

# Estimation for Non-negative Lévy-driven Ornstein-Uhlenbeck Processes

Peter J. Brockwell, Richard A. Davis and Yu Yang,  
Department of Statistics, Colorado State University,  
Fort Collins, CO 80523-1877

## Abstract

Continuous-time autoregressive moving average (CARMA) processes with a non-negative kernel and driven by a non-decreasing Lévy process constitute a very general class of stationary, non-negative continuous-time processes. An example is the Lévy-driven stationary Ornstein-Uhlenbeck (or CAR(1)) process, introduced by Barndorff-Nielsen and Shephard (2001) as a model for stochastic volatility. For such processes we take advantage of the non-negativity of the increments of the driving Lévy process to develop a highly efficient estimation procedure for the parameters when observations are available at uniformly spaced times  $0, h, \dots, Nh$ . We also show how to reconstruct the background driving Lévy process from a continuously observed realization of the process and use this result to estimate the increments of the Lévy process itself when  $h$  is small. Asymptotic properties of the coefficient estimator are derived and the results illustrated using a simulated gamma-driven Ornstein-Uhlenbeck process.

*AMS 2000 Mathematics Subject Classification* : Primary : 62M10, 60H10.

Secondary: 62M09.

*Keywords and phrases*: Continuous-time autoregression, Ornstein-Uhlenbeck process, Lévy process, stochastic differential equation, sampled process.

## 1. Introduction.

This paper is concerned with estimation of the parameters of a non-negative Lévy-driven Ornstein-Uhlenbeck process and of the parameters of the background driving Lévy process, based on observations made at uniformly and closely-spaced times. The idea is to obtain a highly efficient estimator of the CAR(1) coefficient by estimating the corresponding coefficient of the sampled AR(1) process using the estimator of Davis and McCormick (1989) for non-negative discrete-time AR(1) processes. This estimator is then used to estimate the corresponding realization of the driving Lévy process using a generalization of an argument due to Pham-Din-Tuan (1977). The exact recovery of the driving Lévy process requires continuous observation of the Ornstein-Uhlenbeck process. The integral expressions determining the driving Lévy process are therefore replaced by approximating sums using the available discrete-time observations.

In Section 2, we define the stationary Lévy-driven Ornstein-Uhlenbeck (or CAR(1)) process,  $\{Y(t), t \geq 0\}$ . In Section 3, we characterize the sampled AR(1) process,  $\{Y_n^{(h)} = Y(nh), n = 0, 1, 2, \dots\}$ , and the distribution of its driving white noise sequence in terms of the parameters of the underlying CAR(1) process and its driving Lévy process. The autoregressive coefficient of the sampled process is then estimated with very high efficiency using the method of Davis and McCormick (1989). From the relation between the sampled and continuous-time processes we then obtain corresponding parameter estimates for the CAR(1) process. The idea of using the sampled process to estimate the parameters of the underlying continuous-time process was first used by Phillips (1954), but in our case the non-decreasing property of the driving Lévy process and the non-negativity of the corresponding discrete-time increments permits a very large efficiency gain. In Section 4, we show how to recover the driving Lévy process under the assumption that the process is observed continuously and then approximate the results using closely-spaced discrete observations. In Section 5, we derive the asymptotic distribution of the coefficient estimator when the driving Lévy process is a gamma process and illustrate with a simulated example the performance of the estimators of both the CAR(1) parameters and the driving Lévy process. When the continuously observed process is available, the autoregression coefficient can be identified with probability 1. This is discussed in Section 6.

## 2. Stationary Lévy-driven Ornstein-Uhlenbeck processes.

In order to define the stationary Lévy-driven Ornstein-Uhlenbeck (or CAR(1)) process, we first record a few essential facts concerning Lévy processes. (For a detailed account of integration with respect to Lévy processes, see Protter (2004)). Suppose we are given a

filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq \infty}, P)$ , where  $\mathcal{F}_0$  contains all the  $P$ -null sets of  $\mathcal{F}$  and  $(\mathcal{F}_t)$  is right-continuous.

**Definition 1 (Lévy Process).**  $\{L(t), t \geq 0\}$  is an  $(\mathcal{F}_t)$ -adapted Lévy process if  $L(t) \in \mathcal{F}_t$  for all  $t \geq 0$  and

- $L(0) = 0$  a.s.,
- $L(t) - L(s)$  is independent of  $\mathcal{F}_s$ ,  $0 \leq s < t < \infty$ ,
- $L(t) - L(s)$  has the same distribution as  $L(t - s)$  and
- $L(t)$  is continuous in probability.

Every Lévy process has a unique modification which is càdlàg (right continuous with left limits) and which is also a Lévy process. We shall therefore assume that our Lévy process has these properties. For non-decreasing Lévy processes the Laplace transform  $\tilde{f}_{L(t)}(s) := E(\exp(-sL(t)))$  has the form

$$\tilde{f}_{L(t)}(s) = \exp(-t\Phi(s)), \quad \Re(s) \geq 0,$$

where

$$\Phi(s) = m + \int_{(0, \infty)} (1 - e^{-sx}) \nu(dx),$$

with  $m \geq 0$  and  $\nu$  a measure on the Borel subsets of  $(0, \infty)$  satisfying

$$\int_{(0, \infty)} \frac{u}{1+u} \nu(du) < \infty.$$

The measure  $\nu$  is called the **Lévy measure** of the process  $L$  and  $m$  is the **drift**. There exists a wealth of possible marginal distributions for  $L(t)$ , attainable by suitable choice of  $m$  and  $\nu$ . (See for example Barndorff-Nielsen and Shephard (2001).) For second-order Lévy processes  $E(L(1))^2 < \infty$  and there exist real constants  $\mu$  and  $\sigma$  such that

$$EL(t) = \mu t \text{ and } \text{Var}(L(t)) = \sigma^2 t, \text{ for } t \geq 0.$$

To avoid problems of parameter identifiability in the CAR(1) process defined below we assume throughout that  $L$  is scaled so that  $\text{Var}(L(1)) = 1$ . Then  $\text{Var}(L(t)) = t$  for  $t \geq 0$  and we shall refer to the process  $L$  as a standardized second-order Lévy process. Throughout this paper we shall be concerned with CAR(1) (or stationary Ornstein-Uhlenbeck) processes driven by standardized second-order non-decreasing Lévy processes. The Lévy-driven CAR(1) process is defined as follows.

**Definition 2 (Lévy-driven CAR(1) process)** A CAR(1) process driven by the Lévy process  $\{L(t), t \geq 0\}$  is defined to be a strictly stationary solution of the stochastic differential equation,

$$dY(t) + aY(t)dt = \sigma dL(t). \quad (2.1)$$

In the special case when  $\{L(t)\}$  is Brownian motion, (2.1) is interpreted as an Itô equation with solution  $\{Y(t), t \geq 0\}$  satisfying

$$Y(t) = e^{-at}Y(0) + \sigma \int_0^t e^{-a(t-u)} dL(u), \quad (2.2)$$

where the integral is defined as the  $L^2$  limit of approximating Riemann-Stieltjes sums. For any second-order driving Lévy process,  $\{L(t)\}$ , the process  $\{Y(t)\}$  can be defined in the same way, and if  $\{L(t)\}$  is non-decreasing (and hence of bounded variation on compact intervals)  $\{Y(t)\}$  can also be defined pathwise as a Riemann-Stieltjes integral by (2.2). We can also write

$$Y(t) = e^{-a(t-s)}Y(s) + \sigma \int_s^t e^{-a(t-u)} dL(u), \text{ for all } t > s \geq 0, \quad (2.3)$$

showing, by independence of the increments of  $\{L(t)\}$ , that  $\{Y(t)\}$  is Markov. (For a general theory of integration with respect to semimartingales, and in particular with respect to Lévy processes see Protter (2004).) The following proposition gives necessary and sufficient conditions for stationarity of  $\{Y(t)\}$ . For a proof see Brockwell and Marquardt (2005).

**Proposition 1.** If  $Y(0)$  is independent of  $\{L(t), t \geq 0\}$  and  $E(L(1)^2) < \infty$ , then  $Y(t)$  is strictly stationary if and only if  $a > 0$  and  $Y(0)$  has the distribution of  $\sigma \int_0^\infty e^{-au} dL(u)$ .

**Remark 1.** By introducing a second Lévy process  $\{M(t), 0 \leq t < \infty\}$ , independent of  $L$  and with the same distribution, we can extend  $\{Y(t), t \geq 0\}$  to a process with index set  $(-\infty, \infty)$ . Define the following extension of  $L$ :

$$L^*(t) = L(t)I_{[0, \infty)}(t) - M(-t-)I_{(-\infty, 0]}(t), \quad -\infty < t < \infty.$$

Then, provided  $a > 0$ , the process  $\{Y(t)\}$  defined by

$$Y(t) = \sigma \int_{-\infty}^t e^{-a(t-u)} dL^*(u), \quad (2.4)$$

is a strictly stationary process satisfying equation (2.3) (with  $L$  replaced by  $L^*$ ) for all  $t > s$  and  $s \in (-\infty, \infty)$ . Henceforth, we refer to  $L^*$  as the *background driving Lévy process* (BDLP) and denote it by  $L$  for simplicity.

**Remark 2.** From (2.4) we have the relation

$$Y(t) = e^{-a(t-s)}Y(s) + \sigma \int_s^t e^{-a(t-u)}dL(u), \quad t \geq s > -\infty. \quad (2.5)$$

Taking  $s = 0$  and using Lemma 2.1 of Eberlein and Raible (1999), we find that

$$Y(t) = e^{-at}Y(0) + \sigma L(t) - a\sigma \int_0^t e^{-a(t-u)}L(u)du, \quad t \geq 0, \quad (2.6)$$

where the last integral is a Riemann integral and the equality holds for all finite  $t \geq 0$  with probability 1.

### 3. Parameter estimation via the sampled process.

Setting  $t = nh$  and  $s = (n-1)h$  in equation (2.5), we see at once that for any  $h > 0$ , the sampled process  $\{Y_n^{(h)}, n = 0, 1, 2, \dots\}$  is the discrete-time AR(1) process satisfying

$$Y_n^{(h)} = \phi Y_{n-1}^{(h)} + Z_n, \quad n = 0, 1, 2, \dots, \quad (3.1)$$

where

$$\phi = e^{-ah}, \quad (3.2)$$

and

$$Z_n = \sigma \int_{(n-1)h}^{nh} e^{-a(nh-u)}dL(u). \quad (3.3)$$

The noise sequence  $\{Z_n\}$  is i.i.d. and positive since  $L$  has stationary, independent and positive increments.

If the process  $\{Y(t), 0 \leq t \leq T\}$  is observed at times  $0, h, 2h, \dots, Nh$ , where  $N = [T/h]$ , i.e.,  $N$  is the integer part of  $T/h$ , then, since the innovations  $Z_n$  of the process  $\{Y_n^{(h)}\}$  are non-negative and  $0 < \phi < 1$ , we can use the highly efficient Davis-McCormick estimator of  $\phi$ , namely

$$\hat{\phi}_N^{(h)} = \min_{1 \leq n \leq N} \frac{Y_n^{(h)}}{Y_{n-1}^{(h)}}. \quad (3.4)$$

To obtain the asymptotic distribution of  $\hat{\phi}_N^{(h)}$  as  $N \rightarrow \infty$  with  $h$  fixed, we need to suppose that the distribution function  $F$  of  $Z_n$  satisfies  $F(0) = 0$  and that  $F$  is regularly varying at zero with exponent  $\alpha$ , i.e., that there exists  $\alpha > 0$  such that

$$\lim_{t \downarrow 0} \frac{F(tx)}{F(t)} = x^\alpha \quad \text{for all } x > 0.$$

(These conditions are satisfied by the gamma-driven CAR(1) process as we shall show in Section 5.) Under these conditions on  $F$ , the results of Davis and McCormick (1989) imply that  $\hat{\phi}_N^{(h)} \rightarrow \phi$  *a.s.* as  $N \rightarrow \infty$  with  $h$  fixed and that

$$\lim_{N \rightarrow \infty} P \left[ k_N^{-1} (\hat{\phi}_N^{(h)} - \phi) c_\alpha \leq x \right] = G_\alpha(x), \quad (3.5)$$

where  $k_N = F^{-1}(N^{-1})$ ,  $c_\alpha = (EY_1^{(h)\alpha})^{1/\alpha}$  and  $G_\alpha$  is the Weibull distribution function,

$$G_\alpha(x) = \begin{cases} 1 - \exp\{-x^\alpha\}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases} \quad (3.6)$$

From the observations  $\{Y_n^{(h)}, n = 0, 1, \dots, N\}$  we thus obtain the estimator  $\hat{\phi}_N^{(h)}$  and, from (3.2), the corresponding estimator,

$$\hat{a}_N^{(h)} = -h^{-1} \log \hat{\phi}_N^{(h)} \quad (3.7)$$

of the CAR(1) coefficient  $a$ . Provided the distribution function  $F$  of the noise terms  $Z_n$  in the discrete-time sampled process satisfies the conditions indicated above, we can also determine the asymptotic distributions of this estimator. In particular, using a Taylor series approximation, we find that

$$\lim_{N \rightarrow \infty} P \left[ (-h) e^{-ah} c_\alpha k_N^{-1} (\hat{a}_N^{(h)} - a) \leq x \right] = G_\alpha(x), \quad (3.8)$$

where  $G_\alpha$  is given in (3.6). Since  $\text{var}(Y^{(h)}) = \sigma^2/(2a)$ , we use the estimator,

$$\hat{\sigma}_N^2 = \frac{2\hat{a}_N^{(h)}}{N} \sum_{i=0}^N (Y_i^{(h)} - \bar{Y}_N^{(h)})^2 \quad (3.9)$$

to estimate  $\sigma^2$ .

#### 4. Estimating the Lévy increments.

So far, we have made no assumptions about the driving Lévy process except for non-negativity and existence of  $EL(1)^2$ . In order to suggest an appropriate parametric model for  $L$  and to estimate the parameters, it is important to recover an approximation to  $L$  from the observed data. If the CAR(1) process is continuously observed on  $[0, T]$ , then the argument of Pham-Din-Tuan (1977) can be used to recover  $\{L(t), 0 \leq t \leq T\}$ . His  $L^2$ -based spectral argument which he applied to Gaussian processes, also applies to the Lévy-driven CAR(1) process to give

$$L(t) = \sigma^{-1} \left[ Y(t) - Y(0) + a \int_0^t Y(s) ds \right]. \quad (4.1)$$

A direct justification of this result can be obtained by *defining*  $L$  as in (4.1) and then showing that  $Y(0)e^{-at} + \sigma \int_0^t e^{-a(t-u)} dL(u) = Y(t)$ . Thus, using Remark 2 of Section 2, we have

$$\begin{aligned}
& Y(0)e^{-at} + \sigma \int_0^t e^{-a(t-u)} dL(u) \\
&= Y(0)e^{-at} + \sigma L(t) - a\sigma \int_0^t e^{-a(t-u)} L(u) du \\
&= Y(0)e^{-at} + \left[ Y(t) - Y(0) + a \int_0^t Y(s) ds \right] - a \int_0^t e^{-a(t-u)} \left[ Y(u) - Y(0) + a \int_0^u Y(s) ds \right] du \\
&= Y(t) + a \int_0^t Y(s) ds - a \int_0^t e^{-a(t-u)} Y(u) du - a^2 \int_0^t Y(s) \int_s^t e^{-a(t-u)} du ds \\
&= Y(t)
\end{aligned}$$

as required.

From (4.1), the increment of the driving Lévy process on the interval  $((n-1)h, nh]$  is given by

$$\Delta L_n^{(h)} := L(nh) - L((n-1)h) = \sigma^{-1} \left[ Y(nh) - Y((n-1)h) + a \int_{(n-1)h}^{nh} Y(u) du \right].$$

Replacing the CAR(1) parameters by their estimators and the integral by a trapezoidal approximation, we obtain the estimated increments,

$$\Delta \hat{L}_n^{(h)} = \hat{\sigma}_N^{-1} \left[ Y_n^{(h)} - Y_{n-1}^{(h)} + \hat{a}_N^{(h)} h (Y_n^{(h)} + Y_{n-1}^{(h)}) / 2 \right]. \quad (4.2)$$

## 5. The gamma-driven CAR(1) process

In this section, we illustrate the preceding estimating procedure in the case when  $L$  is a standardized gamma process. Thus  $L(t)$  has the gamma density  $f_{L(t)}$  with exponent  $\gamma t$ , scale-parameter  $\gamma^{-1/2}$ , mean  $\gamma^{1/2}t$  and variance  $t$ . The Laplace transform of  $L(t)$  is

$$\tilde{f}_{L(t)}(s) := E \exp(-sL(t)) = \exp\{-t\Phi(s)\}, \quad \Re(s) \geq 0, \quad (5.1)$$

where  $\Phi(s) = \gamma \log(1 + \beta s)$ ,  $\beta = \gamma^{-1/2}$  and  $\gamma > 0$ .

Based on the  $h$ -spaced observations  $\{Y_n^{(h)}, n = 0, 1, \dots, N\}$ , we estimate the discrete-time autoregression coefficient  $\phi$  and the CAR(1) parameters  $a$  and  $\sigma^2$  using (3.4), (3.7) and (3.9) respectively. We then estimate the Lévy increments as in (4.2) and use them to estimate the parameter  $\gamma$  of the standardized gamma process  $L$ . To obtain the asymptotic distributions of  $\hat{\phi}_N^{(h)}$  and  $\hat{a}_N^{(h)}$  as  $N \rightarrow \infty$  with  $h$  fixed, we first show that the

distribution function  $F$  of  $Z_n$  in (3.1) is regularly varying at zero with exponent  $\gamma h$  and then determine the coefficients  $k_N = F^{-1}(N^{-1})$  and  $c_\alpha = (EY_1^{(h)\alpha})^{1/\alpha}$  in (3.5). To do so, we use the Laplace transform (5.1) to investigate the behavior of the density of  $Z_1 = \sigma \int_0^h e^{-a(h-t)} dL(t) = \sigma \int_0^h e^{-at} dL(t)$  near zero.

Define  $W_h := Z_1/\sigma = \int_0^h e^{-at} dL(t)$ . The Laplace transform of  $W_h$  is

$$\begin{aligned} \tilde{f}_{W_h}(s) &= \exp \left[ - \int_0^h \Phi(se^{-at}) dt \right] \\ &= \exp \left[ - \int_0^h \gamma \log(1 + \beta se^{-at}) dt \right]. \end{aligned} \quad (5.2)$$

The exponent in (5.2) has the power series expansion,

$$\begin{aligned} - \int_0^h \gamma \log(1 + \beta se^{-at}) dt &= -\gamma \int_0^h \log \left[ s\beta e^{-at} \left( 1 + \frac{1}{\beta se^{-at}} \right) \right] dt \\ &\approx \log(s\beta)^{-\gamma h} + \frac{1}{2} \gamma ah^2 + \gamma \left[ \frac{1}{\beta sa} (1 - e^{ah}) - \frac{1}{4\beta^2 s^2 a} (1 - e^{2ah}) + \dots \right], \end{aligned}$$

as  $s \rightarrow \infty$ . Hence  $\tilde{f}_{W_h}(s)$  has the corresponding expansion,

$$\tilde{f}_{W_h}(s) \approx \frac{\beta^{-\gamma h}}{s^{\gamma h}} e^{\frac{1}{2} \gamma ah^2} + \frac{C_1}{s^{\gamma h+1}} + \frac{C_2}{s^{\gamma h+2}} + \dots,$$

where  $C_1, C_2, \dots$  are constants depending on  $\gamma, \beta, h$  and  $a$ . Since  $\tilde{f}_{Z_1}(s) = \tilde{f}_{W_h}(\sigma s)$ ,

$$s^{\gamma h} \tilde{f}_{Z_1}(s) \rightarrow (\sigma\beta)^{-\gamma h} e^{\frac{1}{2} \gamma ah^2}, \text{ as } s \rightarrow \infty.$$

By Theorem 30.2 of Doetsch (1974), the density  $f_{Z_1}$  of  $Z_1$  has the expansion, in a neighbourhood of zero,

$$f_{Z_1}(x) = \frac{(\sigma\beta)^{-\gamma h} x^{\gamma h-1}}{\Gamma(\gamma h)} e^{\frac{1}{2} \gamma ah^2} + \frac{(\sigma x)^{\gamma h} C_1}{\sigma \Gamma(\gamma h + 1)} + \frac{(\sigma x)^{\gamma h+1} C_2}{\sigma \Gamma(\gamma h + 2)} + \dots.$$

So

$$\frac{f_{Z_1}(x)}{x^{\gamma h-1}} \rightarrow (\sigma\beta)^{-\gamma h} e^{\frac{1}{2} \gamma ah^2} / \Gamma(\gamma h), \text{ as } x \rightarrow 0,$$

and

$$F_{Z_1}(x) \sim x^{\gamma h} (\sigma\beta)^{-\gamma h} e^{\frac{1}{2} \gamma ah^2} / \Gamma(\gamma h + 1), \text{ as } x \rightarrow 0. \quad (5.3)$$

Thus the distribution  $F$  of  $Z_n$  is regularly varying at zero with exponent  $\gamma h$ .

From the definition of  $k_N$  in (3.5) we have  $\frac{1}{N} = \int_0^{k_N} F_{Z_1}(du)$ . This equation, together with (5.3), gives

$$k_N^{-1} \sim (\sigma\beta)^{-1} [\Gamma(\gamma h + 1)]^{-1/(\gamma h)} e^{\frac{1}{2} \gamma ah^2} N^{1/(\gamma h)}, \text{ as } N \rightarrow \infty. \quad (5.4)$$



In order to calculate  $c_{\gamma h}$ , we need to find  $E[Y_n^{(h)}]^{\gamma h}$ , where  $Y_n^{(h)} = \sum_{j=0}^{\infty} \phi^j Z_{n-j}$ . The Laplace transform of  $Y_n^{(h)}$  is

$$\tilde{f}_{Y_n^{(h)}}(s) = E e^{-s Y_n^{(h)}} = \prod_{j=0}^{\infty} E e^{-s \phi^j Z_{n-j}}.$$

So

$$\begin{aligned} \log \tilde{f}_{Y_n^{(h)}}(s) &= \sum_{j=0}^{\infty} \log \tilde{f}_{Z_1}(s \phi^j) \\ &= \sum_{j=0}^{\infty} \log \tilde{f}_{W_h}(s \sigma \phi^j) \\ &= -\gamma \sum_{j=0}^{\infty} \int_0^h \log(1 + \beta s \sigma \phi^j e^{-ay}) dy, \end{aligned}$$

and hence

$$\begin{aligned} \tilde{f}_{Y_n^{(h)}}(s) &= \exp \left[ \frac{\gamma}{a} \sum_{j=0}^{\infty} [\text{dilog}(1 + \beta s \sigma \phi^j) - \text{dilog}(1 + \beta s \sigma \phi^j e^{-ah})] \right] \\ &= \exp \left( \frac{\gamma}{a} \text{dilog}(1 + \beta s \sigma) \right), \end{aligned}$$

where  $\text{dilog}$  is the dilogarithm function,  $\text{dilog}(x) = \int_1^x \log(u)/(1-u) du$ . Using Theorem 2.1 of Brockwell and Brown (1978), we get

$$\begin{aligned} E[Y_n^{(h)}]^{\gamma h} &= \frac{1}{\Gamma(1 - \gamma h)} \int_0^{\infty} s^{-\gamma h} \left| D \tilde{f}_{Y_n^{(h)}}(s) \right| ds \\ &= \frac{\gamma}{a \Gamma(1 - \gamma h)} \int_0^{\infty} s^{-\gamma h - 1} \exp \left( \frac{\gamma}{a} \text{dilog}(1 + \beta s \sigma) \right) \log(1 + \beta s \sigma) ds, \end{aligned} \quad (5.5)$$

where  $Df$  denotes the derivative of  $f$ . Then  $c_{\gamma h} = \left[ E[Y_n^{(h)}]^{\gamma h} \right]^{1/(\gamma h)}$  can be numerically evaluated from (5.5) for fixed  $h$ .

**Theorem 1.** For a sequence of observations  $\{Y_n^{(h)}, n = 0, 1, \dots, N\}$  from a gamma-driven CAR(1) process, we have  $\hat{a}_N^{(h)} \rightarrow a$  a.s. and

$$\lim_{N \rightarrow \infty} P \left[ (-h) e^{-ah} k_N^{-1} (\hat{a}_N^{(h)} - a) c_{\alpha} \leq x \right] = G_{\alpha}(x),$$

where  $G_{\alpha}$  is as in (3.6),  $\alpha = \gamma h$ ,  $\hat{a}_N^{(h)}$  is defined in (3.7),  $k_N^{-1}$  is given in (5.4), and  $c_{\alpha}$  is evaluated through (5.5).

**Proof.** At the beginning of Section 3, we have shown that  $Y_n^{(h)}$  is a stationary discrete-time AR(1) with autoregression coefficient  $\phi \in (0, 1)$  and i.i.d. noise sequence  $\{Z_n\}$ . According

to (5.3), the distribution function  $F$  of  $Z_n$  is regularly varying at zero with exponent  $\alpha = \gamma h$  and satisfies the condition  $F(0) = 0$ . Since  $0 \leq Z_n \leq \sigma(L(nh) - L((n-1)h))$ ,  $\int u^\xi F(du) < \infty$  for all  $\xi > 0$ . By Corollary 2.4 of Davis and McCormick (1989), we have  $\hat{\phi}_N^{(h)} \rightarrow \phi$  a.s., which implies  $\hat{a}_N^{(h)} \rightarrow a$  a.s.. From the same corollary, we also conclude that

$$\lim_{N \rightarrow \infty} P \left[ k_N^{-1} (\hat{\phi}_N^{(h)} - \phi) c_{\gamma h} \leq x \right] = G_{\gamma h}(x),$$

where  $\hat{\phi}_N^{(h)}$  and  $k_N^{-1}$  are given in (3.4) and (5.4) respectively, and  $c_{\gamma h}$  is evaluated through (5.5). Using a Taylor series expansion, we find from this result that

$$\lim_{N \rightarrow \infty} P \left[ (-h) e^{-ah} k_N^{-1} (\hat{a}_N^{(h)} - a) c_{\gamma h} \leq x \right] = G_{\gamma h}(x). \quad \square$$

Theorem 1 gives the limiting distribution of  $\hat{\phi}_N^{(h)}$  for fixed  $h$  as  $N \rightarrow \infty$ . It is of interest also to consider the behaviour of the estimator as  $h$  also goes to zero. For any non-negative random variable  $Y$  with density function  $f(u)$ , we have

$$\begin{aligned} [EY^s]^{1/s} &= \left[ \int_0^\infty u^s f(u) du \right]^{1/s} = \left[ 1 + s \int_0^\infty u^{s-1} f(u) du \right]^{1/s} \\ &\rightarrow \exp \left( \int_0^\infty u^{-1} f(u) du \right) = \exp(EY^{-1}) \quad \text{as } s \rightarrow 0, \end{aligned}$$

as long as  $EY^{-1}$  is finite. Applying this result to  $Y_n^{(h)}$  and using Theorem 2.1 of Brockwell and Brown (1978), we obtain

$$\begin{aligned} \lim_{h \rightarrow 0} c_{\gamma h} &= \exp(E(Y_n^{(h)})^{-1}) = \exp \left( \int_0^\infty \tilde{f}_{Y_n^{(h)}}(s) ds \right) \\ &= \exp \left( \int_0^\infty e^{\frac{\gamma}{\alpha} \text{dilog}(1+\beta s \sigma)} ds \right). \end{aligned} \quad (5.6)$$

The behaviour of  $k_N^{-1}$ , defined in (5.4), is more complicated. Using L'Hospital's Rule, we have

$$\lim_{s \rightarrow 0} -\frac{\log \Gamma(s+1)}{s} = -\lim_{s \rightarrow 0} \frac{\Gamma'(s+1)}{\Gamma(s+1)} = -\Gamma'(1) = \gamma_E,$$

where  $\gamma_E$  is the Euler-Mascheroni constant, with numerical value of  $0.5772 \dots$ . Hence  $\lim_{h \rightarrow 0} [\Gamma(\gamma h + 1)]^{-1/(\gamma h)} = e^{\gamma_E}$  and

$$k_N^{-1} \sim (\sigma \beta)^{-1} e^{\gamma_E} N^{1/(\gamma h)} \quad \text{as } N \rightarrow \infty \text{ and } h \rightarrow 0. \quad (5.7)$$

When  $h$  is small,  $k_N^{-1}$  and  $c_{\gamma h}$  can be well approximated by (5.6) and (5.7). Since the rate of convergence in Theorem 1, as indicated by  $k_N^{-1}$ , increases as  $h$  decreases and since the limiting distribution  $G_{\gamma h}$  becomes degenerate as  $h \rightarrow 0$ , this suggests the possibility of

super-convergence of  $\hat{a}_N^{(h)}$  to  $a$  as  $N \rightarrow \infty$  and  $h \rightarrow 0$ . In fact, in Section 6, we show that for any fixed  $T > 0$ ,  $\hat{a}_{T/h}^{(h)} \rightarrow a$  a.s. as  $h \rightarrow 0$ .

**Example 1.** We now illustrate the estimation procedure with a simulated example. The gamma-driven CAR(1) process defined by,

$$DY(t) + 0.6Y(t) = DL(t), \quad t \in [0, 5000], \quad (5.8)$$

was simulated at times  $0, 0.001, 0.002, \dots, 5000$ , using an Euler approximation. The parameter  $\gamma$  of the standardized gamma process was 2. The process was then sampled at intervals  $h = 0.01$ ,  $h = 0.1$  and  $h = 1$  by selecting every  $10^{\text{th}}$ ,  $100^{\text{th}}$  and  $1000^{\text{th}}$  value respectively. We generated 100 such realizations of the process and applied the above estimation procedure to generate 100 independent estimates, for each  $h$ , of the parameters  $a$  and  $\sigma$ . The sample means and standard deviations of these estimators are shown in Table 1, which illustrates the remarkable accuracy of the estimators.

**Table 1.** Estimated parameters based on 100 replicates on  $[0, 5000]$  of the gamma-driven CAR(1) process (5.8) with  $\gamma = 2$ , observed at times  $nh, n = 0, \dots, [T/h]$ .

Spacing	Parameter	Gamma increments	
		Sample mean of estimators	Sample std deviation of estimators
$h=1$	$a$	0.59269	0.00381
	$\sigma$	0.99796	0.01587
$h=0.1$	$a$	0.59999	0.00000
	$\sigma$	1.00011	0.01281
$h=0.01$	$a$	0.60000	0.00000
	$\sigma$	0.99990	0.01175

To estimate the parameter  $\gamma$  of the driving standardized gamma process, the following procedure was used. For each  $h$  and each realization, the estimated CAR(1) parameters were used in (4.2) to generate the estimated increments  $\Delta L_n^{(h)}$ ,  $n = 1, \dots, 5000/h$ . These were then added in blocks of length  $1/h$  to obtain 5000 independent estimated increments of  $L$  in one time unit. The histogram of the increments for one realization with  $h = .01$  is shown, together with the true probability density of  $L(1)$ , in Figure 1. Even if we did not know that the background driving Lévy process is a gamma process, the histogram strongly suggests that this is the case. For each  $h$  and for each realization of the process, the sample mean  $\hat{\gamma}$  of the estimated increments per unit time was then used to estimate the parameter  $\gamma$  of the driving standardized Lévy process, giving a set of 100 independent estimates of  $\gamma$  for each  $h$ . The sample means and standard deviations of these estimators

are shown in Table 2.

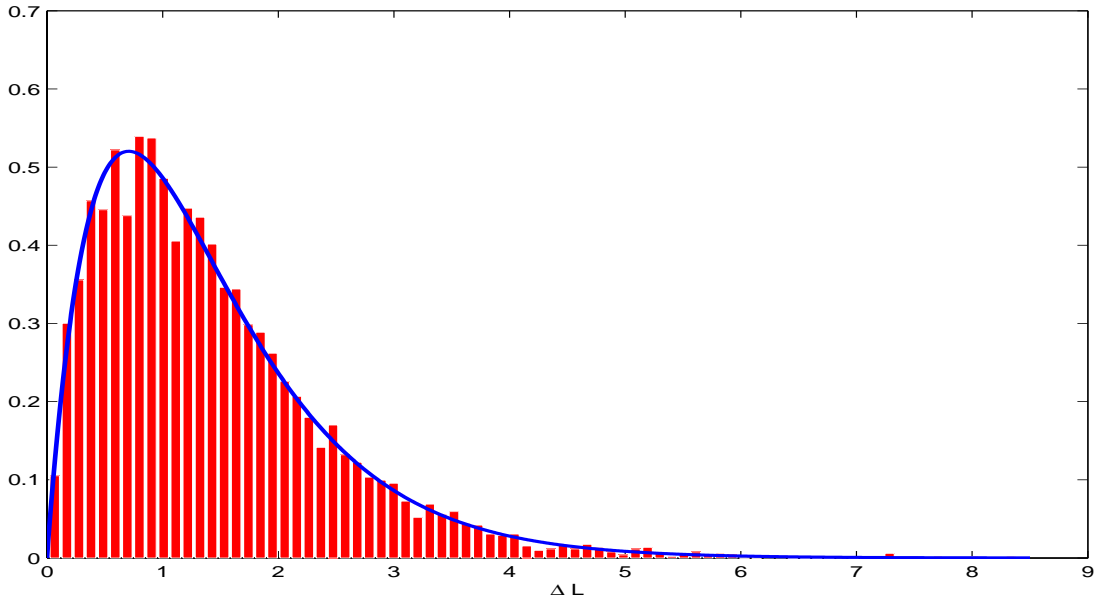


Figure 1: The probability density of the increments per unit time of the standardized Lévy process with  $\gamma = 2$  and the histogram of the estimated increments from a realization of the CAR(1) process (5.8), obtained by computing  $\Delta L_n^{(0.01)}$ ,  $n = 1, \dots, 500000$ , from (4.2) and adding successive values in blocks of 100 to give estimated increments per unit time.

**Table 2.** Estimated parameter of the standardized driving Lévy process.

Spacing	Parameter	Sample mean of estimators	Sample std deviation of estimators
$h = 1$	$\gamma$	1.99598	0.05416
$h = 0.1$	$\gamma$	2.00529	0.03226
$h = 0.01$	$\gamma$	2.00547	0.02762

## 6. Estimation for the continuously observed process.

It is interesting to note that from a continuously observed realization on  $[0, T]$  of a CAR(1) process driven by a non-decreasing Lévy process with drift  $m = 0$ , the value of  $a$  can be identified exactly with probability 1. This contrasts strongly with the case of a Gaussian CAR(1) process. The result is a corollary of the following theorem.

**Theorem 2.** If the CAR(1) process  $\{Y(t), t \geq 0\}$  defined by (2.1) is driven by a non-

decreasing Lévy process  $L$  with drift  $m$  and Lévy measure  $\nu$ , then for each fixed  $t$ ,

$$\frac{Y(t+h) - Y(t)}{h} + aY(t) \rightarrow m \text{ a.s. as } h \downarrow 0.$$

**Proof.** From (2.6) we find that

$$\begin{aligned} Y(t+h) - Y(t) &= Y(0)(e^{-a(t+h)} - e^{-at}) + \sigma(L(t+h) - L(t)) \\ &\quad - a\sigma \int_0^t e^{-a(t-u)}(e^{-ah} - 1)L(u)du - a\sigma \int_t^{t+h} e^{-a(t+h-u)}L(u)du. \end{aligned}$$

Dividing each side by  $h$ , letting  $h \downarrow 0$ , and using the fact that  $\lim_{h \downarrow 0} (L(t+h) - L(t))/h = m$  (Shtatland (1965)), we see that

$$\frac{Y(t+h) - Y(t)}{h} \rightarrow m - aY(0)e^{-at} + a^2\sigma \int_0^t e^{-a(t-u)}L(u)du - a\sigma L(t) = m - aY(t). \quad \square$$

**Corollary 1.** If  $m = 0$  in Theorem 2 (this is the case if the point zero belongs to the closure of the support of  $L(1)$ ), then for each fixed  $t$ , with probability 1,

$$a = \lim_{h \downarrow 0} \frac{\log Y(t) - \log Y(t+h)}{h}. \quad (6.1)$$

For each fixed  $T > 0$ ,  $a$  is also expressible, with probability 1, as

$$a = \sup_{0 \leq s < t \leq T} \frac{\log Y(s) - \log Y(t)}{t - s}. \quad (6.2)$$

**Proof.** By setting  $L(t) = 0$  for all  $t$  in the defining equation (2.1) we obtain the inequality, for all  $s$  and  $t$  such that  $0 \leq s < t \leq T$ ,

$$\log Y(s) - \log Y(t) < a(t - s),$$

from which it follows that

$$a \geq \sup_{0 \leq s < t \leq T} \frac{\log Y(s) - \log Y(t)}{t - s}. \quad (6.3)$$

From Theorem 2 with  $m = 0$  we find that

$$\frac{Y(t) - Y(t+h)}{hY(t)} \rightarrow a \text{ as } h \downarrow 0.$$

From the inequalities (6.3) and  $1 - x \leq -\log x$  for  $0 < x \leq 1$ , we obtain the inequalities,

$$\frac{Y(t) - Y(t+h)}{hY(t)} \leq \frac{\log Y(t) - \log Y(t+h)}{h} \leq a,$$

and letting  $h \downarrow 0$  gives (6.1). But this implies that

$$a \leq \sup_{0 \leq s < t \leq T} \frac{\log Y(s) - \log Y(t)}{t - s},$$

which, with (6.3), gives (6.2). □

**Remark 3.** If observations are available only at times  $\{nh : n = 0, 1, 2, \dots, [T/h]\}$ , and if the driving Lévy process has zero drift, Corollary 1 suggests the estimator,

$$\hat{a}_T^{(h)} = \sup_{0 \leq n < [T/h]} \frac{\log Y(nh) - \log Y((n+1)h)}{h}.$$

This estimator is precisely the same as the estimator (3.7). Its remarkable accuracy has already been illustrated in Table 1. The analogous estimator, based on closely but irregularly spaced observations at times  $t_1, t_2, \dots, t_N$  such that  $0 \leq t_1 < t_2 < \dots < t_N \leq T$ , is

$$\hat{a}_T = \sup_n \frac{\log Y(t_n) - \log Y(t_{n+1})}{t_{n+1} - t_n}.$$

By Corollary 1, both estimators converge almost surely to  $a$  as the maximum spacing between successive observations converges to zero.

## 7. Conclusions

In Section 3 of this paper, we developed a highly efficient method, based on observations at times  $0, h, 2h, \dots, Nh$ , for estimating the parameters of a stationary Ornstein-Uhlenbeck process  $\{Y(t)\}$  driven by a non-decreasing Lévy process  $\{L(t)\}$ . If  $h$  is small, we used a discrete approximation to the exact integral representation of  $L(t)$  in terms of  $\{Y(s), s \leq t\}$  to estimate the increments of the driving Lévy process, and hence to estimate the parameters of the Lévy process. Under specified conditions on the driving Lévy process we obtained the asymptotic distribution of the estimator of the CAR(1) coefficient as  $N \rightarrow \infty$  with  $h$  fixed. The accuracy of the procedure was illustrated with a simulated example of a gamma-driven process. We also showed that the CAR(1) coefficient  $a$  is determined almost surely by a continuously observed realization of  $Y$  on any interval  $[0, T]$ . The expression for  $a$  suggests an estimator based on discrete observations of  $Y$  which, for uniformly spaced observations, is the same as the estimator developed in Section 3. The generalization of the procedure to non-negative Lévy-driven continuous-time ARMA processes is currently in progress.

## Acknowledgments

The authors are indebted to the National Science Foundation for support of this work under Grant DMS-0308109 and PB for the additional support of Deutsche Forschungsgemeinschaft, SFB 386, at Technische Universität München.

## References

- Barndorff-Nielsen, O.E. and Shephard, N. (2001). Non-Gaussian Ornstein-Uhlenbeck based models and some of their uses in financial economics (with discussion). *J. Roy. Statist. Soc. Ser. B*, **63**, 167–241.
- Brockwell, P. J. and Brown, B.M. (1978). Expansions for the positive stable laws. *Z. Wahrsch. Verw. Gebiete* **45**, 213–224.
- Brockwell, P.J. (2001). Lévy-driven CARMA processes. *Ann. Inst. Stat. Mat.* **53**, 113–124.
- Brockwell, P.J. and Marquardt, T. (2005). Fractionally integrated continuous-time ARMA processes. *Statistica Sinica* **15**, 477–494.
- Davis, R. and McCormick, W. (1989). Estimation for first-order autoregressive processes with positive and bounded innovations, *Stochastic Process. Appl.* **31**, 273–250.
- Doetsch, G. (1974). *Introduction to the theory and application of the Laplace transformation*, Springer-Verlag, New York.
- Eberlein, E. and Raible, S. (1999). Term structure models driven by general Lévy processes, *Mathematical Finance* **9**, 31–53.
- Pham-Din-Tuan (1977). Estimation of parameters of a continuous-time Gaussian stationary process with rational spectral density function. *Biometrika* **64**, 385–399.
- Protter, P.E. (2004). *Stochastic Integration and Differential Equations*. 2nd edition, Springer, New York.
- Shtatland, E.S. (1965). On local properties of processes with independent increments. *Theory of Prob. and Applications* **10**, 317–322.