Limit Theory for Some Non-Linear Time Series Models Including GARCH and SV

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### Outline

- **Characteristics of some financial time series**
  - IBM returns
- **Linear processes**
- **Background results**
  - Multivariate regular variation
  - Point process convergence
  - Application to sample ACVF and ACF
- **Applications**
  - Stochastic recurrence equations (GARCH)
  - Stochastic volatility models
- **More on multivariate regular variation (Mikosch)**
Characteristics of Some Financial Time Series

Define $X_t = 100*(\ln (P_t) - \ln (P_{t-1}))$  (log returns)

• heavy tailed

$$P(|X_1| > x) \sim C x^{-\alpha}, \quad 0 < \alpha < 4.$$  

• uncorrelated

$$\hat{\rho}_x (h) \text{ near } 0 \text{ for all lags } h > 0 \text{ (MGD sequence)}$$

• $|X_t|$ and $X_t^2$ have slowly decaying autocorrelations

$$\hat{\rho}_{|X|} (h) \text{ and } \hat{\rho}_x^2 (h) \text{ converge to } 0 \text{ slowly as } h \text{ increases.}$$

• process exhibits ‘stochastic volatility’.
Sample ACF IBM (a) 1962-1981, (b) 1982-2000

(a) ACF of IBM (1st half)

(b) ACF of IBM (2nd half)
Sample ACF of abs values for IBM (a) 1961-1981, (b) 1982-2000

(a) ACF, Abs Values of IBM (1st half)

(b) ACF, Abs Values of IBM (2nd half)
Sample ACF of squares for IBM (a) 1961-1981, (b) 1982-2000

(a) ACF, Squares of IBM (1st half)

(b) ACF, Squares of IBM (2nd half)
Sample ACF of original data and squares for IBM 1962-2000
Plot of $M_t(4)/S_t(4)$ for IBM

![Graph of $M_t(4)/S_t(4)$ for IBM](image-url)
500-daily log-returns of NZ/US exchange rate
ACF of log-returns of NZ/US exchange rate

ACF of log-returns of NZ/US exchange rate

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Plot of $M_t(4)/S_t(4)$
Hill’s plot of tail index
Models for Log(returns)

Basic model

\[ X_t = 100 \times (\ln (P_t) - \ln (P_{t-1})) \quad \text{(log returns)} \]
\[ = \sigma_t Z_t, \]

where

- \( \{Z_t\} \) is IID with mean 0, variance 1 (if exists). (e.g. N(0,1) or a \( t \)-distribution with \( \nu \) df.)
- \( \{\sigma_t\} \) is the volatility process
- \( \sigma_t \) and \( Z_t \) are independent.
Models for Log(returns)-cont

\[ X_t = \sigma_t Z_t \]  
(\text{observation eqn in state-space formulation})

Examples of models for volatility:

(i) GARCH(p,q) process (observation-driven specification)

\[ \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_p X_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_q \sigma_{t-q}^2 . \]

Special case: ARCH(1),  \[ X_t^2 = (\alpha_0 + \alpha_1 X_{t-1}^2) Z_t^2. \]

\[ \rho_{X^2}(h) = \alpha_1^h, \text{ if } \alpha_1^2 < 1/3. \]

(ii) stochastic volatility process (parameter-driven specification)

\[ \log \sigma_t^2 = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}, \quad \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty, \{\varepsilon_t\} \sim \text{IID } N(0, \sigma^2) \]

\[ \rho_{X^2}(h) = Cor(\sigma_t^2, \sigma_{t+h}^2) / EZ_1^4 \]
Linear Processes

Model: \( X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j} \{Z_t\} \sim \text{IID}, \ P(|Z_t|>x) \sim Cx^{-\alpha}, \ 0<\alpha<2. \)

Properties:

- \( P(|X_t|>x) \sim C_2 x^{-\alpha} \)
- Define \( \rho(h) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} / \sum_{j=-\infty}^{\infty} \psi_j^2 \).

Case \( \alpha > 2: \)
\[
n^{1/2} (\hat{\rho}(h) - \rho(h)) \xrightarrow{d} \sum_{j=1}^{\infty} (\rho(h+j)+\rho(h-j)-2\rho(j)\rho(h)) N_j, \ {N_t} \sim \text{IIDN} \]

Case \( 0 < \alpha < 2: \)
\[
(n / \ln n)^{1/\alpha} (\hat{\rho}(h) - \rho(h)) \xrightarrow{d} \sum_{j=1}^{\infty} (\rho(h+j)+\rho(h-j)-2\rho(j)\rho(h)) S_j / S_0, \ {S_t} \sim \text{IID stable (\alpha)}, \ S_0 \text{ stable (\alpha/2)} \]
Background Results—multivariate regular variation

Multivariate regular variation of $X=(X_1, \ldots, X_m)$: There exists a random vector $\theta \in S^{m-1}$ such that

$$P(|X| > t \ x, X/|X| \in \bullet) / P(|X| > t) \rightarrow_v x^{-\alpha} P(\theta \in \bullet)$$

($\rightarrow_v$ vague convergence on $S^{m-1}$).

- $P(\theta \in \bullet)$ is called the spectral measure
- $\alpha$ is the index of $X$.

Equivalence: There exist positive constants $a_n$ and a measure $\mu$,

$$nP(X/ a_n \in \bullet) \rightarrow_v \mu(\bullet)$$

In this case, one can choose $a_n$ and $\mu$ such that

$$\mu((x, \infty) \times B) = x^{-\alpha} P(\theta \in B)$$
Another equivalence?

MRV ⇔ all linear combinations of $X$ are regularly varying

i.e., if and only if

$$P(c^T X > t)/P(1^T X > t) \to w(c),$$

exists for all real-valued $c$, in which case,

$$w(tc) = t^{-\alpha}w(c).$$

(⇒) true

(⇐) established by Basrak, Davis and Mikosch (2000) for $\alpha$ not an even integer—case of even integer is unknown.
Background Results—point process convergence

**Theorem** (Davis & Hsing `95, Davis & Mikosch `97). Let \( \{X_t\} \) be a stationary sequence of random \( m \)-vectors. Suppose

(i) finite dimensional distributions are jointly regularly varying (let \((\theta_{-k}, \ldots, \theta_k)\) be the vector in \( S^{(2k+1)m-1} \) in the definition).

(ii) mixing condition \( A (a_n) \) or strong mixing.

(iii) \( \lim \limsup_{k \to \infty, n \to \infty} P( \bigvee_{k \leq |t| \leq r_n} |X_t| > a_n y \mid |X_0| > a_n y) = 0. \)

Then

\[
\gamma = \lim_{k \to \infty} E(\| \theta_0^{(k)} \|^\alpha - \bigvee_{j=1}^k |\theta_j^{(k)}|_+) / E |\theta_0^{(k)}|^\alpha
\]

exists. If \( \gamma > 0 \), then

\[
N_n := \sum_{t=1}^n \varepsilon_{X_t/a_n} \xrightarrow{d} N := \sum_{i=1}^\infty \sum_{j=1}^\infty \varepsilon_{P_i Q_{ij}},
\]
where

- \((P_i)\) are points of a Poisson process on \((0,\infty)\) with intensity function \(\nu(dy) = \gamma \alpha y^{-\alpha-1}dy\).
- \(\sum_{j=1}^{\infty} \varepsilon_{Q_{ij}}\), \(i \geq 1\), are iid point process with distribution \(Q\), and \(Q\) is the weak limit of

\[
\lim_{k \to \infty} E\left( \left| \theta_0^{(k)} \right|^{\alpha} - \bigvee_{j=1}^{k} \left| \theta_j^{(k)} \right| \right) + I \left( \sum_{|t| \leq k} \varepsilon_{\theta^{(k)}_{t}} \right) / E\left( \left| \theta_0^{(k)} \right|^{\alpha} - \bigvee_{j=1}^{k} \left| \theta_j^{(k)} \right| \right) +
\]
Background Results—application to ACVF & ACF

Set-up: Let \( \{X_t\} \) be a stationary sequence and set
\[
X_t = X_t(m) = (X_t, \ldots, X_{t+m}).
\]
Suppose \( X_t \) satisfies the conditions of previous theorem. Then
\[
N_n := \sum_{t=1}^{n-1} \epsilon_{X_t} / a_n \quad \text{d} \rightarrow N := \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \epsilon_{P_iQ_{ij}},
\]

Sample ACVF and ACF:
\[
\hat{\gamma}_X(h) = n^{-1} \sum_{t=1}^{n-h} X_t X_{t+h}, \quad h \geq 0, \quad \text{ACVF}
\]
\[
\hat{\rho}_X(h) = \hat{\gamma}_X(h) / \hat{\gamma}_X(0), \quad h \geq 1, \quad \text{ACF}
\]
If \( E X_0^2 < \infty \), then define \( \gamma_X(h) = E X_0 X_h \) and \( \rho_X(h) = \gamma_X(h) / \gamma_X(0) \).
Background Results—application to ACVF & ACF

(i) If \( \alpha \in (0,2) \), then

\[
\left( n a_n^{-2} \hat{\gamma}_X (h) \right)_{h=0,\ldots,m} \xrightarrow{d} (V_h)_{h=0,\ldots,m} \\
\left( \hat{\rho}_X (h) \right)_{h=1,\ldots,m} \xrightarrow{d} (V_h / V_0)_{h=1,\ldots,m},
\]

where

\[
V_h = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P_i^2 Q_{ij}^{(0)} Q_{ij}^{(h)}, \quad h = 0,\ldots,m. \quad \text{(jointly stable)}
\]

(ii) If \( \alpha \in (2,4) + \) additional condition, then

\[
\left( n a_n^{-2} (\hat{\gamma}_X (h) - \gamma_X (h)) \right)_{h=0,\ldots,m} \xrightarrow{d} (V_h)_{h=0,\ldots,m} \\
\left( n a_n^{-2} (\hat{\rho}_X (h) - \rho_X (h)) \right)_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1} (0) \left( V_h - \rho_X (h) V_0 \right)_{h=1,\ldots,m}.
\]
Applications—stochastic recurrence equations

\[ Y_t = A_t Y_{t-1} + B_t, \quad (A_t, B_t) \sim \text{IID}, \]

\( A_t \) \( d \times d \) random matrices, \( B_t \) random \( d \)-vectors

Examples

ARCH(1): \( X_t = (\alpha_0 + \alpha_1 X_{t-1}^2)^{1/2} Z_t, \quad \{Z_t\} \sim \text{IID}. \) Then the squares follow an SRE with \( Y_t = X_t^2, \quad A_t = \alpha_1 Z_{t-1}^2, \quad B_t = \alpha_0 Z_t^2. \)

GARCH(2,1): \( X_t = \sigma_t Z_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \alpha_2 X_{t-2}^2 + \beta_1 \sigma_{t-1}^2. \) Then \( Y_t = (\sigma_t^2, X_{t-1}^2)' \) follows the SRE given by

\[
\begin{bmatrix}
\sigma_t^2 \\
X_{t-1}^2
\end{bmatrix}
= \begin{bmatrix}
\alpha_1 Z_{t-1}^2 + \beta_1 & \alpha_2 \\
Z_{t-1}^2 & 0
\end{bmatrix}
\begin{bmatrix}
\sigma_{t-1}^2 \\
X_{t-2}^2
\end{bmatrix}
+ \begin{bmatrix}
\alpha_0 \\
0
\end{bmatrix}
\]
Examples (tricks)

GARCH(1,1): \( X_t = \sigma_t Z_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \).

Although this process does not have a 1-dimensional SRE representation, the process \( \sigma_t^2 \) does. Iterating, we have

\[
\begin{align*}
\sigma_t^2 &= \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \\
&= \alpha_0 + \alpha_1 \sigma_{t-1}^2 Z_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \\
&= (\alpha_1 Z_{t-1}^2 + \beta_1) \sigma_{t-1}^2 + \alpha_0.
\end{align*}
\]

Bilinear(1): \( X_t = aX_{t-1} + bX_{t-1}Z_{t-1} + Z_t, \quad \{Z_t\} \sim \text{iid} \)

\[
= Y_{t-1} + Z_t,
\]

\[
Y_t = A_t Y_{t-1} + B_t, \quad A_t = a + bZ_t, \quad B_t = A_t Z_t
\]
Stochastic Recurrence Equations (cont)

\[ Y_t = A_t Y_{t-1} + B_t, \quad (A_t, B_t) \sim \text{IID} \]

Existence of stationary solution

- \( E \ln^+ \| A_1 \| < \infty \)
- \( E \ln^+ \| B_1 \| < \infty \)
- \( \inf n^{-1} E \ln \| A_1 \ldots A_n \| =: \gamma < 0 \) (\( \gamma \) – Lyapunov exponent)

Ex. (d=1) \( E \ln |A_1| < 0 \).

Strong mixing

If \( E \| A_1 \|^\varepsilon < \infty, E |B_1|^\varepsilon < \infty \) for some \( \varepsilon > 0 \), then the SRE \((Y_t)\) is geometrically ergodic \(\Rightarrow\) strong mixing with geometric rate (Meyn and Tweedie `93).
Regular variation of the marginal distribution (Kesten)

Assume $A$ and $B$ have non-negative entries and

- $\mathbb{E} \|A_1\|^\varepsilon < 1$ for some $\varepsilon > 0$
- $A_1$ has no zero rows a.s.
- W.P. 1, $\{\ln \rho(A_1 \ldots A_n):$ is dense in $\mathbb{R}$ for some $n, A_1 \ldots A_n > 0\}$
- There exists a $\kappa_0 > 0$ such that $\mathbb{E} \|A\|^\kappa_0 \ln^+ \|A\| < \infty$ and
  $$\mathbb{E} \left( \min_{i=1, \ldots, d} \sum_{j=1}^d A_{ij} \right)^{\kappa_0} \geq d^{\kappa_0/2}$$

Then there exists a $\kappa_1 \in (0, \kappa_0]$ such that all linear combinations of $Y$ are regularly varying with index $\kappa_1$. (Also need $\mathbb{E} |B|^{\kappa_i} < \infty$.)
Application to GARCH

**Proposition:** Let \((Y_t)\) be the soln to the SRE based on the *squares* of a GARCH model. Assume

- Lyapunov exponent \(\gamma < 0.\) *(See Bougerol and Picard`92)*
- \(Z\) has a positive density on \((-\infty, \infty)\) with all moments finite or \(E|Z|^h = \infty, \) for all \(h \geq h_0\) and \(E|Z|^h < \infty\) for all \(h < h_0.\)
- Not all the GARCH parameters vanish.

Then \((Y_t)\) is *strongly mixing* with geometric rate *(Boussama `98)* and all finite dimensional distributions are *multivariate regularly varying* with index \(\kappa_1.\)

**Corollary:** The corresponding GARCH process is strongly mixing and has all finite dimensional distributions that are MRV with index \(\kappa = 2\kappa_1.\)
Application to GARCH (cont)

Remarks:
1. Kesten’s result applied to an iterate of $Y_t$, i.e., $Y_{tm} = \tilde{A}_t Y_{(t-1)m} + \tilde{B}_t$

2. Determination of $\kappa$ is difficult. Explicit expressions only known in two(?) cases.
   
   • ARCH(1): $E|\alpha_1 Z^2|^{\kappa/2} = 1.$
     
     | $\alpha_1$ | .312 | .577 | 1.00 | 1.57 |
     | $\kappa$   | 8.00 | 4.00 | 2.00 | 1.00 |

   • GARCH(1,1): $E|\alpha_1 Z^2 + \beta_1|^{\kappa/2} = 1$ (Mikosch and Stãricã)

   • For IGARCH ($\alpha_1 + \beta_1 = 1$), then $\kappa = 2 \Rightarrow$ infinite variance.

   • Can estimate $\kappa$ empirically by replacing expectations with sample moments.
Summary for GARCH(p,q)

κ ∈ (0,2):
\[
\left( \hat{\rho}_X(h) \right)_{h=1,\ldots,m} \xrightarrow{d} \left( V_h / V_0 \right)_{h=1,\ldots,m},
\]

κ ∈ (2,4):
\[
\left( n^{1-2/\kappa} \hat{\rho}_X(h) \right)_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1}(0) \left( V_h \right)_{h=1,\ldots,m}.
\]

κ ∈ (4,∞):
\[
\left( n^{1/2} \hat{\rho}_X(h) \right)_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1}(0) \left( G_h \right)_{h=1,\ldots,m}.
\]

Remark: Similar results hold for the sample ACF based on |X_t| and X_t^2.
Results for IBM Data

Fitted GARCH(1,1) model IBM data with t-noise:

<table>
<thead>
<tr>
<th></th>
<th>1st half:</th>
<th>2nd half:</th>
<th>both:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$X_t = \sigma_t Z_t$, $\sigma_t^2 = 0.0342 + 0.0726 \sigma_{t-1}^2 + 0.9107 \sigma_{t-1}^2$, df = 7.185, $\kappa = 4.56$</td>
<td>$X_t = \sigma_t Z_t$, $\sigma_t^2 = 0.0513 + 0.0472 \sigma_{t-1}^2 + 0.9364 \sigma_{t-1}^2$, df = 5.398, $\kappa = 4.16$</td>
<td>$X_t = \sigma_t Z_t$, $\sigma_t^2 = 0.0320 + 0.0557 X_{t-1}^2 + 0.9319 \sigma_{t-1}^2$, df = 6.089, $\kappa = 4.96$</td>
</tr>
</tbody>
</table>
Fitted GARCH(1,1) model for NZ-USA exchange:

\[ X_t = \sigma_t Z_t, \quad \sigma_t^2 = (6.70)10^{-7} + .1519X_{t-1}^2 + .772\sigma_{t-1}^2 \]

\[ (Z_t) \sim \text{IID } t\text{-distr with 5 df. } \kappa \text{ is approximately 3.8} \]
ACF of Fitted GARCH(1,1) Process

ACF of squares of realization 1

ACF of squares of realization 2
ACF of 2 realizations of an (ARCH)$^2$: $X_t = (0.001 + 0.7 X_{t-1})^{1/2} Z_t$
ACF of 2 realizations of an ARCH: \( X_t = (0.001 + X_{t-1})^{1/2} Z_t \)
Stochastic Volatility Models

SVM: \( X_t = \sigma_t Z_t \)

- \((Z_t) \sim \text{IID with mean 0 (if it exists)}\)
- \((\sigma_t)\) is a stationary process (\(2 \log \sigma_t\) is a linear process) given by

\[
\log \sigma_t^2 = \sum_{j=-\infty}^{\infty} \psi_j \epsilon_{t-j}, \quad \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty, (\epsilon_t) \sim \text{IID N}(0, \sigma^2)
\]

Heavy tails: Assume \(Z_t\) has Pareto tails with index \(\alpha\), i.e.,

\[
P(|Z_t| > z) \sim C z^{-\alpha} \Rightarrow P(|X_t| > z) \sim C E\sigma^\alpha z^{-\alpha}.
\]

Then if \(\alpha \in (0, 2)\),

\[
\left(\frac{n}{\ln n}\right)^{1/\alpha} \hat{\rho}_X(h) \xrightarrow{d} \frac{\|\sigma_1 \sigma_{h+1}\|_\alpha}{\|\sigma_1\|^2_\alpha} \frac{S_h}{S_0}.
\]
Other powers:

1. Absolute values: \( \alpha \in (1, 2) \),

\[
E|X_t| = E|\sigma_t E|Z_t|, \quad E|X_t X_{t+h}| = (E|\sigma_t \sigma_{t+h}|)(E|Z_t E|Z_{t+h} |
\]

\[
Cov(|X_t|, |X_{t+h}|) = Cov(\sigma_t, \sigma_{t+h})(E|Z|)^2
\]

\[
Cor(|X_t|, |X_{t+h}|) = Cor(\sigma_t, \sigma_{t+h})(E|Z|)^2/ EZ^2
= 0 (\text{?}).
\]

We obtain

\[
n(n \ln n)^{-1/\alpha} (\hat{\gamma}_{|X|}(h) - \gamma_{|X|}(h)) \xrightarrow{d} \|\sigma \sigma_{h+1}\|_\alpha S_h
\]

and

\[
(n / \ln n)^{1/\alpha} \hat{\rho}_{|X|}(h) \xrightarrow{d} \begin{pmatrix} \sigma_1 \sigma_{h+1} \|_\alpha S_h \\ \|\sigma_1\|_\alpha^2 S_0 \end{pmatrix}.
\]
2. Higher order: $\alpha \in (0,2)$

The squares are again a SV process and the results of the previous proposition apply. Namely,

\[
\left( \frac{n}{\ln n} \right)^{2/\alpha} \hat{\rho}_{X^2}(h) \xrightarrow{d} \frac{\left\| \sigma_1^2 \sigma_{h+1}^2 \right\|_{\alpha/2}}{\left\| \sigma_1^2 \right\|_{\alpha/2}^2} \frac{S_h}{S_0}.
\]

In particular,

\[
\hat{\rho}_{X^2}(h) \xrightarrow{p} 0.
\]
Stochastic Volatility Models (cont)

(log $X^2$) - mean for NZ-USA exchange rates
Stochastic Volatility Models (cont)

ACF/PACF for \( (\log X^2) \) suggests ARMA \((1,1)\) model:

\[ \mu = -11.5403, \quad Y_t = .9646 Y_{t-1} + \epsilon_t - .8709 \epsilon_{t-1}, \quad (\epsilon_t) \sim WN(0, 4.6653) \]
The ARMA (1,1) model for log $X^2$ leads to the SV model

$$X_t = \sigma_t Z_t$$

with

$$2 \ln \sigma_t = -11.5403 + v_t + \varepsilon_t$$

$$v_t = .9646 v_{t-1} + \gamma_t, \quad (\gamma_t) \sim \text{WN}(0,.07253)$$

$$(\varepsilon_t) \sim \text{WN}(0,4.2432).$$
Simulation of SVM model.

Took $Z_t$ to be distributed as a $t$ with 3 df (suitable normalized).

ACF: abs(realization)
Stochastic Volatility Models (cont)

$A C F: \ (r e a l i z a t i o n)^2$

$A C F: \ (r e a l i z a t i o n)^4$
Sample ACF for GARCH and SV Models (1000 reps)

(a) GARCH(1,1) Model, n=10000

(b) SV Model, n=10000
Sample ACF for Squares of GARCH and SV (1000 reps)

(a) GARCH(1,1) Model, n=10000

(b) SV Model, n=10000
Sample ACF for Squares of GARCH and SV (1000 reps)

(c) GARCH(1,1) Model, n=100000

(d) SV Model, n=100000

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