots + \varepsilon_{(0,\ldots,0,P_i)} \right]\,.$$

The space $\widehat{M}((\overline{\overline{\mathbb{D}}}_0)^m)$ consists of the point measures on $(\overline{D}_0)^m$ endowed with the topology generated by $\widehat{w}$-convergence, and $\sum_{i=1}^{\infty} \varepsilon_{P_i}$ is $\mathrm{PRM}(m_Z)$ on $\overline{\overline{\mathbb{D}}}_0$.

Proof. Consider the class $\mathcal{S}$ of bounded sets of the form

$$B = \{(x_1, \ldots, x_m) \in (\overline{\mathbb{D}}_0)^m\} : (|x_i|_\infty, \widetilde{x}_i) \in B_i \times C_i\,, i = 1, \ldots, m\}\,,$$

where $C_i \subset \mathbb{S}_\mathbb{D}$, $\sigma_Z(\partial C_i) = 0$, and $B_i = (b_i, c_i]$ or $B_i = [0, c_i]$, $0 \le b_i < c_i \le \infty$, $i = 1, \ldots, m$. It is easy to verify that this class of sets is a DC-semiring in the sense of Kallenberg [16]. Moreover, since $B \in \mathcal{S}$ is bounded away from $\mathbf{0}$, either $B = B_1 \times \cdots \times B_m$ has empty intersection with all the coordinate axes or intersects only one axis in an interval. That is, with $\mathbf{e}_i$ being the basis element with ith component equal to 1 and the rest zero,

$$(5.4) \qquad\qquad B_1 \times \cdots \times B_m \cap \{y\mathbf{e}_i : y \ge 0\} = \emptyset \quad \text{for } i = 1, \ldots, m\,,$$

or

$$(5.5) \quad B_1 \times \cdots \times B_m \cap \{y\mathbf{e}_i : y \ge 0\} = \begin{cases} \{0\} \times \cdots \times \{0\} \times B_j \times \{0\} \cdots \times \{0\} & i = j\,, \\ \emptyset & i \ne j\,. \end{cases}$$

In the latter case, we must have $B_i = [0, c_i]$ for $i \ne j$. We next show that

$$(5.6) \qquad\qquad \widetilde{I}_n(B) - I_n(B) \overset{P}{\to} 0 \quad \text{for all } B \in \mathcal{S} \text{ such that } P(I(\partial B) = 0) = 1\,,$$

where

$$\widetilde{I}_n(B) = \sum_{t=1}^{n} \sum_{i=1}^{m} \varepsilon_{a_n^{-1} Z_t \mathbf{e}_i}\,.$$

For continuity sets B satisfying (5.4), $\widetilde{I}_n(B) = 0$ a.s. and

$$EI_n(B) \le n\, P(a_n^{-1}(|Z_m|, \ldots, |Z_1|) \in B) = n \prod_{i=1}^{m} P(a_n^{-1}|Z_1| \in B_i) \to 0\,.$$

The limit is zero since in order for B to be a continuity set, $b_i > 0$ for at least two values of i. Hence (5.6) follows. For B satisfying (5.5), so that $0 \in B_i$ for all $i \ne j$ and $b_j > 0$, we have $I_n(B) \le \widetilde{I}_n(B)$ and

$$
\begin{aligned}
P(\widetilde{I}_n(B) - I_n(B) > \epsilon) &\le P\left(\bigcup_{t=1}^{n} \{a_n^{-1}|Z_{t-j}| \in B_j\,, a_n^{-1}|Z_{t-i}| \notin B_i \text{ for some } i \ne j\}\right) \\
&\le n \sum_{i \ne j} P(|Z_1| > a_n\, b_j\,, |Z_2| > a_n\, c_i) \\
&\le n \sum_{i \ne j} P(|Z_1| > a_n\, b_j)\, P(|Z_2| > a_n\, c_i) \to 0\,.
\end{aligned}
$$

This proves that (5.6) holds for all $B \in \mathcal{S}$, as was to be shown.

To complete the proof, it suffices to show (see Daley and Vere-Jones [6], Corollary 9.1.VIII) that for any continuity sets $S_1, \ldots, S_k \in \mathcal{S}$,

$$(I_n(S_1), \ldots, I_n(S_k)) \overset{d}{\to} (I(S_1), \ldots, I(S_k))\,.$$

However, in view of (5.6), this will be implied by

$$(\widetilde{I}_n(S_1), \ldots, \widetilde{I}_n(S_k)) \overset{d}{\to} (I(S_1), \ldots, I(S_k))\,.$$

The proof of this result follows by an application of the continuous mapping theorem; see e.g. the proof of Theorem 2.2 in Davis and Resnick [7]. This proves the proposition. $\qquad\square$

Proposition 5.4. *For each $m \geq 1$, the sequence of point processes*

$$N_n^{(m)} = \sum_{t=1}^n \varepsilon_{a_n^{-1} X_t^{(m)}} \xrightarrow{d} N^{(m)} = \sum_{i=1}^\infty \sum_{j=0}^m \varepsilon_{\psi_j P_i} \,,$$

where $X_t^{(m)} = \sum_{j=0}^m \psi_j Z_{t-j}$ is the finite order moving average process, and $\min_{j=0,\dots,m} |\psi_j|_\infty > 0$.

Proof. By the characterization of weak convergence on $\widehat{M}(\overline{\mathbb{D}}_0)$, we need to show that $N_n^{(m)}(f) \xrightarrow{d} N^{(m)}(f)$ for all continuous bounded functions f that vanish off a bounded set. But $N_n^{(m)}(f) = I_n(f \circ T)$, where $T : (\overline{\mathbb{D}}_0)^{m+1} \to \overline{\mathbb{D}}_0$ is the mapping $T(\mathbf{u}) = \sum_{j=0}^m \psi_j u_j$. The composition function is continuous on the support E of the point process I in Proposition 5.3, i.e.,

$$E = \{\overline{\mathbb{D}}_0 \times \{0\} \times \cdots \times \{0\}\} \cup \{\{0\} \times \overline{\mathbb{D}}_0 \times \{0\} \times \cdots \times \{0\}\} \cup \cdots$$

$$\cup \{\{0\} \times \cdots \times \{0\} \times \overline{\mathbb{D}}_0\} \,.$$

To see this, suppose $\mathbf{u}^{(n)} \to \mathbf{u}$ with respect to the J_1-metric d in $(\overline{\mathbb{D}}_0)^{m+1}$. If the limit vector is in E then $u_j \neq 0$ for some j and $u_i = 0$ for $i \neq j$. It follows that $|u_i^{(n)}|_\infty \to 0$ for all $i \neq j$. Hence

$$d\left(\sum_{i=0}^m \psi_i u_i^{(n)}, \sum_{i=0}^m \psi_i u_i\right) = d\left(\sum_{i=0}^m \psi_i u_i^{(n)}, \psi_j u_j\right)$$

$$\leq d\left(\sum_{i=0}^m \psi_i u_i^{(n)}, \psi_j u_j^{(n)}\right) + d\left(\psi_j u_j^{(n)}, \psi_j u_j\right) \,.$$

The first term converges to zero since $|\sum_{i=0 \neq j}^m \psi_i u_i^{(n)}|_\infty \to 0$ while the second term converges to zero since $\max_j |\psi_j|_\infty < \infty$.

In addition, the continuous mapping $f \circ T$ has bounded support. Suppose the support of f is contained in $\{x : |x|_\infty > c\}$ for some $c > 0$. Then the support of $f \circ T$ is contained in the set

$$\left\{\mathbf{u} : \left|\sum_{i=0}^m \psi_i u_i\right|_\infty > c\right\} \subset \left\{\mathbf{u} : \sum_{i=0}^m |\psi_i|_\infty |u_i|_\infty > c\right\} \,,$$

which is a bounded set on $(\overline{\mathbb{D}}_0)^m$.

Now, applying the characterization of $\widehat{w}$-convergence, it follows that

$$I_n(f \circ T) = N_n^{(m)} \xrightarrow{d} N^{(m)}(f) = I(f \circ T) \,,$$

which proves the proposition. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We are now ready to state and prove the main result of this section which extends the above result to the infinite moving average case.

Theorem 5.5. *Under the assumptions of Proposition 5.4,*

$$N_n = \sum_{t=1}^n \varepsilon_{a_n^{-1} X_t} \xrightarrow{d} N = \sum_{i=1}^\infty \sum_{j=0}^\infty \varepsilon_{\psi_j P_i} \,.$$

Proof. To transfer the point process convergence result of Proposition 5.4 onto N_n, it suffices to show, by Theorem 4.2 in Billingsley [3], that for any $\eta > 0$,

$$(5.7) \qquad\qquad \lim_{m \to \infty} \limsup_{n \to \infty} P\left(\widetilde{\rho}\left(N_n^{(m)}, N_n\right) > \eta\right) = 0$$

and

$$(5.8) \qquad \sum_{i=1}^{\infty}\sum_{j=0}^{m} \varepsilon_{\psi_j P_i} \xrightarrow{d} \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \varepsilon_{\psi_j P_i}\,,$$

where $\widetilde{\rho}$ is a metric on $\widehat{M}(\overline{\mathbb{D}}_0)$. Relation (5.8) is immediate; see e.g. (5.1). For (5.7), we show that

$$(5.9) \qquad \lim_{m\to\infty}\limsup_{n\to\infty} P\left(\sum_{t=1}^{n}\left|f(X_t/a_n) - f(X_t^{(m)}/a_n)\right| > \eta\right) = 0\,,$$

for every bounded continuous function f which vanishes off a bounded set. By the form of the metric $\widetilde{\rho}$, it will then follow that (5.7) holds. The probability in (5.9) is bounded by

$$\eta^{-1}\, n\, E\left|f(X_1/a_n) - f(X_1^{(m)}/a_n)\right|\,,$$

which converges to zero by first letting $n \to \infty$ and then $m \to \infty$, in view of (5.2). This proves the theorem. $\qquad\square$

5.3. Some applications.

Example 5.6. Let $A \subset [0,1]^d$ be a Borel set and define $f_A : \mathbb{D} \to \mathbb{R}$ by

$$f_A(x) = \int_A x(\mathbf{s})\,d\mathbf{s}\,, \quad x \in \mathbb{D}\,.$$

Although we only consider simple averages here, one could also study averages relative to a kernel function given by $\int_A K(\mathbf{t} - \mathbf{s})\,x(\mathbf{s})d\mathbf{s}$. The functional f_A is continuous with respect to the J_1-metric. Indeed, consider a sequence (x_n) in $\mathbb{D}$ converging to x in $\mathbb{D}$ in the J_1-sense. Then there exist continuous bijections $\lambda_n : [0,1]^d \to [0,1]^d$ which are increasing in every component and such that $\lambda_n(\mathbf{0}) = \mathbf{0}$, $\lambda_n(\mathbf{1}) = \mathbf{1}$, and

$$d(x_n, x) = \sup_{\mathbf{s}\in[0,1]^d} |x_n(\lambda_n(\mathbf{s})) - x(\mathbf{s})| \vee |\lambda_n(\mathbf{s}) - \mathbf{s}| \to 0\,, \quad n \to \infty\,.$$

Then

$$\begin{aligned}
|f_A(x_n) - f_A(x)| &= \left|\int_A (x_n(\lambda_n(\mathbf{s})) - x(\mathbf{s}))\,d\mathbf{s} + \int_A (x_n(\mathbf{s}) - x_n(\lambda_n(\mathbf{s})))\,d\mathbf{s}\right| \\
&\leq |A|\,d(x_n, x) + \int_{A_c} |x_n(\mathbf{s}) - x_n(\lambda_n(\mathbf{s}))|\,d\mathbf{s}\,,
\end{aligned}$$

where A_c denotes the set of continuity points of x in A. A dominated convergence argument ensures that the right-hand side converges to zero as $n \to \infty$. Hence f_A is a continuous mapping.

For $\epsilon > 0$, note that

$$\{x \in \overline{\mathbb{D}}_0 : |f_A(x)| > \epsilon\} \subseteq \{x \in \overline{\mathbb{D}}_0 : |x|_\infty |A| > \epsilon\}\,.$$

Hence $f_A(x)$ transforms bounded sets in $\overline{\mathbb{R}}\backslash\{0\}$ into bounded sets in $\overline{\mathbb{D}}_0$ provided A has positive Lebesgue measure. Thus, if X is a regularly varying random field with index $\alpha > 0$ and limiting measure m in $\overline{\mathbb{D}}_0$, and if $A \subset [0,1]^d$ has positive Lebesgue measure, then the continuous mapping theorem for regularly varying $\mathbb{D}$-valued random fields ensures that $f_A(X)$ is regularly varying as well with limiting measure $m \circ f_A^{-1}$. In particular, if X_t is a linear random field as defined in (1.1),

then for $y > 0$,

$$n P \left(a_n^{-1} \int_A X_t(\mathbf{s}) \, d\mathbf{s} > y \right) \quad \to \quad \sum_{i=0}^{\infty} (m_Z \circ \psi_i^{-1} \circ f_A^{-1})(y, \infty)$$

$$= \quad \sum_{i=0}^{\infty} m_Z(\{x \in \overline{\mathbb{D}}_0 : \int_A \psi_i(\mathbf{s}) x(\mathbf{s}) \, d\mathbf{s} > y\})$$

$$= \quad y^{-\alpha} \sum_{i=0}^{\infty} m_Z(\{x \in \overline{\mathbb{D}}_0 : \int_A \psi_i(\mathbf{s}) x(\mathbf{s}) \, d\mathbf{s} > 1\}).$$

Further, if $\psi_i(\mathbf{s}) = \psi_i$ for some constants ψ_i and all $\mathbf{s} \in A$, then we obtain

$$y^{-\alpha} \left[m_Z(\{x \in \overline{\mathbb{D}}_0 : \int_A x(\mathbf{s}) \, d\mathbf{s} > 1\}) \sum_{i=0}^{\infty} (\psi_i)_+^{\alpha} + m_Z(\{x \in \overline{\mathbb{D}}_0 : \int_A x(\mathbf{s}) \, d\mathbf{s} < -1\}) \sum_{i=0}^{\infty} (\psi_i)_-^{\alpha} \right].$$

The same result can be derived by observing that $\int_A X_t(\mathbf{s}) \, d\mathbf{s}$ has representation as a one-dimensional linear process with iid regularly varying noise $\int_A Z_i(\mathbf{s}) \, d\mathbf{s}$. Results of this kind can be found e.g. in Embrechts et al. [8], Appendix 3.3.

If we further specify the noise Z to be a regularly varying Lévy random field as considered in Section 3.2, the latter expression further simplifies. For example, for a random vector $\mathbf{V}$ with uniform distribution on $(0, 1)^d$, θ independent of $\mathbf{V}$ with distribution as described in Section 3.2,

$$m_Z(\{x \in \overline{\mathbb{D}}_0 : \int_A x(\mathbf{s}) \, d\mathbf{s} > 1\}) \quad = \quad \alpha \int_1^{\infty} \int_{[0,1]^d} P(\theta I_{[\mathbf{v},\mathbf{1}] \cap A} > 1) \, d\mathbf{v} \, \theta^{-\alpha-1} \, d\theta$$

$$= \quad \alpha \int_1^{\infty} \int_{[0,1]^d} I_{[\mathbf{v},\mathbf{1}] \cap A} \, d\mathbf{v} \, \theta^{-\alpha-1} \, d\theta$$

$$= \quad |A| .$$

Example 5.7. In this example, we consider the limiting distribution of the space-time maxima $\max_{t=1,\dots,n} |X_t|_{\infty}$. From Theorem 5.5 and the continuous mapping theorem we have

$$N_n^{(1)} = \sum_{i=1}^{\infty} \varepsilon_{a_n^{-1}|X_t|_{\infty}} \xrightarrow{d} N^{(1)} = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \varepsilon_{|\psi_j P_i|_{\infty}} .$$

Hence

$$P(a_n^{-1} \max_{t=1,\dots,n} |X_t|_{\infty} \le y) \quad = \quad P(N_n^{(1)}((y, \infty]) = 0)$$

$$\to \quad P(N^{(1)}((y, \infty]) = 0)$$

(5.10) $$= \quad P(\sup_{i \ge 1, \, j \ge 0} |\psi_j P_i|_{\infty} \le y) = G(y) .$$

In order to get explicit formulas for the limit distribution, we assume that $\psi_j(\mathbf{s}) = \psi_j$ for some constants ψ_j and all $j \ge 0$. Then

$$G(y) = P(\sup_i |P_i|_{\infty} \le y / \sup_j |\psi_j|) .$$

The points P_i, $i = 1, 2, \dots$, constitute a PRM on $\overline{\mathbb{D}}_0$ with mean measure m_Z, hence the points $|P_i|_{\infty}$, $i = 1, 2, \dots$, constitute a PRM on $(0, \infty]$ with mean measure of $(z, \infty]$ given by

$$m_Z(\{x \in \overline{\mathbb{D}}_0 : |x|_{\infty} > z\}) = z^{-\alpha} m_Z(\{x \in \overline{\mathbb{D}}_0 : |x|_{\infty} > 1\}) = c_Z \, z^{-\alpha} .$$

Hence

$$G(y) = e^{-c_Z \, (\sup_i |\psi_i|/y)^\alpha} .$$

A straightforward calculation shows that

$$n \, P(|X|_\infty > a_n y) \sim y^{-\alpha} \, c_Z \sum_{j=0}^\infty |\psi_j|^\alpha .$$

The extremal index of the process $(|X_t|_\infty)$ can then be read off from G as

(5.11)
$$\max_{j=0,1,\dots} |\psi_j|^\alpha / \sum_{j=0}^\infty |\psi_j|^\alpha .$$

We refer to Leadbetter et al. [19] and Embrechts et al. [8], Section 8.1, for the definition and properties of the extremal index of a strictly stationary sequence. We also mention that the extremal index of $(|X_t|_\infty)$ coincides with the extremal index of the absolute value sequence of a one-dimensional linear process $Y_t = \sum_{j=0}^\infty \psi_j \eta_{t-j}$, where (η_j) is an iid sequence of real-valued regularly varying random variables with index $\alpha > 0$. Alternatively, the value (5.11) coincides with the extremal index of the sequence $\sum_{j=0}^\infty |\psi_j| \, |Z_{t-j}|_\infty$. These facts about the extremal index of a linear process follow from Davis and Resnick [7].

It is a rather surprising fact that the extremal indices of the sequences $(|X_t|_\infty)$, $(\sum_{j=0}^\infty |\psi_j| \, |Z_{t-j}|_\infty)$ and $(|X_t(\mathbf{s})|)$, $\mathbf{s} \in [0,1]^d$, coincide. A particular consequence is that the extremal index (5.11) can be estimated from the time series of observations $|X_t(\mathbf{s})|$ at any site $\mathbf{s}$.

Example 5.8. A result analogous to (5.10) can be obtained for the sequence of maxima at a finite number of sites $\mathbf{s}_i \in [0,1]^d$. We illustrate the case with two distinct sites $\mathbf{s}_1$ and $\mathbf{s}_2$. Then the continuous mapping theorem yields

$$\sum_{i=1}^\infty \varepsilon_{(X_i(\mathbf{s}_1), X_i(\mathbf{s}_2))/a_n} \xrightarrow{d} \sum_{i=1}^\infty \sum_{j=0}^\infty \varepsilon_{(\psi_j(\mathbf{s}_1)P_i(\mathbf{s}_1), \psi_j(\mathbf{s}_2)P_i(\mathbf{s}_2))}$$

We conclude that

$$P(a_n^{-1} \max_{t=1,\dots,n} X_t(\mathbf{s}_1) \le y_1 \,, a_n^{-1} \max_{t=1,\dots,n} X_t(\mathbf{s}_2) \le y_2)$$

$$\rightarrow \quad P(\sup_{i \ge 1, j \ge 0} \psi_j(\mathbf{s}_1)P_i(\mathbf{s}_1) \le y_1 \,, \sup_{i \ge 1, j \ge 0} \psi_j(\mathbf{s}_2)P_i(\mathbf{s}_2) \le y_2)$$

$$= \quad P(\sup_{j \ge 0} \psi_j(\mathbf{s}_1) \sup_{i \ge 1} P_i(\mathbf{s}_1) \le y_1 \,, \sup_{j \ge 0} \psi_j(\mathbf{s}_2) \sup_{i \ge 1} P_i(\mathbf{s}_2) \le y_2) = p(y_1, y_2) \,.$$

We assume that both $\Psi_i = \sup_{j \ge 0} \psi_j(\mathbf{s}_i)$, $i = 1, 2$, are positive. Then

$$-\log p(y_1, y_2) \quad = \quad m_Z(\{x \in \overline{\mathbb{D}}_0 : (x(\mathbf{s}_1), x(\mathbf{s}_2)) \notin [0, y_1/\Psi_1] \times [0, y_2/\Psi_2]\})$$

$$= \quad (\Psi_1/y_1)^\alpha \, m_Z(\{x \in \overline{\mathbb{D}}_0 : x(\mathbf{s}_1) > 1\}) + (\Psi_2/y_2)^\alpha \, m_Z(\{x \in \overline{\mathbb{D}}_0 : x(\mathbf{s}_2) > 1\})$$

$$- m_Z(\{x \in \overline{\mathbb{D}}_0 : x(\mathbf{s}_1) > y_1/\Psi_1 \,, x(\mathbf{s}_2) > y_2/\Psi_2\}) \,.$$

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