Point Process Theory for Bilinear and Stochastic Volatility Models

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I. Point process theory for bilinear models

The Model

Linear model: $X_t = \sum_{k=0}^{\infty} c_k Z_{t-k}$, where $\{Z_t\}$ is an IID sequence.

Simple first-order bilinear model:

$$X_t = cZ_{t-1}X_{t-1} + Z_t$$

where $\{Z_t\}$ is an IID sequence.

Note:

$$
\begin{aligned}
X_t &= cZ_{t-1}X_{t-1} + Z_t \\
&= Z_t + cZ_{t-1}^2 + c^2 Z_{t-1}Z_{t-2}X_{t-2} \\
&= Z_t + cZ_{t-1}^2 + c^2 Z_{t-1}Z_{t-2}^2 + c^3 Z_{t-1}Z_{t-2}Z_{t-3}X_{t-3} \\
&= \sum_{k=0}^{\infty} c^k Y_t^{(k)}
\end{aligned}
$$

where

$$
\begin{aligned}
Y_t^{(0)} &= Z_t \\
Y_t^{(1)} &= Z_{t-1}^2 \\
Y_t^{(k)} &= \left(\prod_{i=1}^{k-1} Z_{t-i} \right) Z_{t-k}^2, \ \text{for } k > 0.
\end{aligned}
$$

Tail Behavior of X_1

Assume $\{Z_t\}$ is an IID sequence of non-negative rv's with cdf

$$1 - F(x) = x^{-\alpha} L(x)$$

where $L(x)$ is a slowly varying function at infinity.

Proposition:

$$\frac{P[Y_t^{(k)} > x]}{P[Z_1^2 > x]} \to \left(E Z_1^{\alpha/2} \right)^{k-1}$$

and

$$\frac{P[Y_t^{(k)} > x, Y_t^{(j)} > x]}{P[Z_1^2 > x]} \to 0 \qquad \text{as } x \to \infty.$$

Corollary: If $c^{\alpha/2} \mathrm{E}\,\mathrm{Z}_1^{\alpha/2} < 1$, then

$$\mathrm{P}[\sum_{k=0}^{m} c^k \mathrm{Y}_t^{(k)} > x] \approx \sum_{k=1}^{m} c^{k\alpha/2} \left(\mathrm{E}\mathrm{Z}_1^{\alpha/2}\right)^{k-1} \mathrm{P}[\mathrm{Z}_1^2 > x]$$

and

$$\lim_{x\to\infty} \frac{\mathrm{P}[\sum_{k=0}^{\infty} c^k \mathrm{Y}_t^{(k)} > x]}{\mathrm{P}[\mathrm{Z}_1^2 > x]} = \sum_{k=1}^{\infty} c^{k\alpha/2} \left(\mathrm{E}\mathrm{Z}_1^{\alpha/2}\right)^{k-1}$$

$$= \frac{c^{\alpha/2}}{1 - c^{\alpha/2} \mathrm{E}\mathrm{Z}_1^{\alpha/2}}.$$

Point process convergence

Assumptions: $X_t = cZ_{t-1}X_{t-1} + Z_t$, $\{Z_t\}$ is IID

- $P[|Z_t| > x] = x^{-\alpha}L(x), \quad P[Z_t > x] / P[|Z_t| > x] \longrightarrow p.$

$$X_t = \sum_{k=0}^{\infty} c^k Y_t^{(k)}, \quad Y_t^{(1)} = Z_{t-1}, \quad Y_t^{(k)} = \left(\prod_{i=1}^{k-1} Z_{t-i}\right) Z_{t-k}^2$$

- $b_n = \inf\{x: P[|Z_t| > x] < n^{-1}\}$

- Define the sequence of point processes

$$N_n(\cdot) = \sum_{t=1}^{n} \varepsilon_{b_n^{-2}X_t}(\cdot)$$

Theorem:

$$\text{(i)}\quad N_n(\cdot) \xrightarrow{d} \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \varepsilon_{j_s^2 c^k W_{s,k}}(\cdot),$$

where $\{ j_s \}$ are points of a Poisson process with intensity measure $\mu(dx) = \alpha(p\, x^{-\alpha-1} 1[x > 0] + (1-p)(-x)^{-\alpha-1} 1[x < 0])dx$, and

$$W_{s,k} = \begin{cases} \prod_{i=1}^{k-1} U_{s,i}, & \text{if } k > 1, \\ 1, & \text{if } k = 1, \\ 0, & \text{if } k < 1, \end{cases}$$

with $\{U_{s,k}\}$ IID (F).

$$\text{(ii)} \quad \sum_{t=1}^{n} \varepsilon_{b_n^{-2}(X_t, X_{t-1}, \ldots, X_{t-h})}(\cdot)$$

$$\xrightarrow{d} \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \varepsilon_{j_s^2 (c^k W_{s,k}, c^{k-1} W_{s,k-1}, \ldots, c^{k-h} W_{s,k-h})}(\cdot).$$

Applications

Extremes: Define $M_n = \max\{X_1, ... , X_n\}$ and observe that

$$\{b_n^{-2} M_n \le x\} = \{N_n(x, \infty] = 0\}$$

where N_n is the point process $\sum_{k=1}^{n} \varepsilon_{b_n^{-2} X_k}$. Thus,

$$P[b_n^{-2} M_n \le x] = P[N_n(x, \infty] = 0]$$

$$\to P[N(x, \infty] = 0]$$

$$= \begin{cases} 0, & \text{if } x \le 0, \\ \exp\{E(V_1^{\alpha/2}) x^{-\alpha}\}, & \text{if } x > 0, \end{cases}$$

and $V_1 = \max\{c^k W_{1,k} \,;\, k > 0\}$.

Partial sums: Assume $\alpha \in (0,4)$ and define

$$S_n = \sum_{k=1}^{n} X_k .$$

Then the partial sums, normalized by b_n^{-2}, are asymptotically stable. For the case $\alpha \in (0,2)$, it follows directly from part (i) of the theorem that

$$b_n^{-2} S_n \xrightarrow{d} S := \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} j_s^2 c^k W_{s,k}$$

$$= \sum_{s=1}^{\infty} j_s^2 A_s ,$$

which has a stable distribution. (See Davis and Hsing `95 for form of characteristic function.)

Sample ACF: Define the sample autocorrelation function w/o mean correction by

$$\hat{\rho}(h) = \frac{\sum_{t=1}^{n-h} X_t X_{t+h}}{\sum_{t=1}^{n} X_t^2}.$$

For heavy-tailed linear processes, $\hat{\rho}(h)$ was shown to be *consistent* by Davis and Resnick `85. (Limit distribution was also derived.) For the bilinear process, $\hat{\rho}(h)$ has a nondegenerate limit distribution w/o any normalization. Proof follows directly from part (ii) of the theorem.

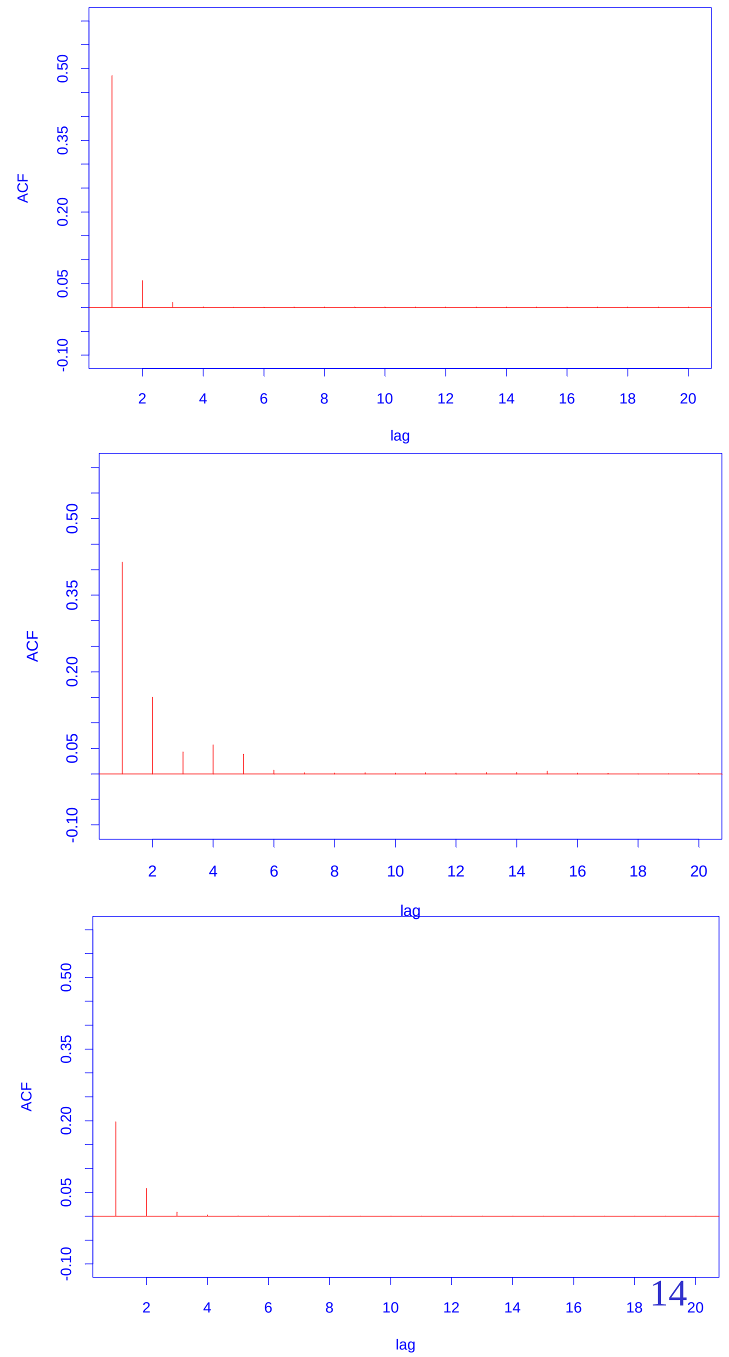
In particular, if $\alpha \in (0,4)$, then

$$\hat{\rho}(h) \xrightarrow{d} \frac{\sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \text{j}_s^4 c^{2k-h} W_{s,k} W_{s,k-h}}{\sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \text{j}_s^4 c^{2k} W_{s,k}^2}.$$

Examples: Sample ACF from 3 realizations, n=5000 from the model:

$$X_t = 0.1 Z_{t-1} X_{t-1} + Z_t,$$

$$\{Z_t\} \text{ IID}, \quad P[Z_t > x] = 1/x, \; x \geq 1.$$



II. Point process theory for stochastic volatility models

The Stochastic Volatility Model

$$Y_t = \exp(\alpha_t / 2)\xi_t, \quad \alpha_t = \sum_{j=0}^{\infty} c_j Z_{t-j} ,$$

where $\{\xi_t\}$ is IID N(0,1), $\{Z_t\}$ is IID N(0, σ^2) independent of $\{\xi_t\}$ and the $\{c_j\}$ are square summable.

Note:

(i) $\{Y_t\}$ is a stationary martingale difference sequence.

(ii) $X_t = \ln Y_t^2 = \alpha_t + \ln \xi_t^2$ is a Gaussian linear process plus an IID log-χ^2 process.

Tail Behavior of X_1

Using a Tauberian argument as in Feigin and Yashchin (1983) and Davis and Resnick (1991), we have

$$P[X_1 > x] \approx \frac{\sigma_\alpha}{\sqrt{\pi}} \exp\{-\frac{x^2}{2\sigma_\alpha^2} + \frac{x \ln x}{\sigma_\alpha^2} + \frac{(k-1)x}{\sigma_\alpha^2} - \frac{(k+\sigma_\alpha^2)\ln x}{2\sigma_\alpha^2} - \frac{\ln^2 x}{2\sigma_\alpha^2} - \frac{k^2}{2\sigma_\alpha^2} + O\left(\frac{\ln^2 x}{x}\right)\},$$

as $x \to \infty$, where $k = \ln(2/\sigma_\alpha^2)$, and $\sigma_\alpha^2 = \mathrm{Var}(\alpha_t)$.

Point Process Convergence

Choose a_n and b_n such that

$$nP[a_n(X_1 - b_n) > x] \longrightarrow e^{-x},$$

i.e. can take

$$a_n = (2/\sigma_\alpha^2)^{1/2} d_n, \quad d_n = (\ln n)^{1/2},$$

and

$$b_n = c_1 d_n + c_2 \ln d_n + c_3 + c_4 \frac{\ln d_n}{d_n} + c_5 \frac{1}{d_n}.$$

Then, if $\mathrm{Cor}(\alpha_t, \alpha_{t+h}) \ln h \to 0$, as $h \to \infty$,

$$P[a_n(M_n - b_n) \le x] \longrightarrow \exp\{-e^{-x}\},$$

$(M_n = \max\{X_1, \ldots, X_n\})$.

More generally, we have the point process convergence,

$$N_n(\cdot) = \sum_{t=1}^{\infty} \varepsilon_{(t/n,\ a_n(X_t - b_n))}(\cdot) \xrightarrow{d} N(\cdot),$$

where, N( ) is a Poisson process on $[0,\infty)\times(-\infty,\infty]$ with intensity measure $dt\times\exp(-x)dx$

Remarks:

- Scaling constants a_n are the same as in the IID Gaussian case. Location constants b_n differ from those in the IID case (call these b_n^*. Then,

$$b_n - b_n^* = c_2 \ln d_n + c_3 + c\frac{1}{d_n},$$

so that b_n is 'slightly larger' than b_n^*.

- The condition, $\mathrm{Cor}(\alpha_t, \alpha_{t+h}) \ln h \to 0$, as $h \to \infty$, is satisfied by both short memory ARMA and long memory fractionally integrated ARMA stochastic volatility models.

- Limit approximation is not as good when $\mathrm{Cor}(\alpha_t, \alpha_{t+h})$ decays slowly or when σ_α^2 is large.