Limit Theory for Some Non-Linear Time Series Models Including GARCH and SV

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Outline

- ☞ Characteristics of some financial time series
  - IBM returns
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- ☞ Background results
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- ☞ Applications
  - Stochastic recurrence equations (GARCH)
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- ☞ More on multivariate regular variation (Mikosch)

Characteristics of Some Financial Time Series

Define $X_t = 100*(\ln(P_t) - \ln(P_{t-1}))$ (log returns)

- heavy tailed

$$P(|X_1| > x) \sim C\, x^{-\alpha}, \qquad 0 < \alpha < 4.$$

- uncorrelated

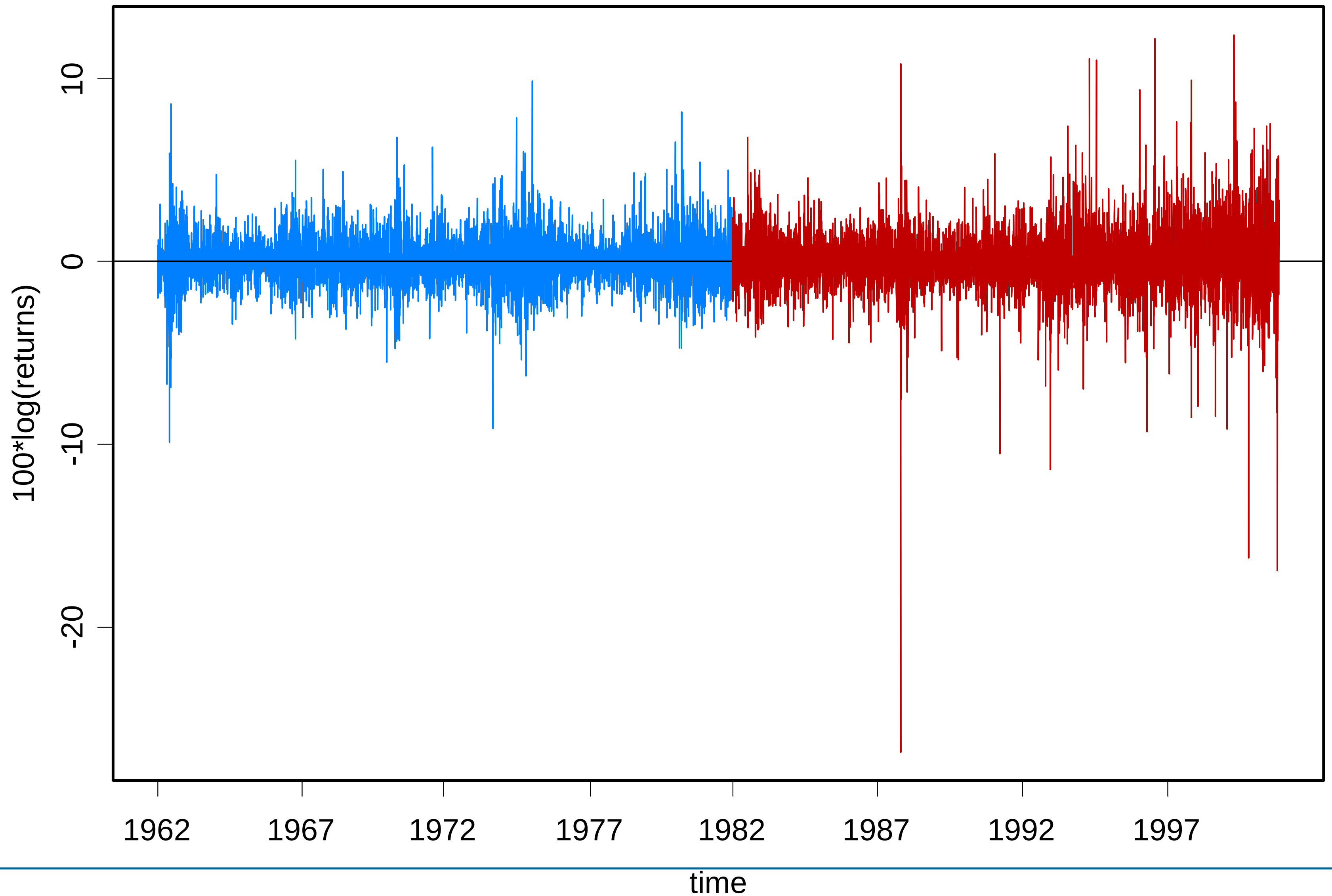
$\hat{\rho}_X(h)$ near 0 for all lags $h > 0$ (MGD sequence)

- $|X_t|$ and X_t^2 have slowly decaying autocorrelations

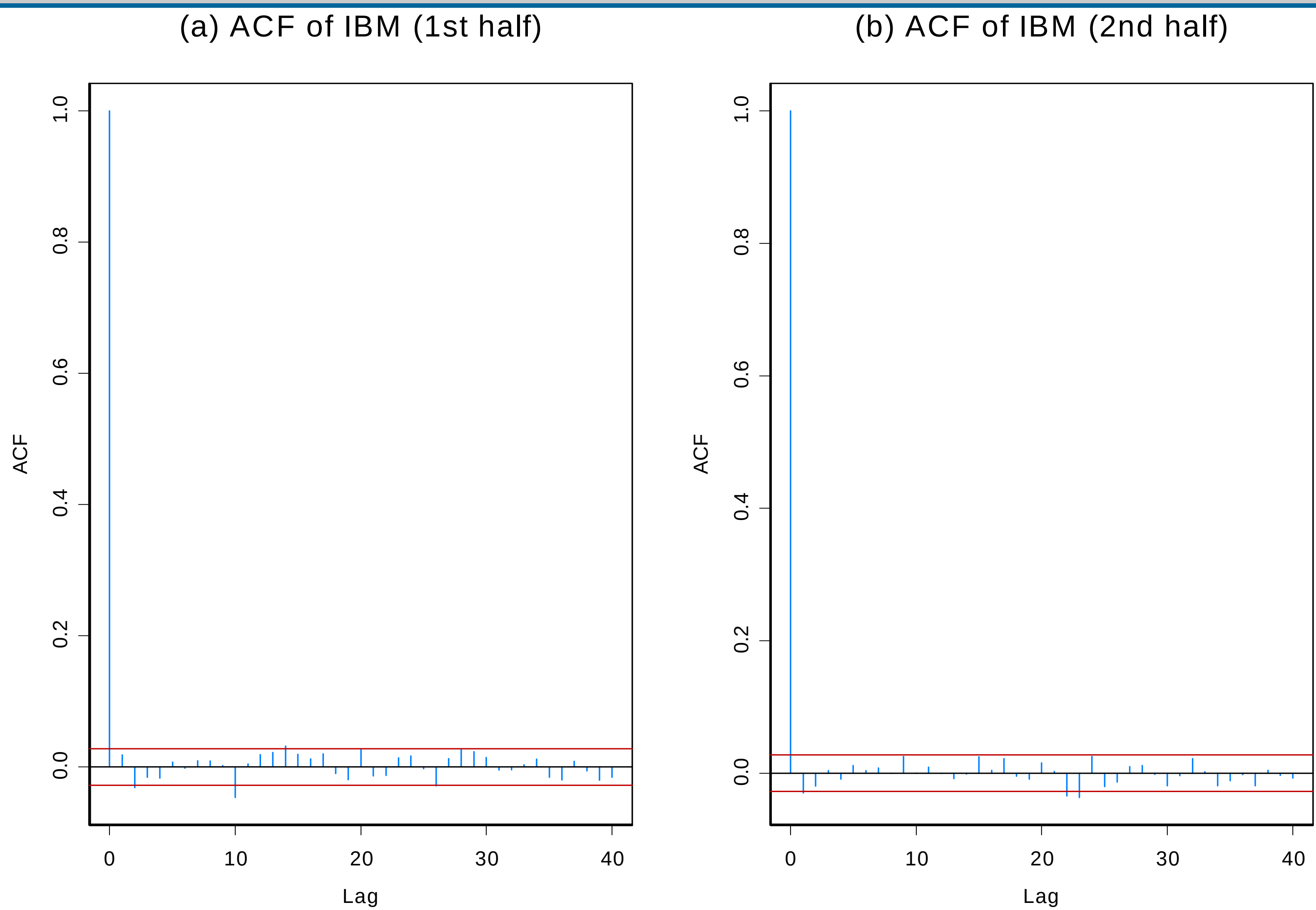
$\hat{\rho}_{|X|}(h)$ and $\hat{\rho}_{X^2}(h)$ converge to 0 *slowly* as h increases.

- process exhibits ‘stochastic volatility’.

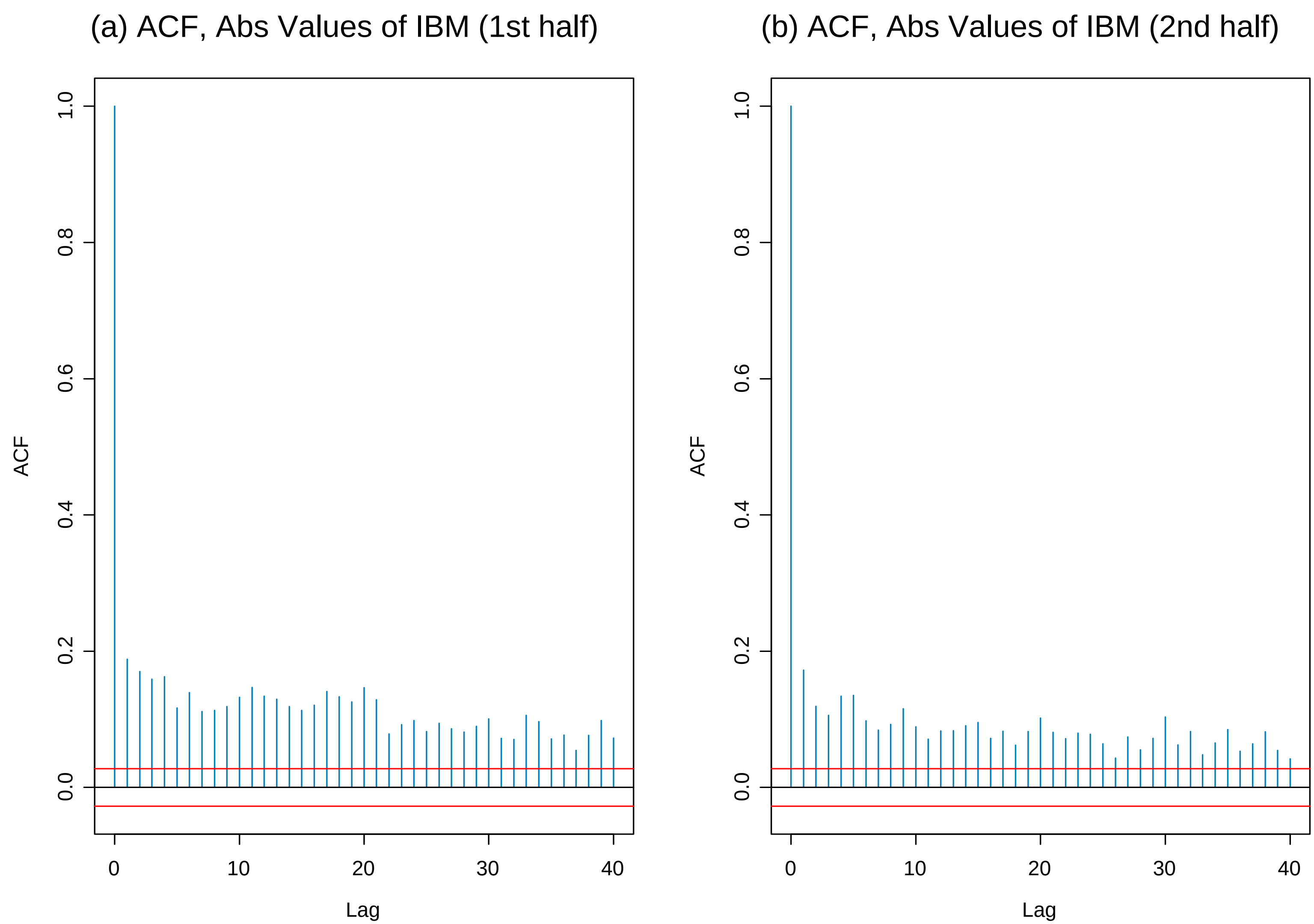
Log returns for IBM 1/3/62-11/3/00 (blue=1961-1981)



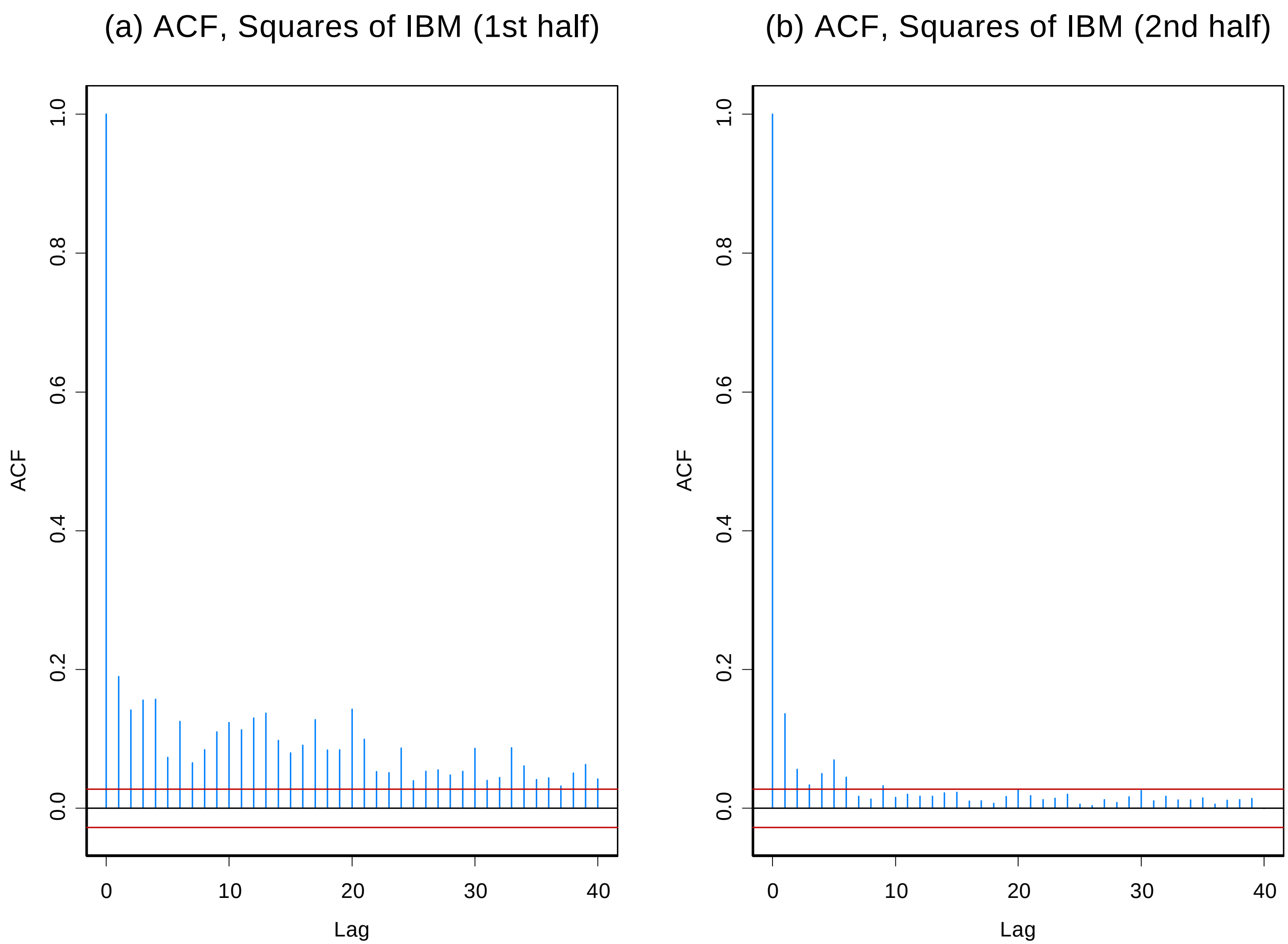
Sample ACF IBM (a) 1962-1981, (b) 1982-2000



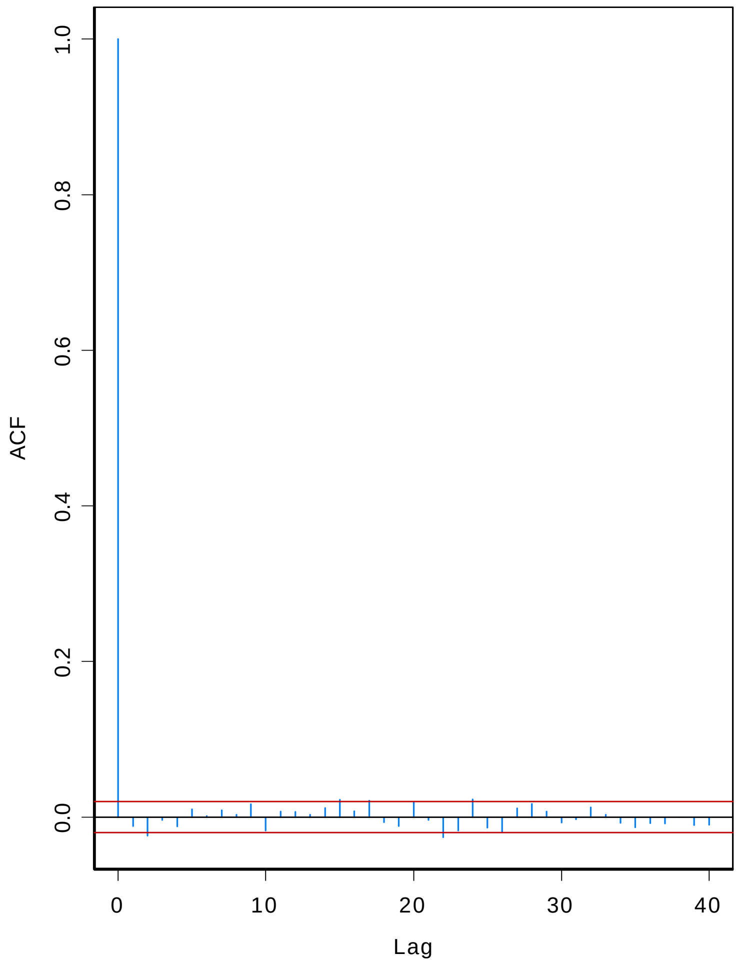
Sample ACF of abs values for IBM (a) 1961-1981, (b) 1982-2000

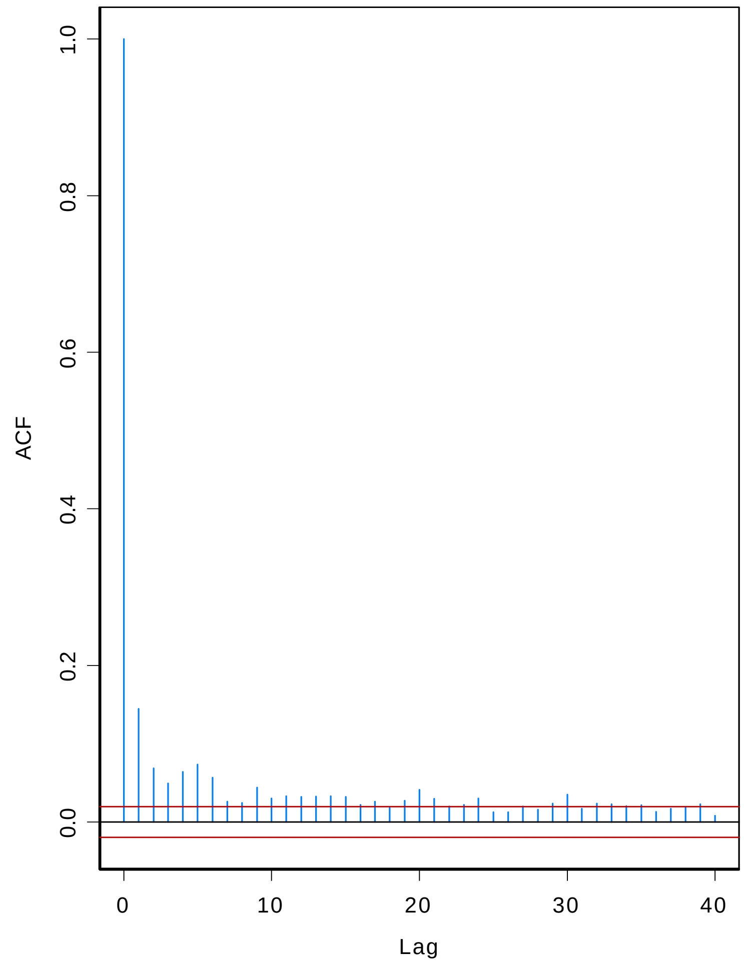


Sample ACF of squares for IBM (a) 1961-1981, (b) 1982-2000

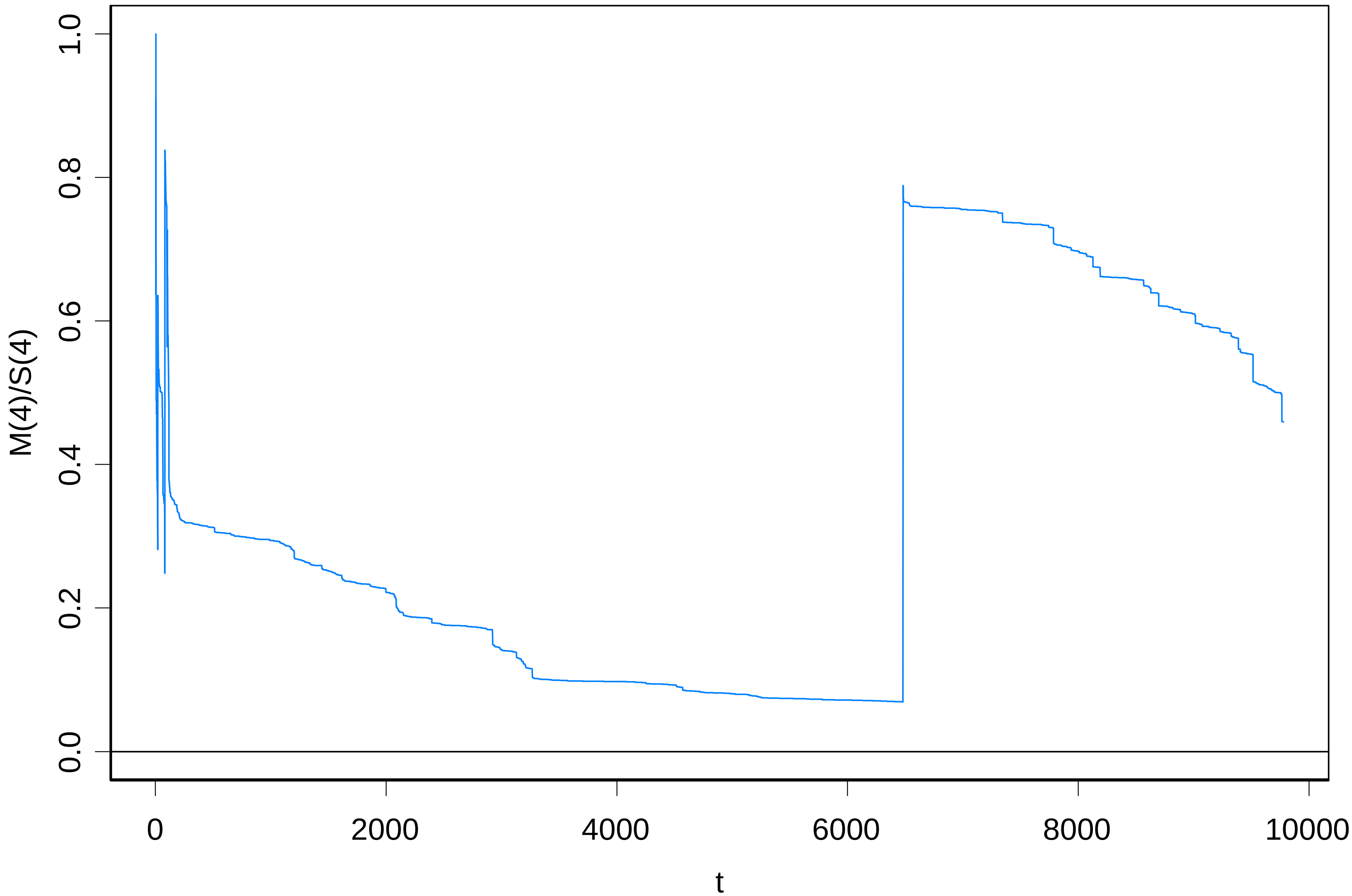


Sample ACF of original data and squares for IBM 1962-2000

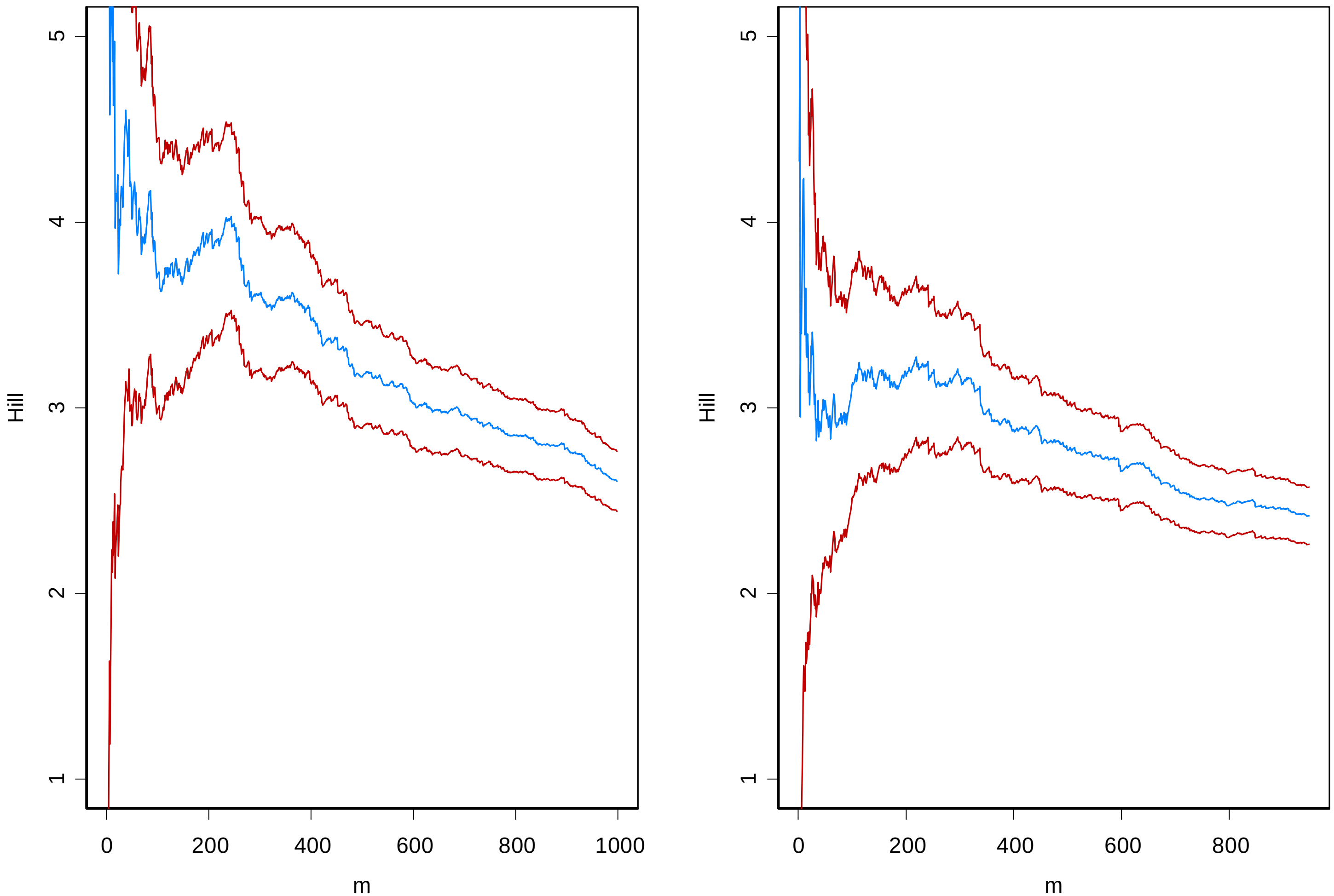




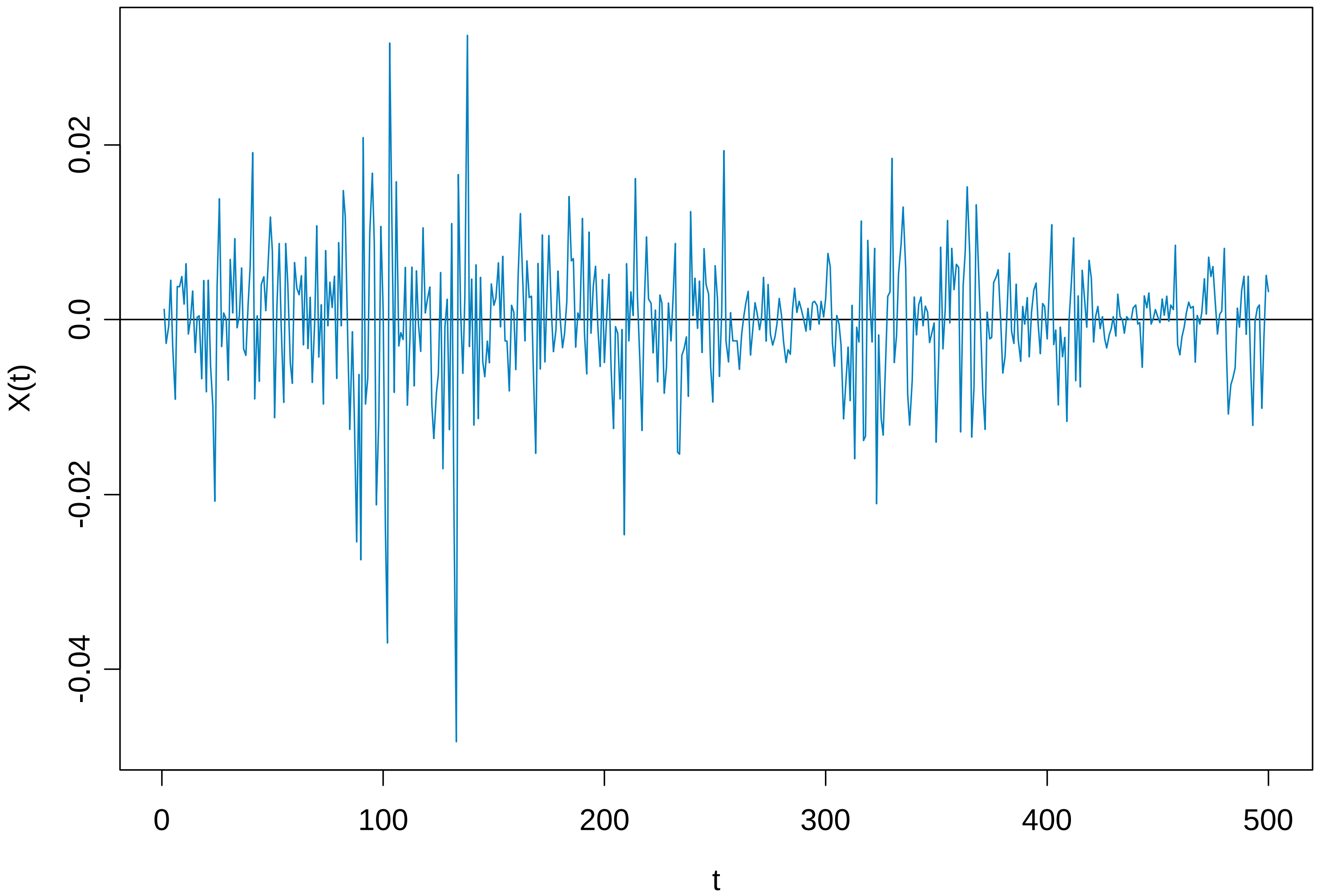
Plot of $M_t(4)/S_t(4)$ for IBM



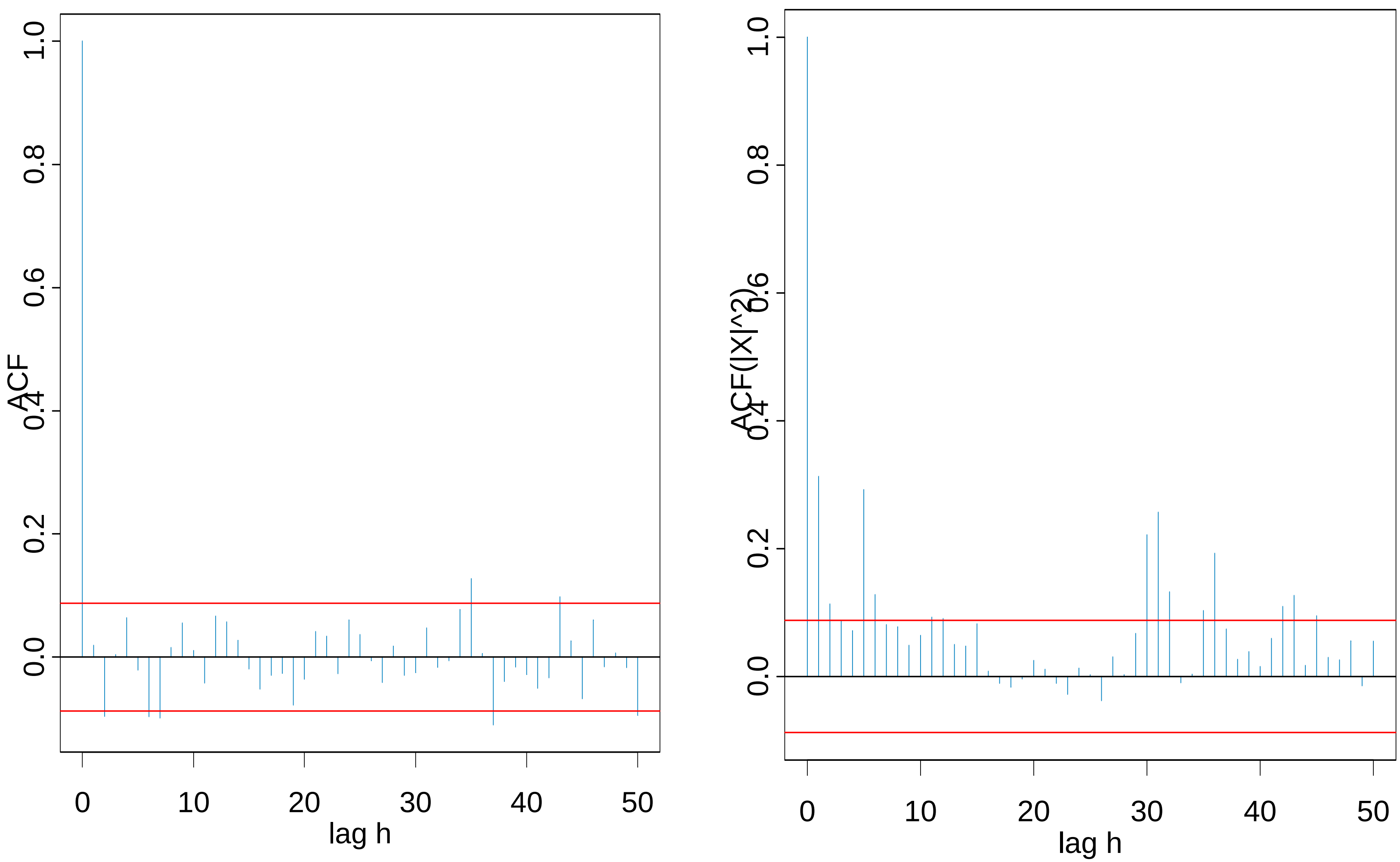
Hill's plot of tail index for IBM (1962-1981, 1982-2000)



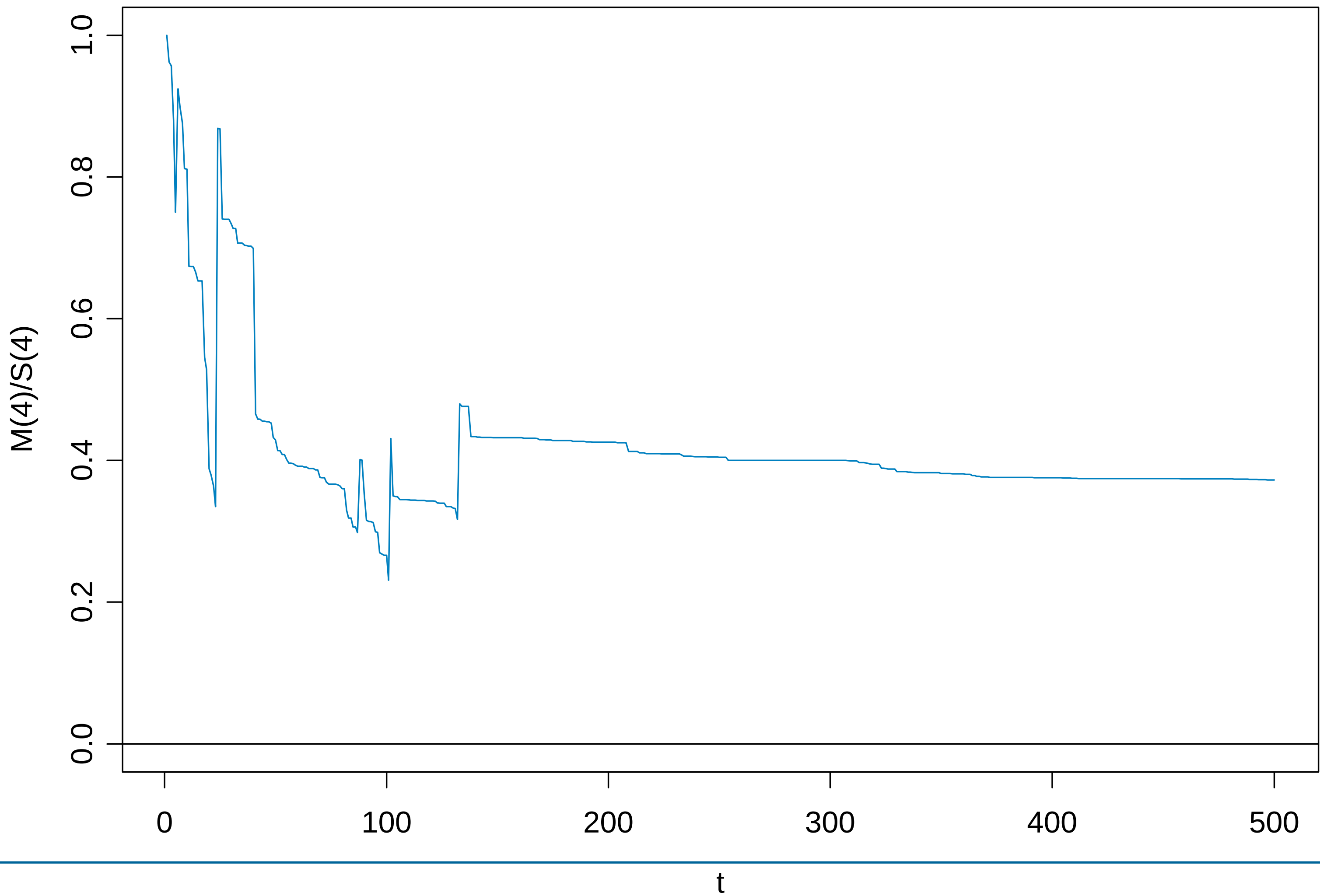
500-daily log-returns of NZ/US exchange rate



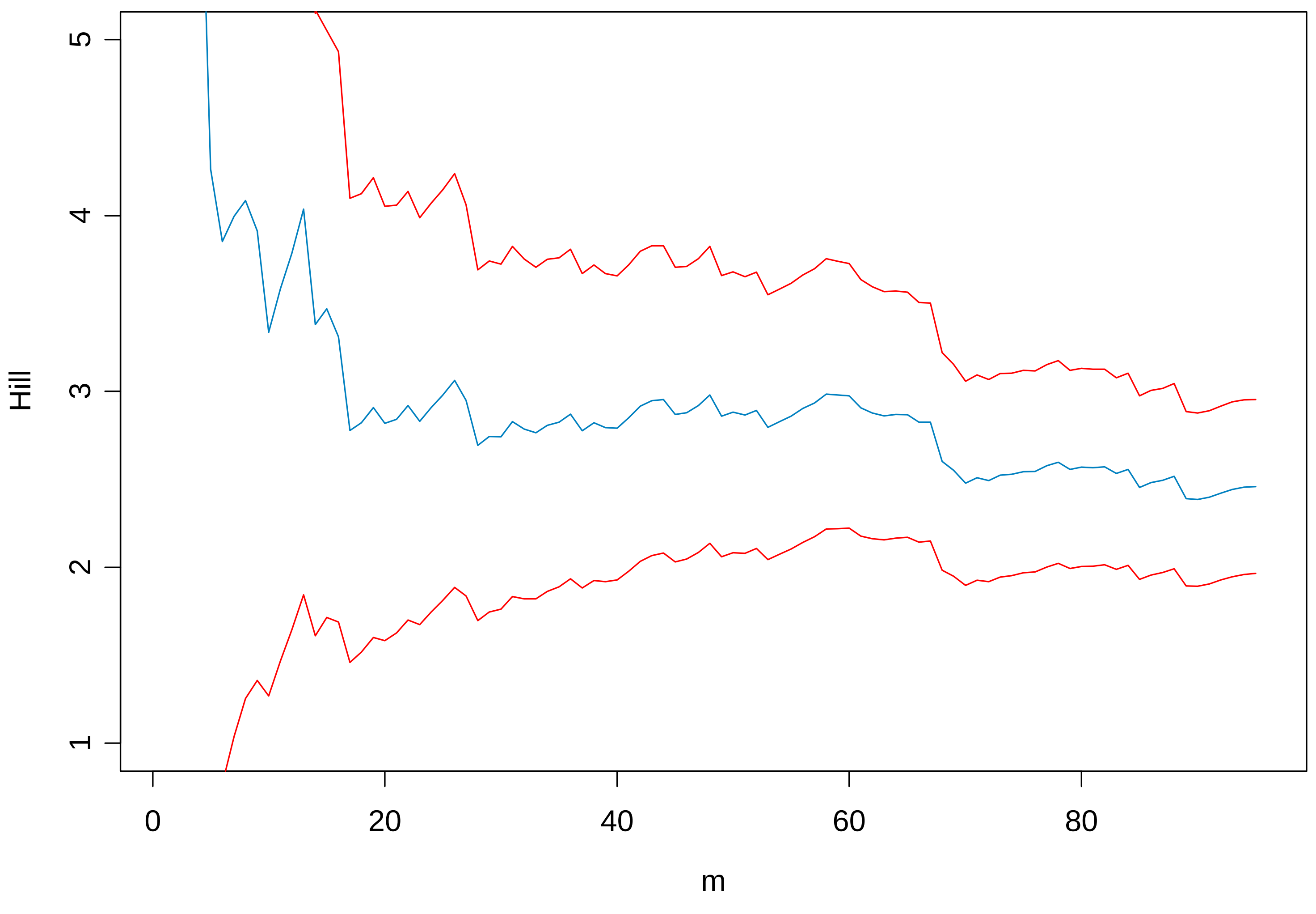
ACF of log-returns of NZ/US exchange rate



Plot of $M_t(4)/S_t(4)$



Hill's plot of tail index



Models for Log(returns)

Basic model

$$X_t = 100*(\ln(P_t) - \ln(P_{t-1})) \quad \text{(log returns)}$$

$$= \sigma_t Z_t \,,$$

where

- $\{Z_t\}$ is IID with mean 0, variance 1 (if exists). (e.g. N(0,1) or a t-distribution with ν df.)
- $\{\sigma_t\}$ is the volatility process
- σ_t and Z_t are independent.

Models for Log(returns)-cont

$X_t = \sigma_t Z_t$ (observation eqn in state-space formulation)

Examples of models for volatility:

(i) GARCH(p,q) process (observation-driven specification)

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_p X_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_q \sigma_{t-q}^2 \ .$$

Special case: ARCH(1), $X_t^2 = (\alpha_0 + \alpha_1 X_{t-1}^2) Z_t^2$.

$$\rho_{X^2}(h) = \alpha_1^h, \text{ if } \alpha_1^2 < 1/3.$$

(ii) stochastic volatility process (parameter-driven specification)

$$\log \sigma_t^2 = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}, \ \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty, \{\varepsilon_t\} \sim \text{IID N}(0, \sigma^2)$$

$$\rho_{X^2}(h) = Cor(\sigma_t^2, \sigma_{t+h}^2) / EZ_1^4$$

Linear Processes

Model: $X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$ $\{Z_t\}$~IID, $P(|Z_t|>x) \sim Cx^{-\alpha}$, $0<\alpha<2$.

Properties:

- $P(|X_t|>x) \sim C_2\, x^{-\alpha}$
- Define $\rho(h) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} / \sum_{j=-\infty}^{\infty} \psi_j^2$.

Case $\alpha > 2$:

$$n^{1/2}(\hat{\rho}(h) - \rho(h)) \xrightarrow{d} \sum_{j=1}^{\infty} (\rho(h+j)+\rho(h-j)-2\rho(j)\rho(h))\, N_j,\ \{N_t\}\text{~IIDN}$$

Case $0 < \alpha < 2$:

$$(n / \ln n)^{1/\alpha} (\hat{\rho}(h) - \rho(h)) \xrightarrow{d} \sum_{j=1}^{\infty} (\rho(h+j)+\rho(h-j)-2\rho(j)\rho(h))\, S_j / S_0,$$

$\{S_t\}$~IID stable (α), S_0 stable ($\alpha/2$)

Background Results—multivariate regular variation

Multivariate regular variation of $\mathbf{X}=(X_1, \ldots, X_m)$: There exists a random vector $\theta \in S^{m-1}$ such that

$$P(|\mathbf{X}|> t\,x,\ \mathbf{X}/|\mathbf{X}| \in \bullet\ )/P(|\mathbf{X}|>t) \to_v x^{-\alpha} P(\ \theta \in \bullet\ )$$

($\to_v$ vague convergence on S^{m-1}) .

- $P(\ \theta \in \bullet)$ is called the spectral measure
- α is the index of $\mathbf{X}$.

Equivalence: There exist positive constants a_n and a measure μ,

$$nP(\mathbf{X}/\ a_n \in \bullet\ ) \to_v \mu(\bullet\ )$$

In this case, one can choose a_n and μ such that

$$\mu((x, \infty) \times B) = x^{-\alpha} P(\ \theta \in B\ )$$

Background Results—multivariate regular variation

Another equivalence?

MRV $\Leftrightarrow$ all linear combinations of $\mathbf{X}$ are regularly varying

i.e., if and only if

$$P(\mathbf{c}^T\mathbf{X}> t)/P(\mathbf{1}^T\mathbf{X}> t) \rightarrow w(\mathbf{c}),$$ exists for all real-valued **c,**

in which case,

$$w(t\mathbf{c}) = t^{-\alpha}w(\mathbf{c}).$$

($\Rightarrow$) true

($\Leftarrow$) established by Basrak, Davis and Mikosch (2000) for α **not** an even integer—case of even integer is unknown.

Background Results—point process convergence

Theorem (Davis & Hsing `95, Davis & Mikosch `97). Let $\{\mathbf{X}_t\}$ be a stationary sequence of random m-vectors. Suppose

(i) finite dimensional distributions are jointly regularly varying (let $(\theta_{-k}, \ldots, \theta_k)$ be the vector in $S^{(2k+1)m-1}$ in the definition).

(ii) mixing condition A (a_n) or strong mixing.

(iii) $\lim_{k\to\infty} \limsup_{n\to\infty} P\left(\bigvee_{k\le|t|\le r_n} |\mathbf{X}_t| > a_n y \,\Big|\, |\mathbf{X}_0| > a_n y\right) = 0.$

Then

$$\gamma = \lim_{k\to\infty} E\left(|\theta_0^{(k)}|^\alpha - \bigvee_{j=1}^{k} |\theta_j^{(k)}|\right)_+ / E|\theta_0^{(k)}|^\alpha$$

exists. If $\gamma > 0$, then

$$N_n := \sum_{t=1}^{n} \varepsilon_{\mathbf{X}_t/a_n} \xrightarrow{d} N := \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varepsilon_{P_i \mathbf{Q}_{ij}},$$

Background Results—point process convergence(cont)

where

- (P_i) are points of a Poisson process on $(0,\infty)$ with intensity function $\nu(dy)=\gamma\alpha y^{-\alpha-1}dy$.
- $\sum_{j=1}^{\infty}\varepsilon_{Q_{ij}}$, $i \geq 1$, are iid point process with distribution Q, and Q is the weak limit of

$$\lim_{k\to\infty} E(|\theta_0^{(k)}|^{\alpha} - \bigvee_{j=1}^{k}|\theta_j^{(k)}|)_+ I_{\bullet}(\sum_{|t|\le k}\varepsilon_{\theta_t^{(k)}}) / E(|\theta_0^{(k)}|^{\alpha} - \bigvee_{j=1}^{k}|\theta_j^{(k)}|)_+$$

Background Results—application to ACVF & ACF

Set-up: Let $\{X_t\}$ be a stationary sequence and set

$$\mathbf{X}_t = \mathbf{X}_t(m) = (X_t, \ldots, X_{t+m}).$$

Suppose $\mathbf{X}_t$ satisfies the conditions of previous theorem. Then

$$N_n := \sum_{t=1}^{n} \varepsilon_{\mathbf{X}_t / a_n} \xrightarrow{d} N := \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varepsilon_{P_i \mathbf{Q}_{ij}},$$

Sample ACVF and ACF:

$$\hat{\gamma}_X(h) = n^{-1} \sum_{t=1}^{n-h} X_t X_{t+h}, \quad h \geq 0, \qquad \text{ACVF}$$

$$\hat{\rho}_X(h) = \hat{\gamma}_X(h) / \hat{\gamma}_X(h), \quad h \geq 1, \qquad \text{ACF}$$

If $EX_0^2 < \infty$, then define $\gamma_X(h) = EX_0X_h$ and $\rho_X(h) = \gamma_X(h) / \gamma_X(0)$.

Background Results—application to ACVF & ACF

(i) If $\alpha \in (0,2)$, then

$$(n a_n^{-2} \hat{\gamma}_X(h))_{h=0,\dots,m} \xrightarrow{d} (V_h)_{h=0,\dots,m}$$

$$(\hat{\rho}_X(h))_{h=1,\dots,m} \xrightarrow{d} (V_h / V_0)_{h=1,\dots,m},$$

where

$$V_h = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P_i^2 Q_{ij}^{(0)} Q_{ij}^{(h)}, \quad h = 0,\dots,m.$$ (jointly stable)

(ii) If $\alpha \in (2,4)$ + additional condition, then

$$\left(n a_n^{-2} (\hat{\gamma}_X(h) - \gamma_X(h)\right)_{h=0,\dots,m} \xrightarrow{d} \left(V_h\right)_{h=0,\dots,m}$$

$$\left(n a_n^{-2} (\hat{\rho}_X(h) - \rho_X(h))\right)_{h=1,\dots,m} \xrightarrow{d} \gamma_X^{-1}(0)\left(V_h - \rho_X(h) V_0\right)_{h=1,\dots,m}.$$

Applications—stochastic recurrence equations

$$\mathbf{Y}_t = \mathbf{A}_t\, \mathbf{Y}_{t-1} + \mathbf{B}_t, \qquad (\mathbf{A}_t\,, \mathbf{B}_t) \sim \text{IID},$$

$\mathbf{A}_t$ $d\times d$ random matrices, $\mathbf{B}_t$ random d-vectors

Examples

ARCH(1): $X_t=(\alpha_0+\alpha_1\, X^2_{t-1})^{1/2}Z_t, \quad \{Z_t\}\sim\text{IID}$. Then the squares follow an SRE with $Y_t = X_t^2,\ \ A_t = \alpha_1 Z_t^2,\ \ B_t = \alpha_0 Z_t^2$.

GARCH(2,1): $X_t = \sigma_t Z_t,\ \ \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \alpha_2 X_{t-2}^2 + \beta_1 \sigma_{t-1}^2$.

Then $\mathbf{Y}_t = (\sigma_t^2, X_{t-1}^2)'$ follows the SRE given by

$$\begin{bmatrix} \sigma_t^2 \\ X_{t-1}^2 \end{bmatrix} = \begin{bmatrix} \alpha_1 Z_{t-1}^2 + \beta_1 & \alpha_2 \\ Z_{t-1}^2 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{t-1}^2 \\ X_{t-2}^2 \end{bmatrix} + \begin{bmatrix} \alpha_0 \\ 0 \end{bmatrix}$$

Stochastic Recurrence Equations (cont)

Examples (tricks)

GARCH(1,1): $X_t = \sigma_t Z_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2$.

Although this process does not have a 1-dimensional SRE representation, the process σ_t^2 does. Iterating, we have

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 Z_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$
$$= (\alpha_1 Z_{t-1}^2 + \beta_1)\sigma_{t-1}^2 + \alpha_0.$$

Bilinear(1): $X_t = aX_{t-1} + bX_{t-1}Z_{t-1} + Z_t, \quad \{Z_t\} \sim \text{IID}$

$$= Y_{t-1} + Z_t\ ,$$

$$Y_t = A_t Y_{t-1} + B_t \quad A_t = a + bZ_t, \quad B_t = A_t Z_t$$

Stochastic Recurrence Equations (cont)

$$\mathbf{Y}_t = \mathbf{A}_t \mathbf{Y}_{t-1} + \mathbf{B}_t, \quad (\mathbf{A}_t, \mathbf{B}_t) \sim \text{IID}$$

Existence of stationary solution

- $E \ln^+ \| \mathbf{A}_1 \| < \infty$
- $E \ln^+ \| \mathbf{B}_1 \| < \infty$
- $\inf n^{-1} E \ln \| \mathbf{A}_1 \ldots \mathbf{A}_n \| =: \gamma < 0$ (γ –Lyapunov exponent)

Ex. (d=1) $E \ln |\mathbf{A}_1| < 0$.

Strong mixing

If $E \|\mathbf{A}_1\|^\varepsilon < \infty$, $E |\mathbf{B}_1|^\varepsilon < \infty$ for some $\varepsilon > 0$, then the SRE $(\mathbf{Y}_t)$ is *geometrically ergodic* $\Rightarrow$ *strong mixing* with geometric rate (Meyn and Tweedie `93).

Stochastic Recurrence Equations (cont)

Regular variation of the marginal distribution (Kesten)

Assume **A** and **B** have non-negative entries and

- $E\,\|\mathbf{A}_1\|^{\varepsilon} < 1$ for some $\varepsilon > 0$
- $\mathbf{A}_1$ has no zero rows a.s.
- W.P. 1, $\{\ln \rho(\mathbf{A}_1 \ldots \mathbf{A}_n)$: is dense in $\mathbf{R}$ for some n, $\mathbf{A}_1 \ldots \mathbf{A}_n > 0\}$
- There exists a $\kappa_0 > 0$ such that $E\|A\|^{\kappa_0} \ln^+\|A\| < \infty$ and

$$E\left(\min_{i=1,\ldots,d} \sum_{j=1}^{d} A_{ij}\right)^{\kappa_0} \geq d^{\kappa_0/2}$$

Then there exists a $\kappa_1 \in (0, \kappa_0]$ such that **all** linear combinations of **Y** are regularly varying with index κ_1. (Also need $E\,|B|^{\kappa_1} < \infty$.)

Application to GARCH

Proposition: Let $(\mathbf{Y}_t)$ be the soln to the SRE based on the *squares* of a GARCH model. Assume

- Lyapunov exponent $\gamma < 0$. (See Bougerol and Picard`92)
- Z has a positive density on $(-\infty, \infty)$ with all moments finite or $E|Z|^h = \infty$, for all $h \geq h_0$ and $E|Z|^h < \infty$ for all $h < h_0$.
- Not all the GARCH parameters vanish.

Then $(\mathbf{Y}_t)$ is *strongly mixing* with geometric rate (Boussama `98) and all finite dimensional distributions are *multivariate regularly varying* with index κ_1.

Corollary: The corresponding GARCH process is strongly mixing and has all finite dimensional distributions that are MRV with index $\kappa = 2\kappa_1$.

Application to GARCH (cont)

Remarks:

1. Kesten's result applied to an iterate of $\mathbf{Y}_t$, i.e., $\mathbf{Y}_{tm} = \tilde{\mathbf{A}}_t \mathbf{Y}_{(t-1)m} + \tilde{\mathbf{B}}_t$

2. Determination of κ is difficult. Explicit expressions only known in two(?) cases.

- ARCH(1): $E|\alpha_1 Z^2|^{\kappa/2} = 1$.

α_1	.312	.577	1.00	1.57
κ	8.00	4.00	2.00	1.00

- GARCH(1,1): $E|\alpha_1 Z^2 + \beta_1|^{\kappa/2} = 1$ (Mikosch and Stărică)
- For IGARCH ($\alpha_1 + \beta_1 = 1$), then $\kappa = 2 \Rightarrow$ infinite variance.
- Can estimate κ empirically by replacing expectations with sample moments.

Summary for GARCH(p,q)

$\kappa \in (0,2)$:

$$(\hat{\rho}_X(h))_{h=1,\ldots,m} \xrightarrow{d} (V_h / V_0)_{h=1,\ldots,m},$$

$\kappa \in (2,4)$:

$$\left(n^{1-2/\kappa}\hat{\rho}_X(h)\right)_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1}(0)\left(V_h\right)_{h=1,\ldots,m}.$$

$\kappa \in (4,\infty)$:

$$\left(n^{1/2}\hat{\rho}_X(h)\right)_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1}(0)\left(G_h\right)_{h=1,\ldots,m}.$$

Remark: Similar results hold for the sample ACF based on $|X_t|$ and X_t^2.

Results for IBM Data

Fitted GARCH(1,1) model IBM data with t-noise:

1st half:

$$X_t = \sigma_t Z_t, \quad \sigma_t^2 = .0342 + .0726_{t-1}^2 + .9107\sigma_{t-1}^2, \, \mathrm{df} = 7.185, \, \kappa = 4.56$$

2nd half:

$$X_t = \sigma_t Z_t, \quad \sigma_t^2 = .0513 + .0472_{t-1}^2 + .9364\sigma_{t-1}^2, \, \mathrm{df} = 5.398, \, \kappa = 4.16$$

both:

$$X_t = \sigma_t Z_t, \quad \sigma_t^2 = .0320 + .0557X_{t-1}^2 + .9319\sigma_{t-1}^2 \, , \, \mathrm{df} = 6.089, \, \kappa = 4.96$$

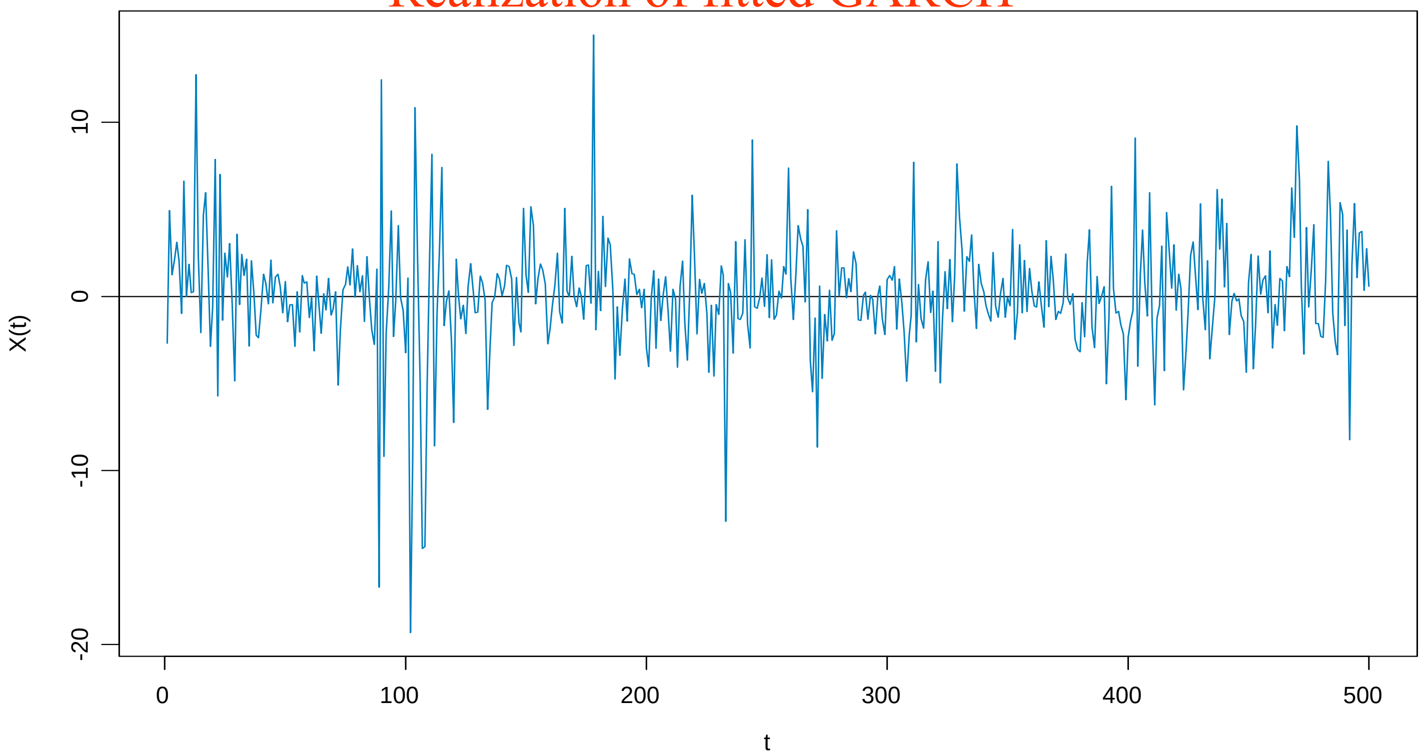
Realization of GARCH Process

Fitted GARCH(1,1) model for NZ-USA exchange:

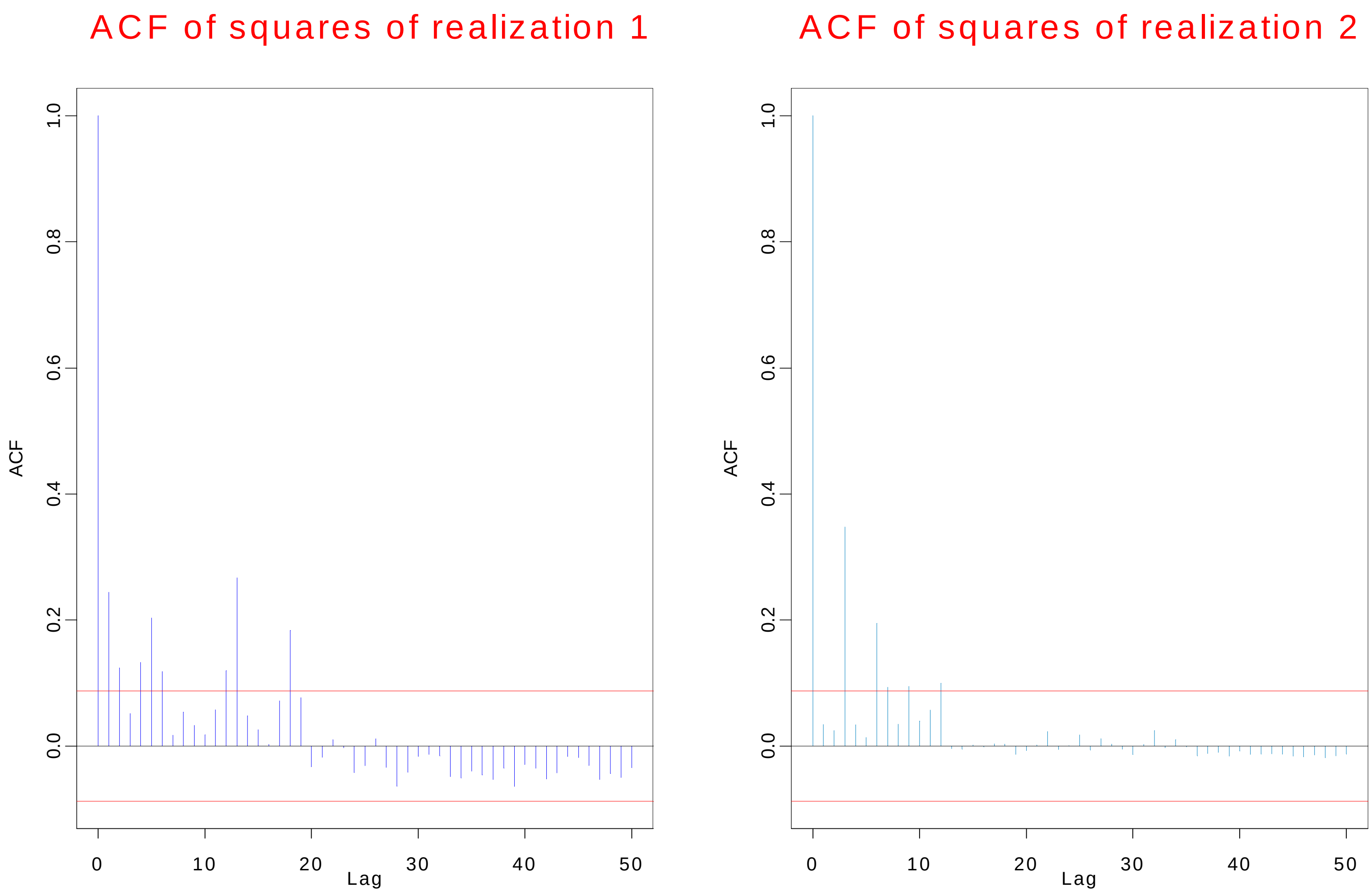
$$X_t = \sigma_t Z_t, \quad \sigma_t^2 = (6.70)10^{-7} + .1519 X_{t\text{-}1}^2 + .772\sigma_{t\text{-}1}^2$$

$(Z_t) \sim$ IID t-distr with 5 df. κ is approximately 3.8

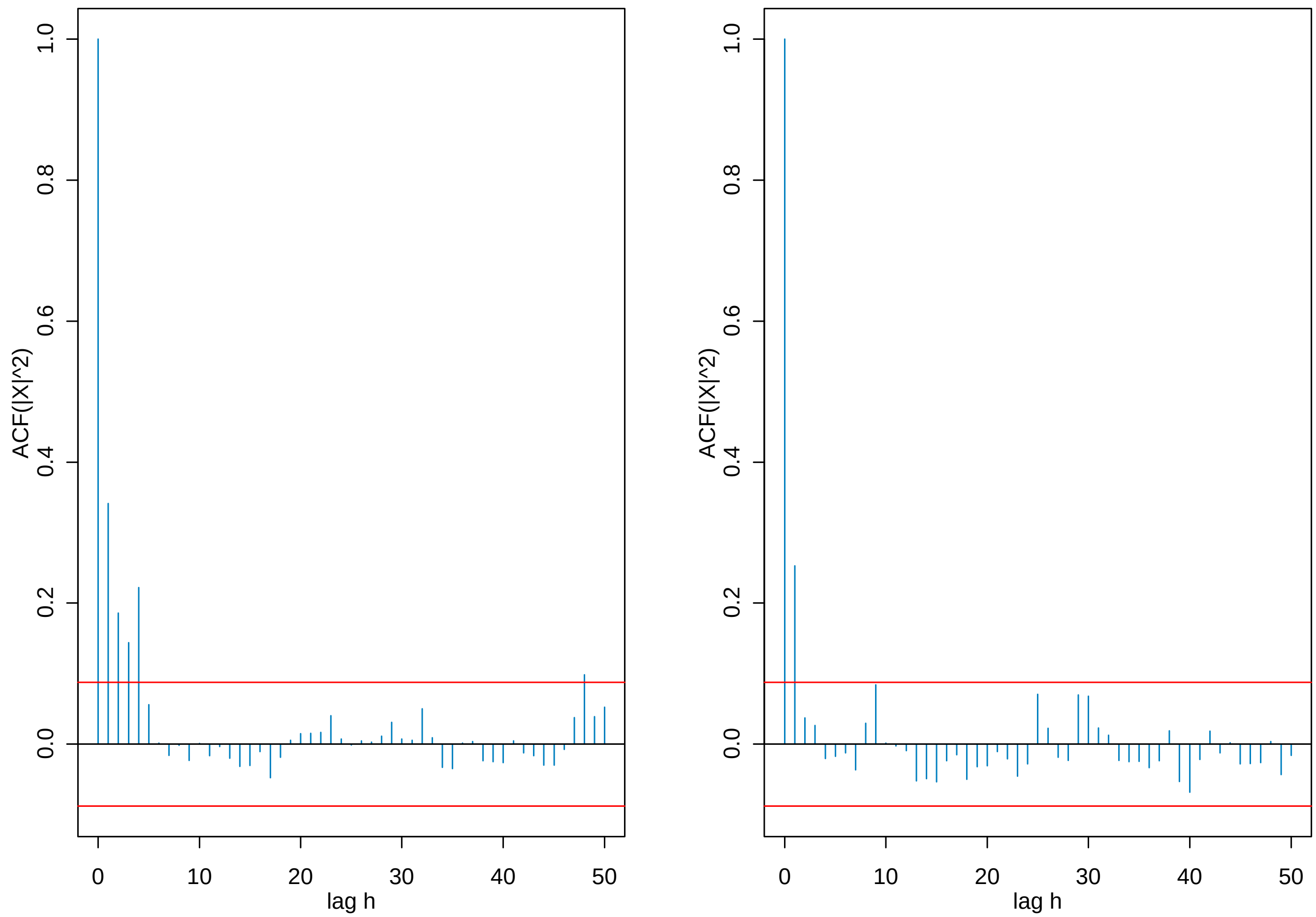
Realization of fitted GARCH



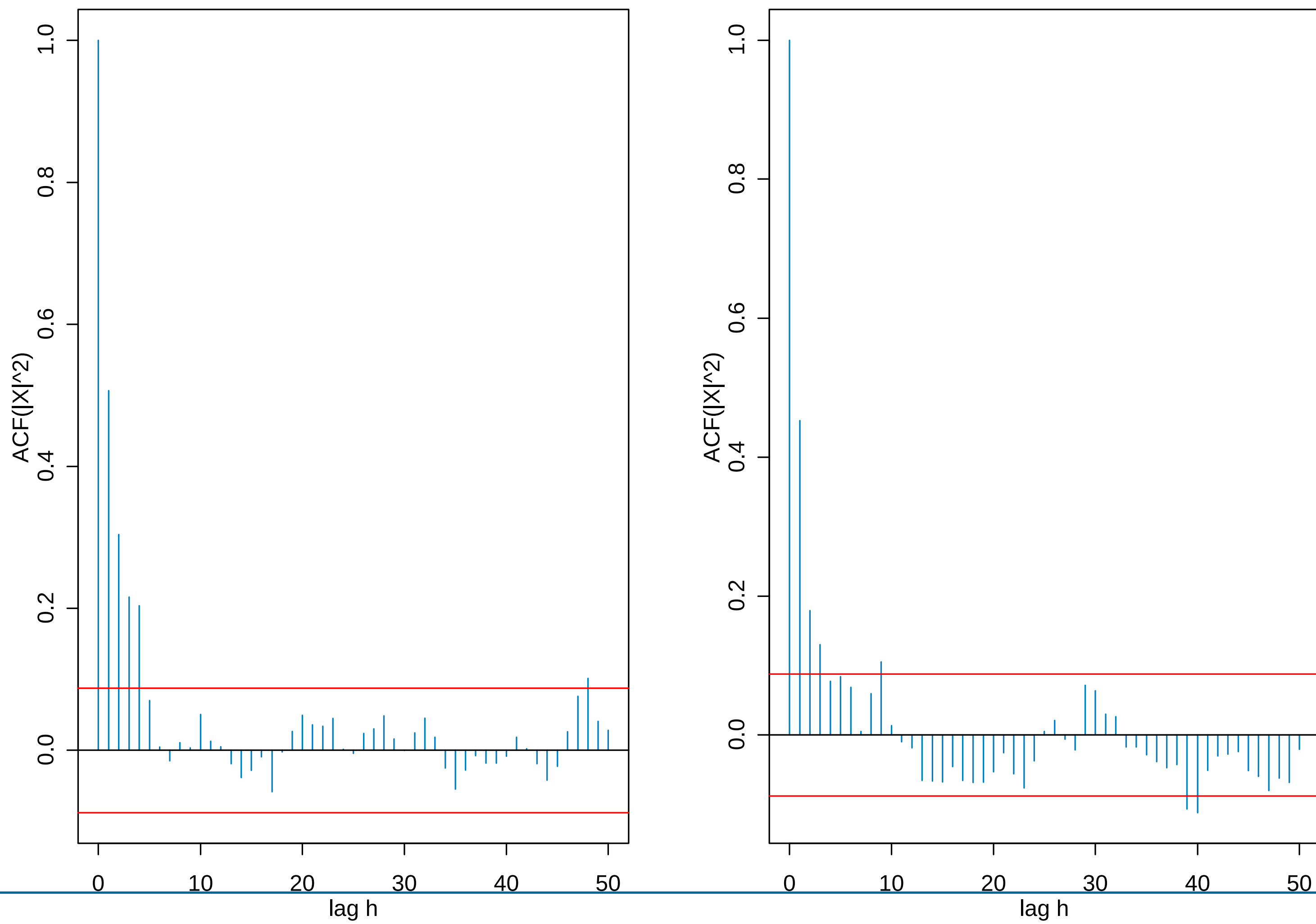
ACF of Fitted GARCH(1,1) Process



ACF of 2 realizations of an $(ARCH)^2$: $X_t=(.001+.7\,X_{t-1})^{1/2}\,Z_t$



ACF of 2 realizations of an |ARCH|: $X_t=(.001+ X_{t-1})^{1/2} Z_t$



Stochastic Volatility Models

SVM: $X_t = \sigma_t Z_t$

- $(Z_t) \sim$ IID with mean 0 (if it exists)
- (σ_t) is a stationary process ($2 \log \sigma_t$ is a linear process) given by

$$\boxed{\log \sigma_t^2 = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}, \quad \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty, (\varepsilon_t) \sim \text{IID N}(0, \sigma^2)}$$

Heavy tails: Assume Z_t has Pareto tails with index α, i.e.,

$$P(|Z_t| > z) \sim C\, z^{-\alpha} \Rightarrow P(|X_t| > z) \sim C\, E\sigma^\alpha\, z^{-\alpha}.$$

Then if $\alpha \in (0,2)$,

$$(n / \ln n)^{1/\alpha} \hat{\rho}_X(h) \xrightarrow{d} \frac{\|\sigma_1 \sigma_{h+1}\|_\alpha}{\|\sigma_1\|_\alpha^2} \frac{S_h}{S_0}.$$

Stochastic Volatility Models (cont)

Other powers:

1. Absolute values: $\alpha \in (1,2)$,

$$E|X_t| = E|\sigma_t|E|Z_t|, \;\; E|X_t X_{t+h}| = (E|\sigma_t \sigma_{t+h}|)(E|Z_t|E|Z_{t+h}|)$$

$$\mathrm{Cov}(|X_t|, |X_{t+h}|) = \mathrm{Cov}(\sigma_t, \sigma_{t+h})(E|Z|)^2$$

$$\mathrm{Cor}(|X_t|, |X_{t+h}|) = \mathrm{Cor}(\sigma_t, \sigma_{t+h})(E|Z|)^2/\, EZ^2$$

$$= 0 \;(?).$$

We obtain

$$n(n \ln n)^{-1/\alpha}\left(\hat{\gamma}_{|X|}(h) - \gamma_{|X|}(h)\right) \xrightarrow{d} \|\sigma_1 \sigma_{h+1}\|_\alpha S_h$$

and

$$(n/\ln n)^{1/\alpha} \hat{\rho}_{|X|}(h) \xrightarrow{d} \frac{\|\sigma_1 \sigma_{h+1}\|_\alpha}{\|\sigma_1\|_\alpha^2} \frac{S_h}{S_0}.$$

Stochastic Volatility Models (cont)

2. Higher order: $\alpha \in (0,2)$

The squares are again a SV process and the results of the previous proposition apply. Namely,

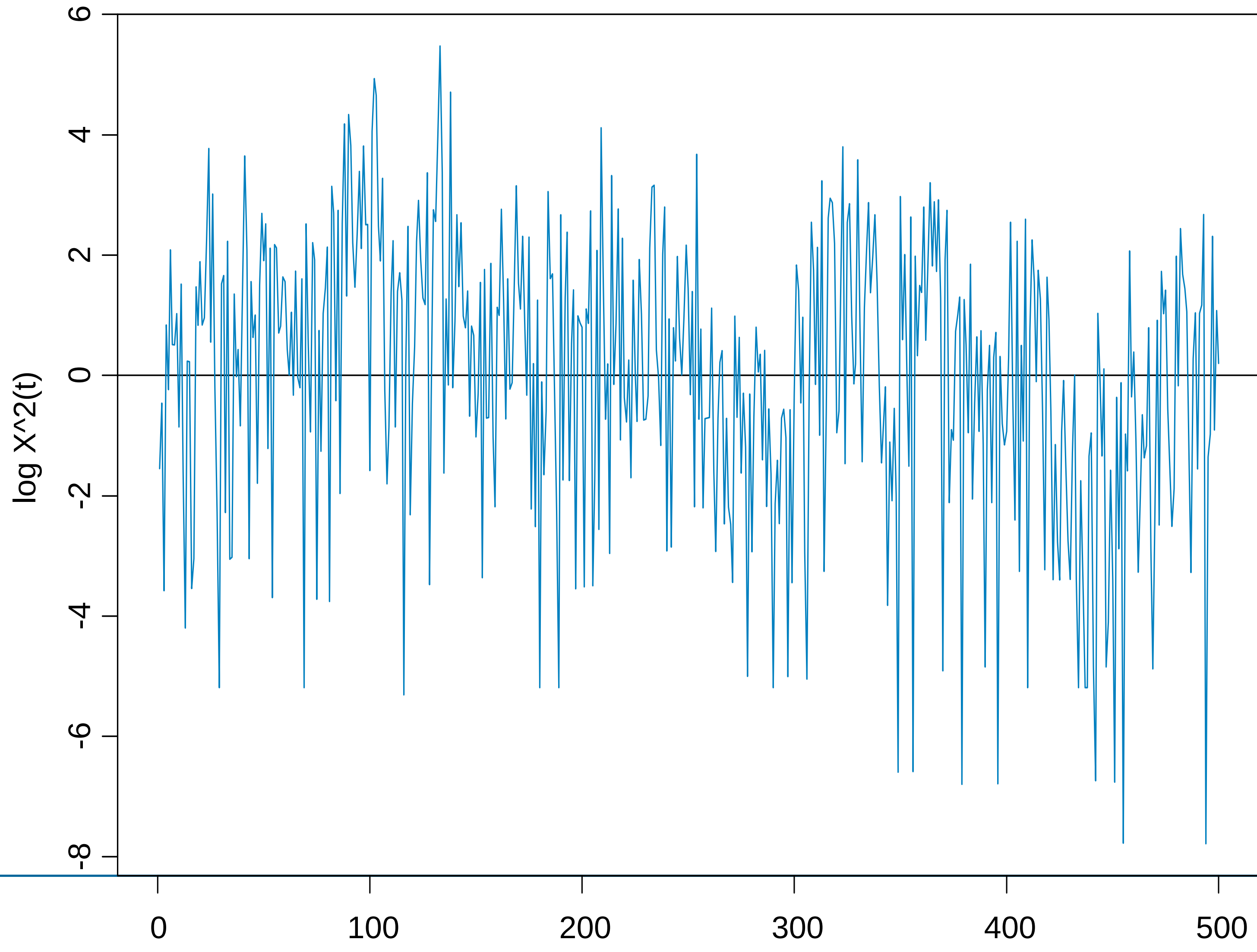
$$(n/\ln n)^{2/\alpha}\hat{\rho}_{X^2}(h) \xrightarrow{d} \frac{\|\sigma_1^2\sigma_{h+1}^2\|_{\alpha/2}}{\|\sigma_1^2\|_{\alpha/2}^2}\frac{S_h}{S_0}.$$

In particular,

$$\hat{\rho}_{X^2}(h) \xrightarrow{P} 0.$$

Stochastic Volatility Models (cont)

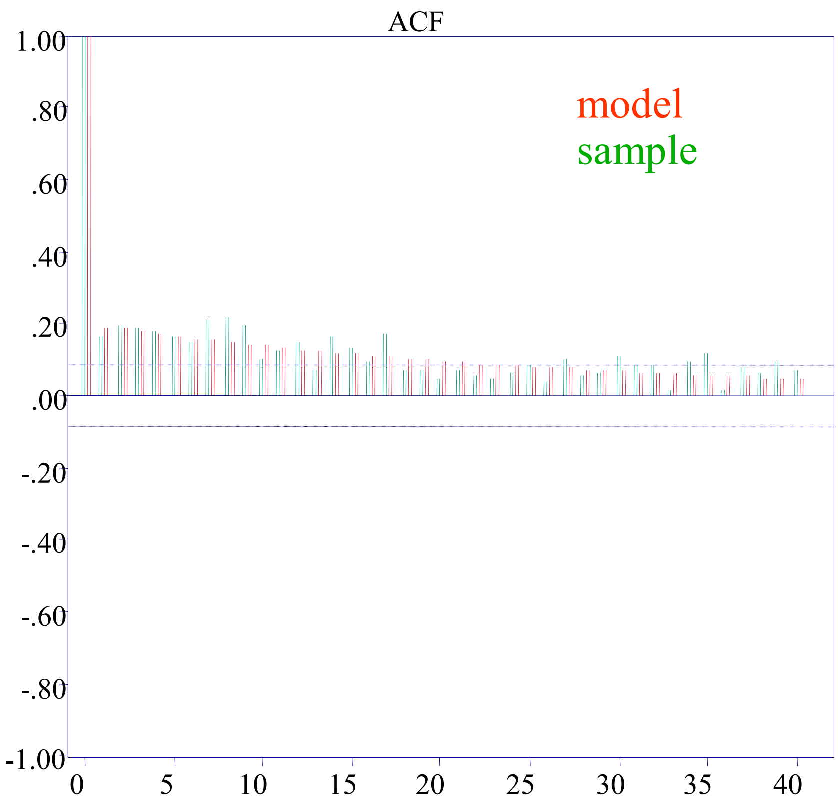
(log X^2) - mean for NZ-USA exchange rates

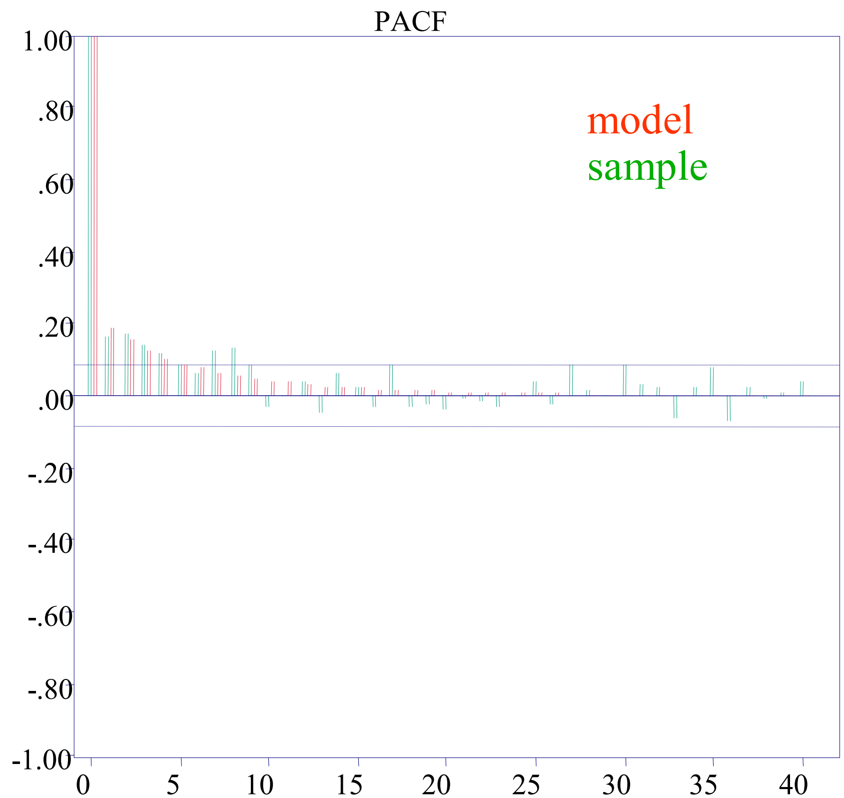


Stochastic Volatility Models (cont)

ACF/PACF for ($\log X^2$) suggests ARMA (1,1) model:

μ= -11.5403, $Y_t = .9646Y_{t-1} + \varepsilon_t - .8709\,\varepsilon_{t-1}$, (ε_t)~WN(0,4.6653)





Stochastic Volatility Models (cont)

The ARMA (1,1) model for log X^2 leads to the SV model

$$X_t = \sigma_t Z_t$$

with

$$2 \ln \sigma_t = -11.5403 + v_t + \varepsilon_t$$

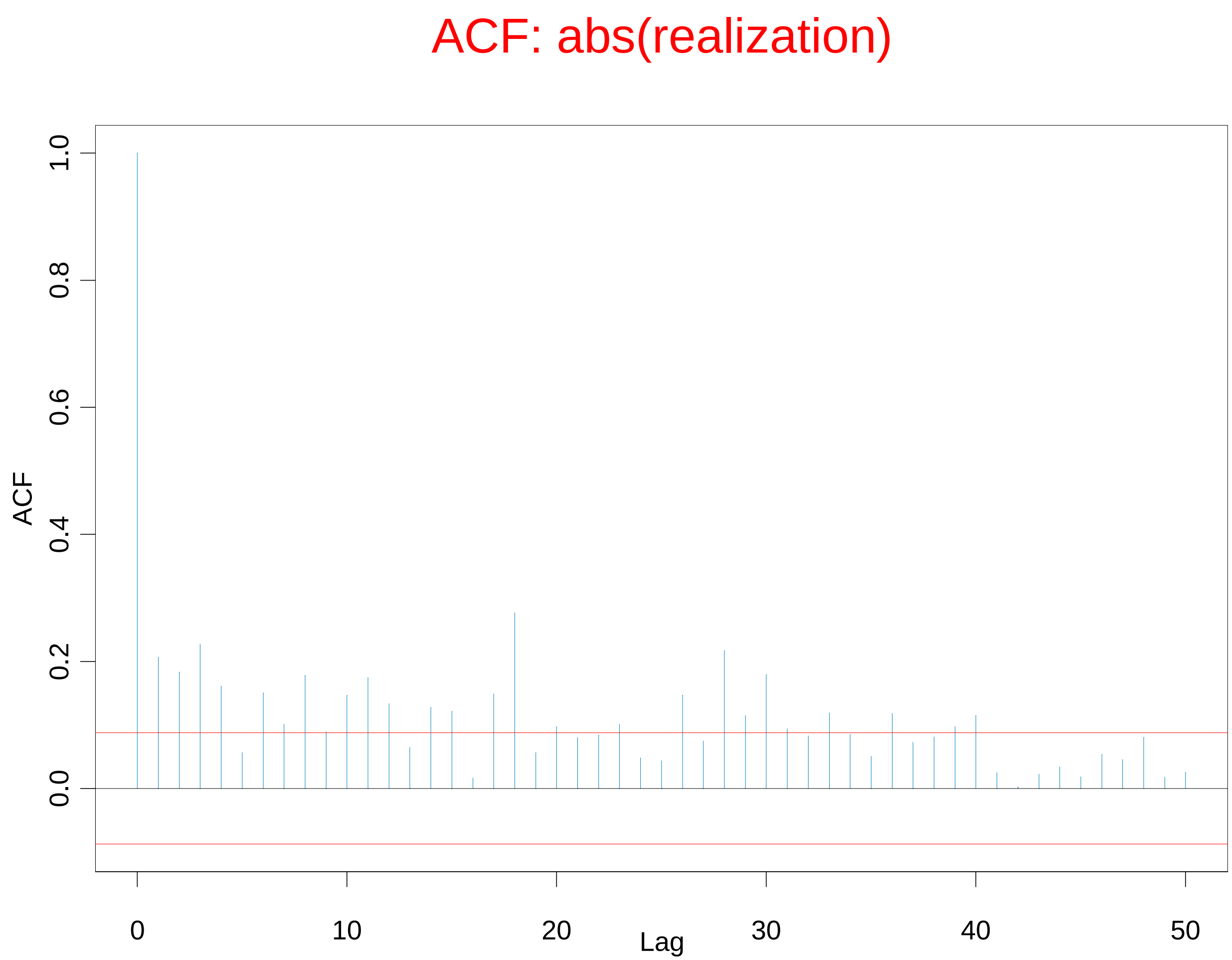
$$v_t = .9646\, v_{t-1} + \gamma_t\,, \quad (\gamma_t) \sim WN(0,.07253)$$

$$(\varepsilon_t) \sim WN(0,4.2432).$$

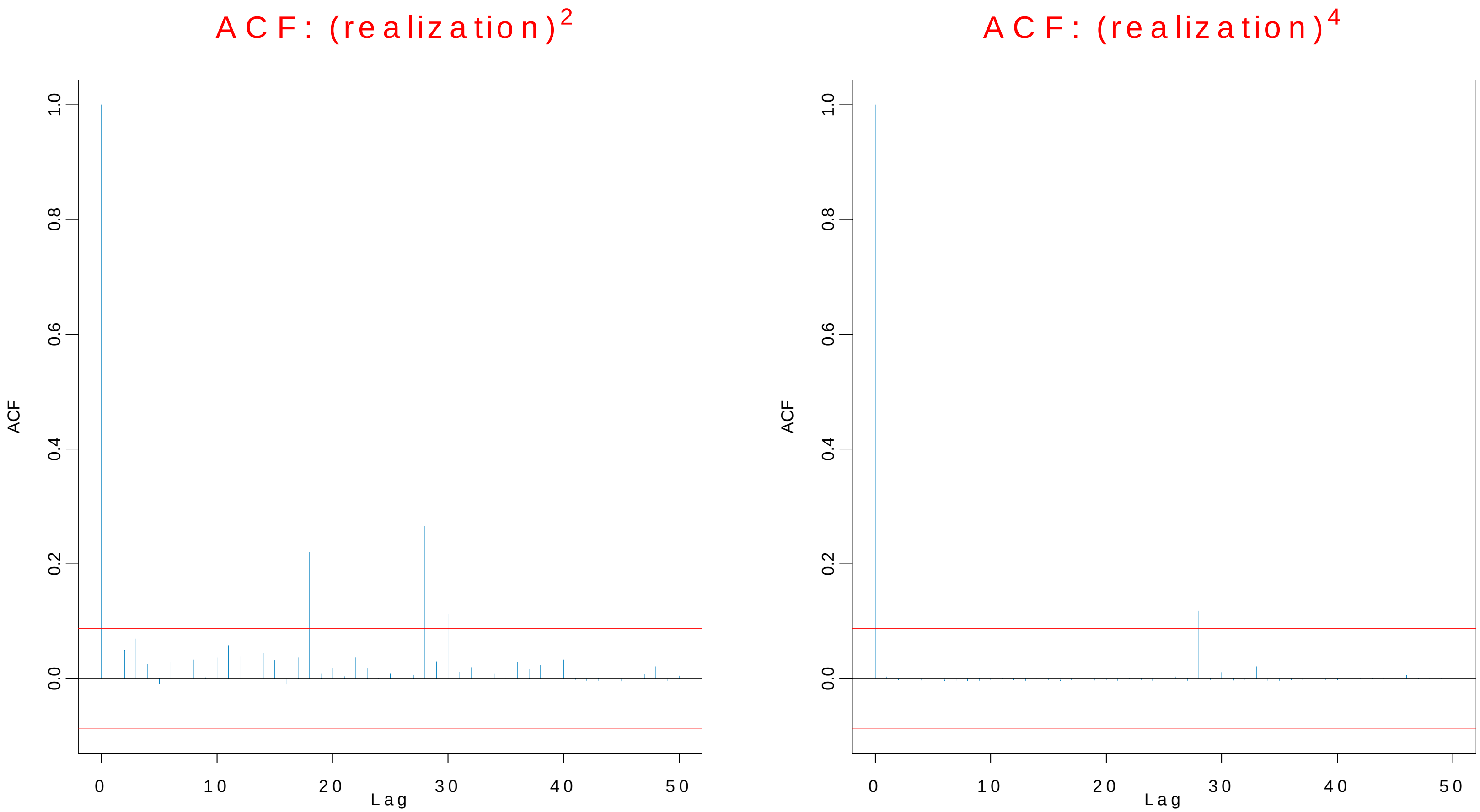
Stochastic Volatility Models (cont)

Simulation of SVM model.

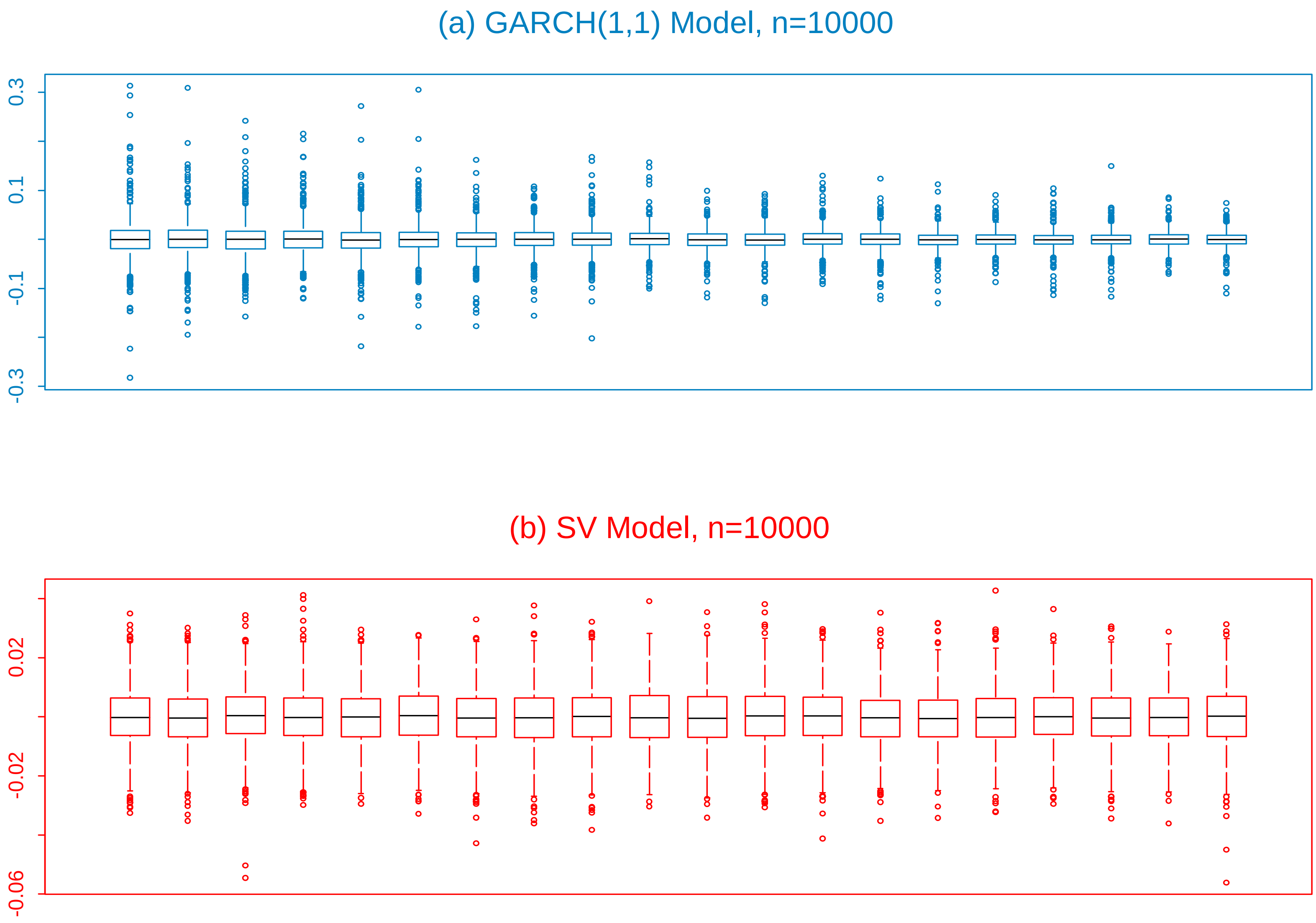
Took Z_t to be distributed as a t with 3 df (suitable normalized).



Stochastic Volatility Models (cont)

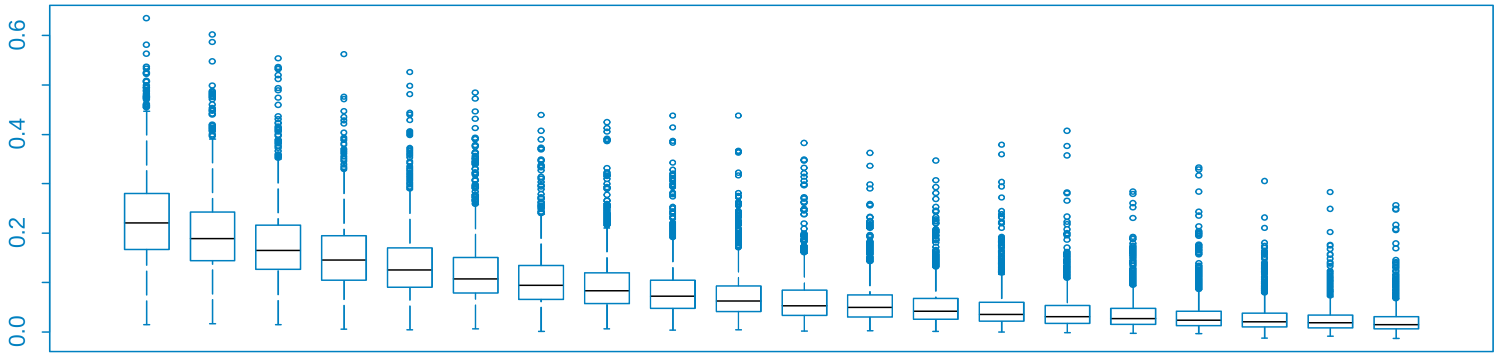


Sample ACF for GARCH and SV Models (1000 reps)

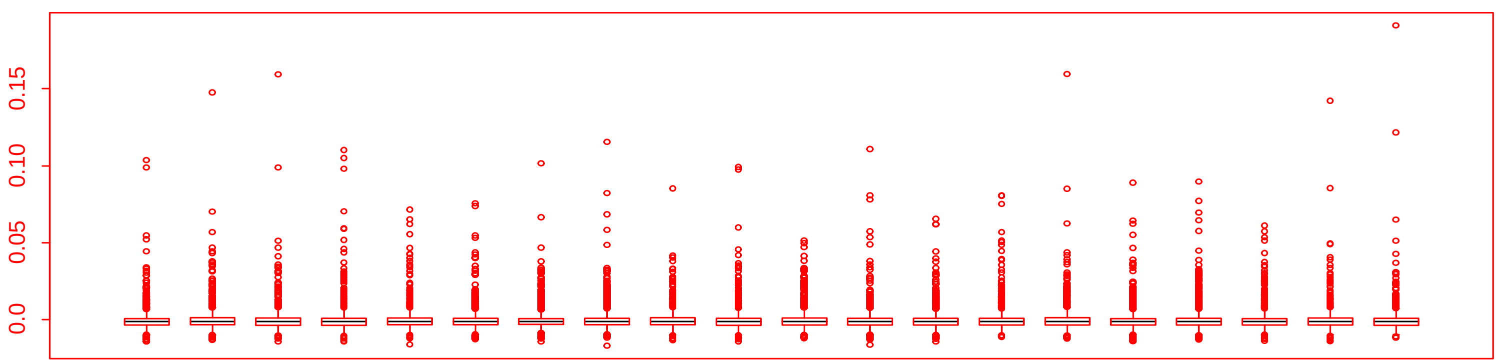


Sample ACF for Squares of GARCH and SV (1000 reps)

(a) GARCH(1,1) Model, n=10000



(b) SV Model, n=10000



Sample ACF for Squares of GARCH and SV (1000 reps)