Recent Developments in the Unit Root Problem for Moving Averages

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+ Non-invertible MA(1) Model w/o Trend

- finite variance noise
- non-normal stable noise

+ Non-invertible MA(1) Model With Trend

- constant trend function
- general trend functions

Basic Problem

MA(1) Model:

$$Y_t = \mu + \varepsilon_t - \theta\, \varepsilon_{t-1}, \quad \{\varepsilon_t\} \sim \text{IID } N(0,\sigma^2)$$

$$|\theta| \le 1 \qquad (|\theta| < 1 \text{ invertible})$$

Observations: $y_1, \ldots, y_T$

$\hat{\theta}, \hat{\mu}$ MLE of θ and μ

For $|\theta| < 1$,

$$\hat{\theta} \text{ is } AN(\theta, (1-\theta^2) / T)$$

and

$$\hat{\mu} \text{ is } AN(\mu, \sigma^2(1-\theta)^2 / T).$$

Question: Are these good approximations if θ is near or equal to 1?

Asymptotics (μ=0, finite variance case)

Idea: build parameter normalization into the likelihood function.

Model: $Y_t = \varepsilon_t - (1-\beta/T)\,\varepsilon_{t-1}$, t=1,…,T.

$$\beta = T(1-\theta),\quad \theta = 1-\beta/T,\ \theta_0 = 1-\gamma/T$$

Likelihood:

$$L_T(\beta) = l_T(1-\beta/T) - l_T(1),$$

where $l_T(\ )$ = reduced log(Gaussian likelihood).

Theorem: Under $\theta_0=1-\gamma\,/\,T$,

$$L_T(\beta) \xrightarrow{d} Z_\gamma(\beta) \quad \text{on } C[0,\infty),$$

where

$$Z_\gamma(\beta) = \sum_{k=1}^{\infty} \frac{\beta^2(\pi^2k^2+\gamma^2)}{(\pi^2k^2+\beta^2)\pi^2k^2} X_k^2 + \sum_{k=1}^{\infty} \ln\left(\frac{\pi^2k^2}{(\pi^2k^2+\beta^2)}\right)$$

Results:

- $T(1-\hat{\theta}_{mle}) \to \hat{\beta}_{mle} = \text{argmax } Z_\gamma(\beta)$
- $T(1-\hat{\theta}_{lm}) \to \hat{\beta}_{lm} = \text{arg}\mathit{local}\text{max } Z_\gamma(\beta)$
- $P(\hat{\theta}_{lm}=1) \to P(\hat{\beta}_{lm}=0) = .6518$ if $\gamma = 0.$

Hypothesis Testing

$H_0: \theta_0 = 1$ vs. $H_1: \theta_0 < 1$

Tests:

LM: Reject if $\hat{\theta}_{lm} < c_{lm}(\alpha)$ (e.g. $\alpha = .05$ and $T = 50$, $\hat{\theta}_{lm} < .87$)

MLE: Reject if $\hat{\theta}_{mle} < c_{mle}(\alpha)$

LRT: Reject if $L_T(\hat{\beta}_{lme}) > b_{glr}(\alpha)$

LBIU (Tanaka): Reject if $S_T > s(\alpha)$

Asymptotics (μ=0 case, non-normal stable case)

Model: $Y_t = \varepsilon_t - \theta_0\varepsilon_{t-1}$, t=1,…,T, $\{\varepsilon_t\}$~IID SαS,

i.e., $E[\exp(it\,\varepsilon_t)] = \exp\{-|t|^{\alpha}\}$, $0 < \alpha < 2$.

Invertible case: ($|\theta_0| < 1$)

Results:

- $(T/\ln T)^{1/\alpha}(\hat{\theta}_G - \theta_0) \to \frac{S_1}{S_0}$ where $\hat{\theta}_G$ is the M(*Gaussian*)LE (Mikosch et al. \`95)
- $T^{1/\alpha}(\hat{\theta}_{lad} - \theta_0) \to \eta$, where $\hat{\theta}_{lad}$ is the LAD (Davis \`96)
- $T^{1/\alpha}(\hat{\theta}_{mle} - \theta_0) \to \xi$, where $\hat{\theta}_{mle}$ is the MLE (Calder and Davis \`97)

Non-invertible case:

Model: $Y_t = \varepsilon_t - (1-\beta/T)\,\varepsilon_{t-1}$, t=1,…,T, $\{\varepsilon_t\}$~IID SαS

$$\beta = T(1-\theta), \quad \theta = 1-\beta/T, \quad \theta_0 = 1-\gamma/T,$$

Likelihood:

$$L_T(\beta) = l_T(1-\beta/T) - l_T(1),$$

where $l_T(\ )$ = reduced log(Gaussian likelihood).

Theorem: Under $\theta_0 = 1-\gamma/T$,

$$L_T(\beta) \xrightarrow{d} Z_\gamma(\beta) \quad \text{on } C[0,\infty),$$

where

$$Z_\gamma(\beta) = \sum_{k=1}^{\infty} \frac{\beta^2(\pi^2k^2+\gamma^2)}{(\pi^2k^2+\beta^2)\pi^2k^2}\tilde{X}_k^2 + \sum_{k=1}^{\infty} \ln\left(\frac{\pi^2k^2}{(\pi^2k^2+\beta^2)}\right),$$

and the $\tilde{X}_t = X_t / X_0$ are defined below.

Definition of X_t:

$$X_0 = \left(2 \sum_{0 \le x \le 1} (\Delta M(x))^2 \right)^{1/2}$$

$$X_t = 2 \int_0^1 \frac{-\pi t \cos(\pi t x) + \gamma \sin(\pi t x)}{(\pi^2 t^2 + \gamma^2)^{1/2}} \, dM(x), \ \mathrm{t} = 1,2,...,$$

where M is an SαS Lévy motion on [0,1] with $M(1) \overset{d}{=} \varepsilon_1$.

Properties of $\tilde{X}_t^{\,2} = X_t^{\,2} / X_0^{\,2}$:

1. Uncorrelated.
2. Mean 1.
3. Exponentially decreasing tail.
4. Chi-square distribution with 1 df when α =2.

Idea of argument

Critical terms:

$$\tilde{\sigma}_T^{\,2} = \frac{1+\theta_0^2}{T+1}\sum_{t=0}^{T}\varepsilon_t^2\,,$$

$$\tilde{U}_{t,T} = (2/(T+1))^{1/2}(1+q_T\cos(\frac{\pi t}{T+1}))^{-1/2}\sum_{s=1}^{T}\frac{Y_s}{\tilde{\sigma}_T}\sin(\frac{\pi t s}{T+1})\,.$$

Result:

- $T^{1-2/\alpha}\tilde{\sigma}_T^{\,2} \xrightarrow{d} X_0^2$
- $\tilde{U}_{t,T} \xrightarrow{d} \tilde{X}_t = X_t / X_0$

Results:

- $T(1-\hat{\theta}_{lm}) \rightarrow \hat{\beta}_{lm} = \arg local \max \mathrm{Z}_{\gamma}(\beta)$
- $\mathrm{P}(\hat{\theta}_{lm} = 1) \rightarrow \mathrm{P}(\hat{\beta}_{lm} = 0)$

Accuracy of the Asymptotic Distribution

Step 1. Simulate $X_0, X_1, \ldots, X_N$ using the approximation:

$$X_0^2 = 2\sum_{s=1}^{K} Z_s^2,$$

$$X_t = 2\sum_{s=1}^{K} \frac{-\pi t\cos(\pi ts) + \gamma\sin(\pi ts)}{(\pi^2 t^2 + \gamma^2)^{1/2}} Z_s, \quad t = 1, 2, \ldots, N,$$

where $Z_1, \ldots, Z_K$ are iid $S\alpha S$ random variables.

Step 2. Truncate the infinite series for $Y_\gamma(\beta) = Z_\gamma'(\beta)/\beta$ at N.

Step 3. If $Y_\gamma(0) < 0$, then put $\hat{\beta}_{lm} = 0$.

Step 4. If $Y_\gamma(0) > 0$, then $\hat{\beta}_{lm}$ is defined as the smallest non-negative zero of $Y_\gamma(\beta)$.

Figure 1. Comparison of sampling cdf's (θ_0=1.0, α =1.0)

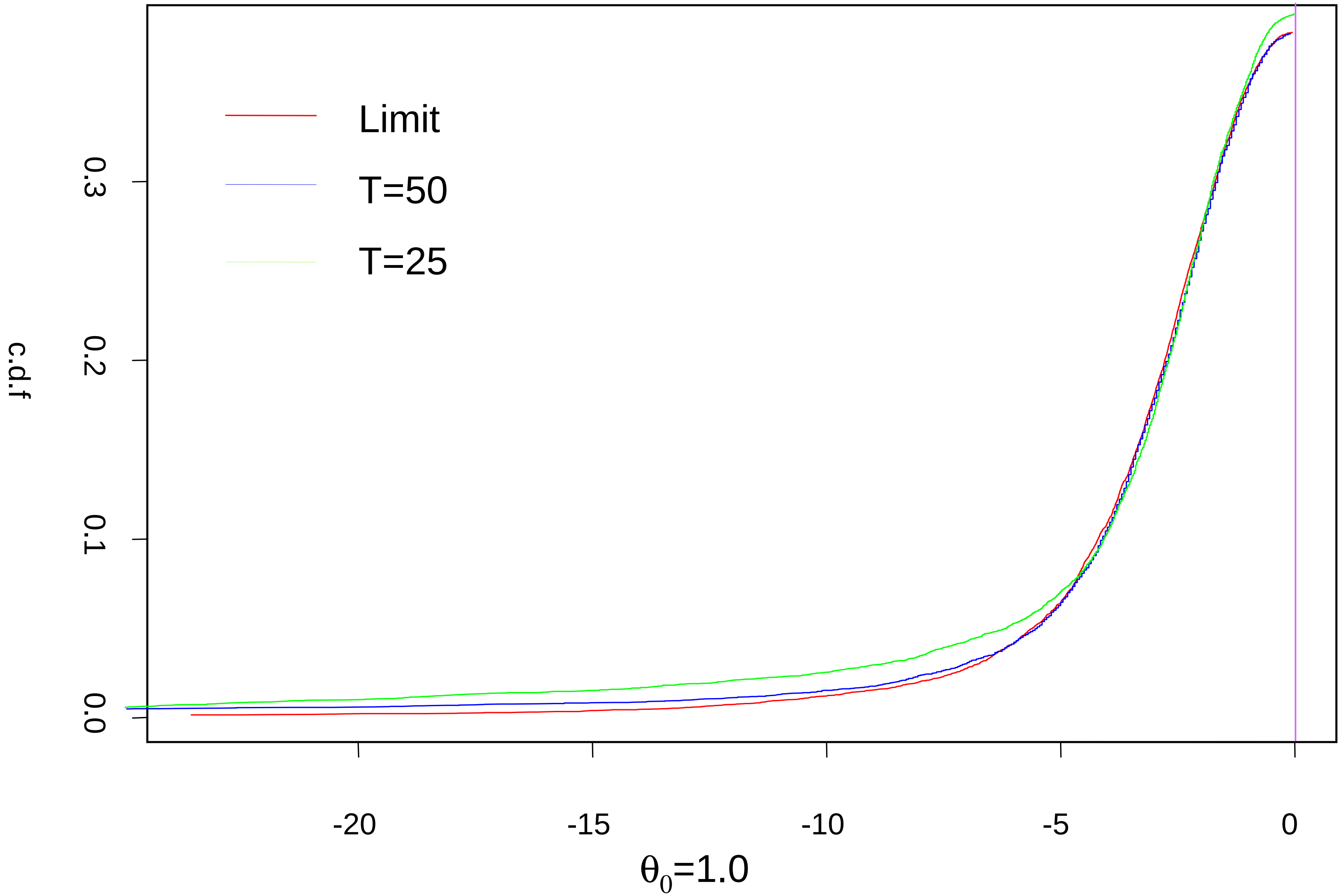


Figure 2. Comparison of limit cdf's for different α's.

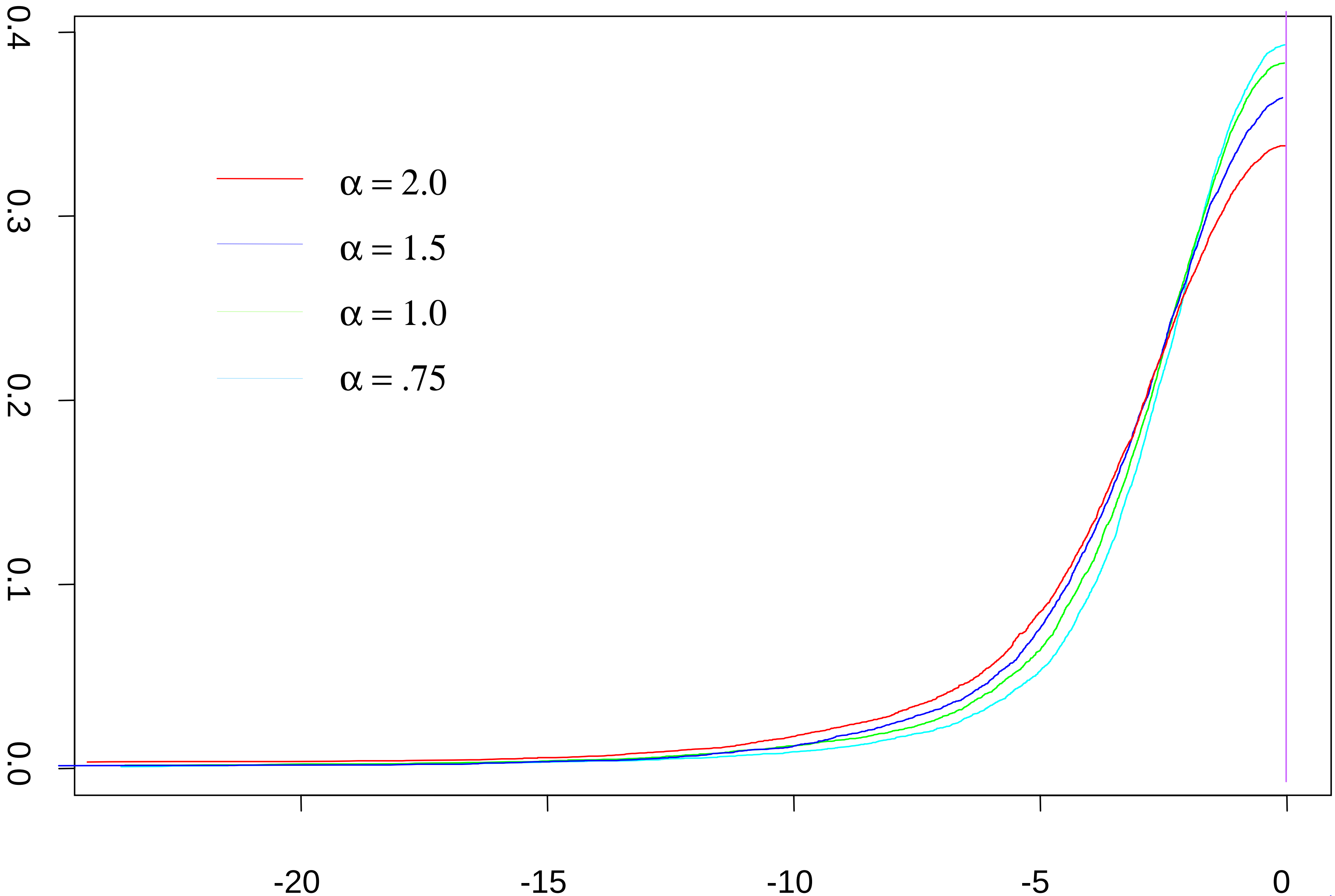


Table 1. Quantiles of $T(\hat{\theta}_{lm} - 1)$ and $-\hat{\beta}_0$ with pile-up,

$$\text{P - U} = P(\hat{\theta}_{lm} = 1)$$

	$\alpha = .75$		$\alpha = 1.0$		$\alpha = 1.5$		$\alpha = 2.0$	
T	.05	P-U	.05	P-U	.05	P-U	.05	P-U
10	-5.63	.597	-7.76	.602	-6.66	.622	-6.48	.637
25	-6.80	.605	-6.17	.606	-6.20	.629	-6.50	.649
50	-5.20	.615	-5.50	.617	-6.25	.630	-6.56	.653
100	-5.20	.615	-5.50	.616	-5.80	.647	-6.50	.655
Limit	-5.13	.607	-5.60	.617	-5.93	.636	-6.52	.658

Table 2. Quantiles of $T(\hat{\theta}_{lm} - 1)$ for T = 50 and θ_0 = .7, .8, .9, .95 (α = 1.0)

θ_0		.05	.1	P-U
.7	T=50	-21.75	-19.40	.052
	Limit	-21.59	-19.65	.058
.8	T=50	-16.15	-14.00	.113
	Limit	-15.72	-14.06	.119
.9	T=50	-10.00	-8.25	.276
	Limit	-9.53	-8.22	.288
.95	T=50	-7.15	-5.60	.465
	Limit	-6.67	-5.47	.464

Asymptotics (constant trend, finite variance)

Idea: build parameter normalization into the likelihood function.

Model: $Y_t = \mu + \varepsilon_t - \theta_0\, \varepsilon_{t-1}$, t=1,…,T, $\{\varepsilon_t\}$~IID $N(0,\sigma^2)$

$$\beta = T(1-\theta), \quad \theta = 1 - \beta/T, \quad \theta_0 = 1 - \gamma/T$$

Likelihood:

$$L_T(\beta) = l_T(1-\beta/T, \hat{\mu}(\beta)) - l_T(1, \hat{\mu}(0)),$$

where $l_T(\ )$ = reduced log(Gaussian likelihood) and $\hat{\mu}(\beta)$ is the MLE of μ with β fixed.

Results:

- $L_T(\beta) \xrightarrow{d} \tilde{Z}_\gamma(\beta)$
- $T(1-\hat{\theta}_{mle}) \xrightarrow{d} \hat{\beta}_{mle} = \text{argmax } \tilde{Z}_\gamma(\beta)$
- $P(\hat{\theta}_{mle} = 1) \to .955$ if $(\gamma = 0)$.
- If $\theta = 1$ known, then

$$T^{3/2}(\hat{\mu}-\mu) \xrightarrow{d} N(0,12\sigma^2).$$

- If $\theta = 1$ and is estimated, then

$T^{3/2}(\hat{\mu}-\mu)$ is NOT asymptotically normal.

Limit distribution is ~$N(0,12\sigma^2)(.955) + N(0,17\sigma^2)(.045)$

Hypothesis Testing

H_0: $\theta_0 = 1$ vs. H_1: $\theta_0 < 1$

Tests:

LRT: Reject if

$$L_T(\hat{\beta}_{mle}) > b_{glr}(\alpha).$$

Because of the large pile-up (.955), have to use randomized test for $\alpha > .045$.

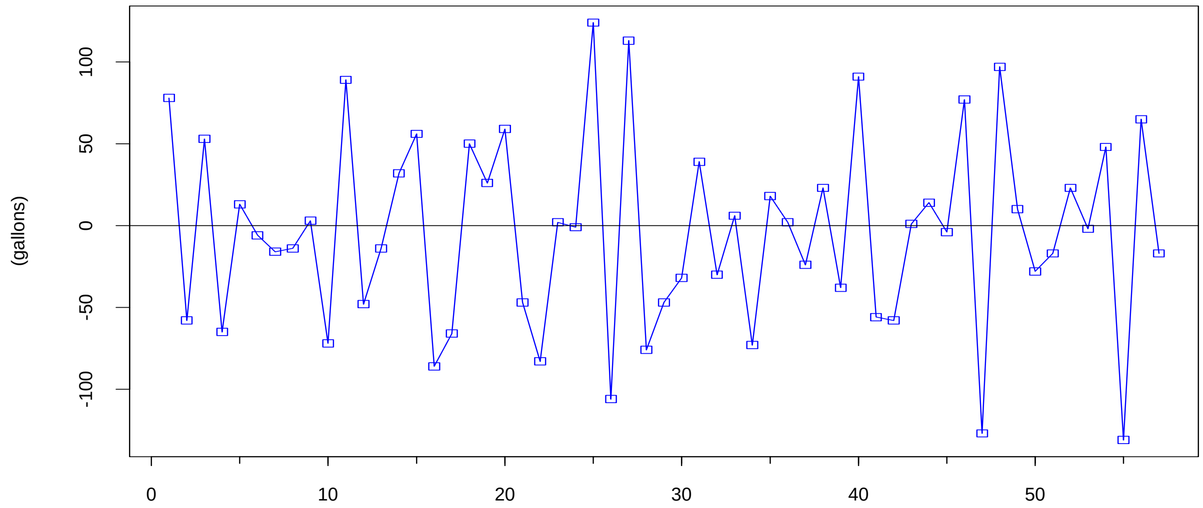
LBIU (Tanaka): Reject if

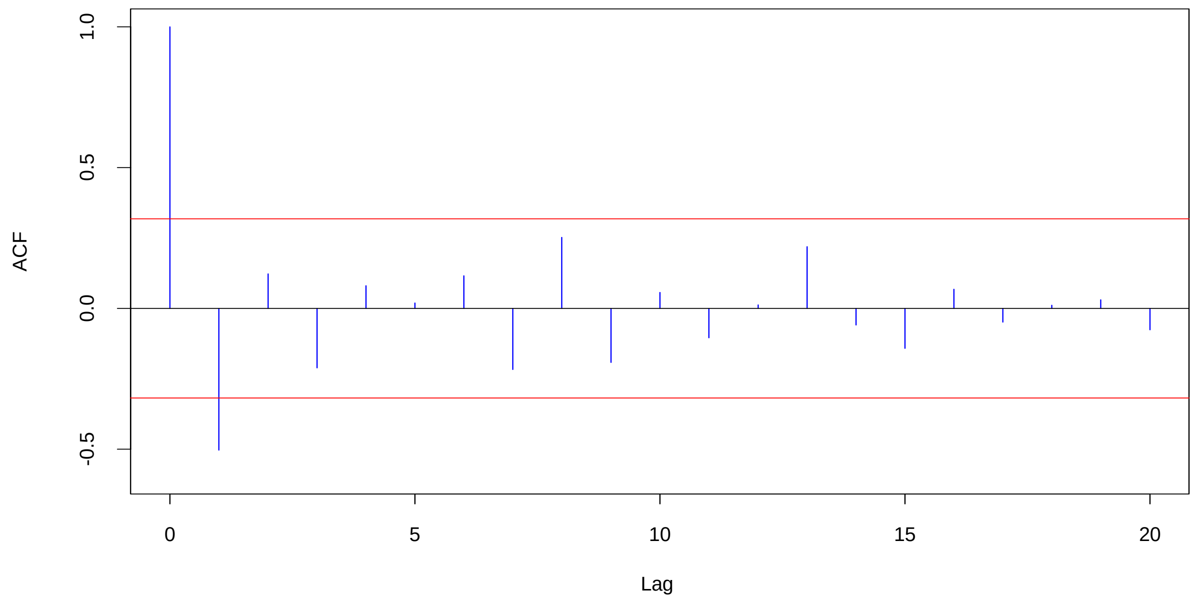
$$S_T > s(\alpha)$$

Power Comparison: (cf. p.390 of Tanaka `96)

	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
γ	LBIU	LR	LBIU	LR	LBIU	LR
1	.011	.011	.053	.054	.105	.104
5	.048	.051	.137	.136	.217	.181
10	.209	.236	.367	.372	.470	.401
20	.603	.692	.748	.784	.816	.796
50	.966	.992	.987	.994	.993	.995
60	.985	.997	.995	.997	.998	.998

An Example : (overshorts $Y_1, \ldots, Y_{57}$ from an underground storage tank.)





Model. $Y_t = \mu + \varepsilon_t - \theta\, \varepsilon_{t-1}$

Problem. Estimate μ and construct a C.I.?

(Is $\mu < -5$ gallons/day?)

Estimation.

Estimates	Asymptotic Var
$\hat{\mu}_{MLE} = -4.78$	$(1-\theta)^2\sigma^2/T = (.905)^2 = .819$
$\hat{\theta}_{MLE} = .849$	$(1-\theta^2)/T = (.070)^2$

Note: If $\theta = 1$ is not estimated then

$\hat{\mu}_{MLE}$ is $AN(\mu_0, 12\sigma^2/T^3) = AN(\mu_0, (.362)^2)$

Asymptotics (general trend, finite variance)

Model: $Y_t = b_0 + x_{t1}b_1 + \cdot\cdot\cdot + x_{tk}b_k + U_t$,

$U_t = \varepsilon_t - (1 - \beta/T)\,\varepsilon_{t-1}$, t=1,…,T, $\{\varepsilon_t\}$~IID $N(0,\sigma^2)$

Linear Model: $\mathbf{Y_T} = \mathbf{X}\,\mathbf{b} + \mathbf{U_T}$

Under growth conditions on the $\{x_{tj}\}$, limit behavior of $\hat{\theta}_{lm}$ and $\hat{\mathbf{b}}$ can be obtained.

Example: $Y_t = b_0 + b_1 t + \varepsilon_t - \varepsilon_{t-1}$.

$$\begin{bmatrix} T^{3/2}(\hat{b}_0 - b_0) \\ T^{5/2}(\hat{b}_1 - b_1) \end{bmatrix} \xrightarrow{d} N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \sigma^2 \begin{bmatrix} 192 & -360 \\ -360 & 720 \end{bmatrix}\right)$$