Maximum Likelihood Estimation for Allpass Time Series Models

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http://www.stat.colostate.edu/~rdavis/lectures/magdeburg02.pdf

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☞ Introduction

- properties of financial time series
- motivating example
- all-pass models and their properties

☞ Estimation

- likelihood approximation
- MLE and LAD
- asymptotic results
- order selection

☞ Empirical results

- simulation
- NZ/USA exchange rates

☞ Noninvertible MA processes

- preliminaries
- a two-step estimation procedure
- Microsoft trading volume

☞ Summary

Financial Time Series

☞ Log returns, $X_t = 100*(\ln(P_t) - \ln(P_{t-1}))$, of financial assets often exhibit:

- heavy-tailed marginal distributions

  $P(|X_1| > x) \sim C\, x^{-\alpha}, \quad 0 < \alpha < 4.$

- lack of serial correlation

  $\hat{\rho}_X(h)$ near 0 for all lags $h > 0$ (MGD sequence)

- $|X_t|$ and X_t^2 have slowly decaying autocorrelations

  $\hat{\rho}_{|X|}(h)$ and $\hat{\rho}_{X^2}(h)$ converge to 0 *slowly* as $h \to \infty$

- process exhibits 'stochastic volatility'

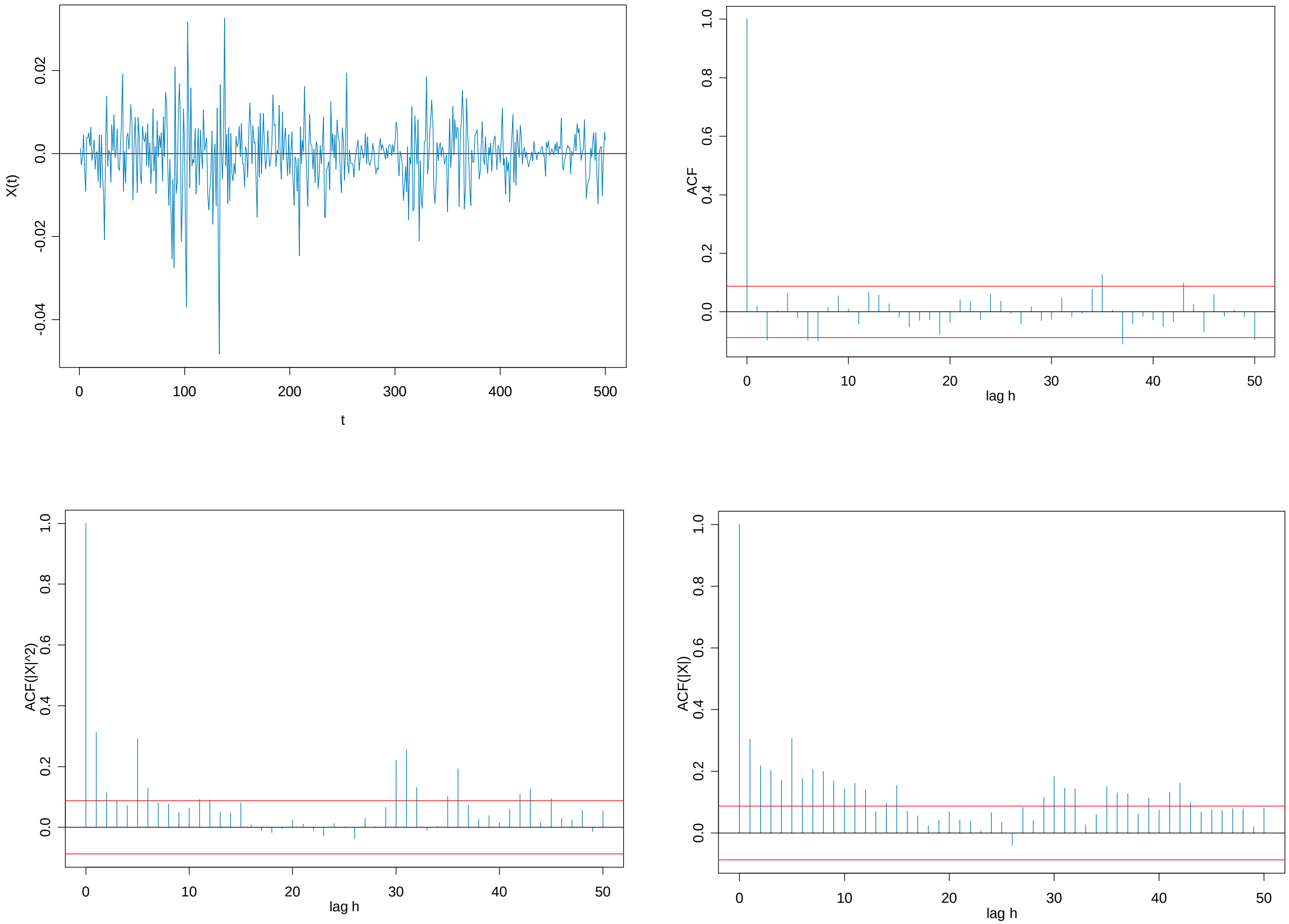
☞ Nonlinear models $X_t = \sigma_t Z_t$, $\{Z_t\} \sim \mathrm{IID}(0,1)$

- ARCH and its variants (Engle `82; Bollerslev, Chou, and Kroner 1992)

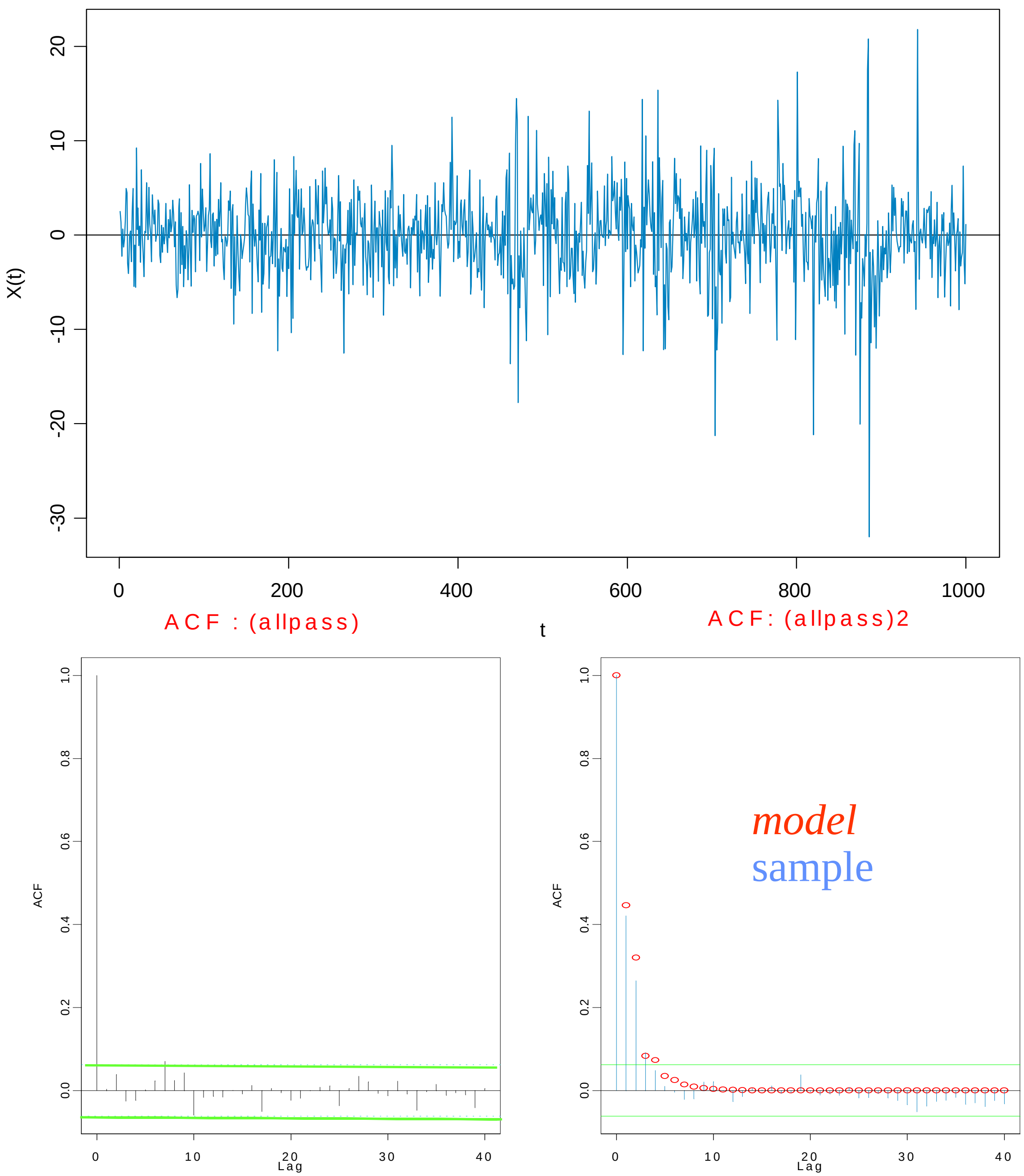
- Stochastic volatility (Clark 1973; Taylor 1986)

Motivating example

500-daily log-returns of NZ/US exchange rate



All-pass model of order 2 (t3 noise)



All-pass Models

Causal AR polynomial: $\phi(z)=1-\phi_1 z - \cdots - \phi_p z^p$, $\phi(z) \neq 0$ for $|z|\leq 1$.

Define MA polynomial:

$$\theta(z) = -z^p\,\phi(z^{-1})/\phi_p = -(z^p - \phi_1 z^{p-1} - \cdots - \phi_p)/\,\phi_p$$

$$\neq 0 \text{ for } |z|\geq 1$$ (MA polynomial is non-invertible).

Model for data $\{X_t\}$: $\phi(B)X_t = \theta(B)\,Z_t$, $\{Z_t\} \sim$ IID (non-Gaussian)

$$B^k X_t = X_{t-k}$$

Examples:

All-pass(1): $X_t - \phi\, X_{t-1} = Z_t - \phi^{-1}\, Z_{t-1}$, $|\,\phi\,| < 1$.

All-pass(2): $X_t - \phi_1\, X_{t-1} - \phi_2\, X_{t-2} = Z_t + \phi_1/\,\phi_2\, Z_{t-1} - 1/\,\phi_2\, Z_{t-2}$

Properties:

- causal, non-invertible ARMA with MA representation

$$X_t = \frac{B^p \phi(B^{-1})}{-\phi_p \phi(B)} Z_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

- uncorrelated (flat spectrum)

$$f_X(\omega) = \frac{|e^{-ip\omega}|^2 |\phi(e^{i\omega})|^2}{\phi_p^2 |\phi(e^{-i\omega})|^2} \frac{\sigma^2}{2\pi} = \frac{\sigma^2}{\phi_p^2 2\pi}$$

- zero mean
- data are dependent if noise is non-Gaussian (e.g. Breidt & Davis 1991).
- squares and absolute values are correlated.
- X_t is heavy-tailed if noise is heavy-tailed.

Estimation for All-Pass Models

☞ Second-order moment techniques do not work

- least squares
- Gaussian likelihood

☞ Higher-order cumulant methods

- Giannakis and Swami (1990)
- Chi and Kung (1995)

☞ Non-Gaussian likelihood methods

- likelihood approximation
- quasi-likelihood
- least absolute deviations
- minimum dispersion

Approximating the likelihood

Data: $(X_1, \ldots, X_n)$

Model: $$X_t = \phi_{01} X_{t-1} + \cdots + \phi_{0p} X_{t-p} - (Z_{t-p} - \phi_{01} Z_{t-p+1} - \cdots - \phi_{0p} Z_t) / \phi_{0r}$$

where ϕ_{0r} is the last non-zero coefficient among the ϕ_{0j}'s.

Noise: $z_{t-p} = \phi_{01} z_{t-p+1} + \cdots + \phi_{0p} z_t - (X_t - \phi_{01} X_{t-1} - \cdots - \phi_{0p} X_{t-p})$,

where $z_t = Z_t / \phi_{0r}$.

More generally define,

$$z_{t-p}(\phi) = \begin{cases} 0, & \text{if } t = n+p, \ldots, n+1, \\ \phi_1 z_{t-p+1}(\phi) + \cdots + \phi_p z_t(\phi) - \phi(B) X_t, & \text{if } t = n, \ldots, p+1. \end{cases}$$

Note: $z_t(\phi_0)$ is a close approximation to z_t (initialization error)

Assume that Z_t has density function f_σ and consider the vector

$$\mathbf{z} = (\underbrace{X_{1-p},\ldots,X_0, z_{1-p}(\phi),\ldots,z_0(\phi)}, \underbrace{z_1(\phi),\ldots,z_{n-p+1}(\phi),\ldots,z_n(\phi)})'$$

independent pieces

Joint density of $\mathbf{z}$:

$$h(\mathbf{z}) = h_1(X_{1-p},\ldots,X_0, z_{1-p}(\phi),\ldots,z_0(\phi))$$

$$\bullet \left(\prod_{t=1}^{n-p} f_\sigma(\phi_q z_t(\phi)) \,|\, \phi_q \,| \right) h_2(z_{n-p+1}(\phi),\ldots,z_n(\phi)),$$

and hence the joint density of the data can be approximated by

$$h(\mathbf{x}) = \left(\prod_{t=1}^{n-p} f_\sigma(\phi_q z_t(\phi)) \,|\, \phi_q \,| \right)$$

where $q = \max\{0 \le j \le p: \phi_j \ne 0\}$.

Log-likelihood:

$$L(\phi,\sigma) = -(n-p)\ln(\sigma / |\phi_q|) + \sum_{t=1}^{n-p} \ln f(\sigma^{-1}\phi_q z_t(\phi))$$

where $f_\sigma(z) = \sigma^{-1} f(z/\sigma)$.

Least absolute deviations: choose Laplace density

$$f(z) = \frac{1}{\sqrt{2}} \exp(-\sqrt{2}\,|z|)$$

and log-likelihood becomes

$$\text{constant} - (n-p)\ln\kappa - \sum_{t=1}^{n-p} \sqrt{2}\,|z_t(\phi)| / \kappa, \;\; \kappa = \sigma / |\phi_q|$$

Concentrated Laplacian likelihood

$$l(\phi) = \text{constant} - (n-p)\ln \sum_{t=1}^{n-p} |z_t(\phi)|$$

Maximizing $l(\phi)$ is equivalent to minimizing the absolute deviations

$$m_{\text{n}}(\phi) = \sum_{t=1}^{n-p} |z_t(\phi)|.$$

Assumptions

☞ Assume $\{Z_t\}$ iid $f_\sigma(z)=\sigma^{-1}f(\sigma^{-1}z)$ with

- σ a scale parameter
- mean 0, variance σ^2

☞ For f known, use maximum likelihood

- further smoothness assumptions (integrability, symmetry, etc.) on f
- Fisher information: $\tilde{I} = \sigma^{-2}\int (f'(z))^2 / f(z)dz$

☞ For f unknown, use quasi-likelihood

☞ Least absolute deviations

- assume f has median 0
- assume f continuous in neighborhood of 0
- act as if f = Laplace to get criterion function

Results

- Let $\gamma(h)$ = ACVF of AR model with AR poly $\phi_0(.)$ and

$$\Gamma_p = [\gamma(j-k)]_{j,k=1}^{p}$$

- Maximum likelihood:

$$\sqrt{n}(\hat{\phi}_{MLE} - \phi_0) \xrightarrow{D} N(0, \frac{1}{2(\sigma^2 \tilde{I} - 1)} \sigma^2 \Gamma_p^{-1})$$

- Least absolute deviations:

$$\sqrt{n}(\hat{\phi}_{LAD} - \phi_0) \xrightarrow{D} N(0, \frac{\mathrm{Var}(|Z_1|)}{2\sigma^4 f_\sigma^2(0)} \sigma^2 \Gamma_p^{-1})$$

Further comments on MLE

Let $\alpha=(\phi_1, \ldots, \phi_p, \sigma/|\phi_1|, \beta_1, \ldots, \beta_q)$, where $\beta_1, \ldots, \beta_q$ are the parameters of pdf f.

Set

☞ $$\hat{I} = \sigma_0^{-2} \int (f'(z;\beta_0))^2 / f(z;\beta_0)\,dz$$

☞ $$\hat{K} = \alpha_{0,p+1}^{-2} \left\{ \int z^2 (f'(z;\beta_0))^2 / f(z;\beta_0)\,dz - 1 \right\}$$

☞ $$L = -\alpha_{0,p+1}^{-1} \int z \frac{f'(z;\beta_0)}{f(z;\beta_0)} \frac{\partial f(z;\beta_0)}{\partial \beta_0}\,dz$$

☞ $$I_f(\beta_0) = \int \frac{1}{f(z;\beta_0)} \frac{\partial f(z;\beta_0)}{\partial \beta_0} \frac{\partial f^T(z;\beta_0)}{\partial \beta_0}\,dz$$ (Fisher Information)

Under smoothness conditions on *f* wrt $\beta_1, \ldots, \beta_q$ we have

$$\sqrt{n}(\hat{\alpha}_{\text{MLE}} - \alpha_0) \xrightarrow{D} N(0, \Sigma^{-1}),$$

where

$$\Sigma^{-1} = \begin{bmatrix} \frac{1}{2(\sigma_0^2 \hat{I} - 1)} \Gamma_p^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\hat{K} - L' I_f^{-1} L)^{-1} & -\hat{K}^{-1} L' (I_f - L\hat{K}^{-1} L')^{-1} \\ \mathbf{0} & -(I_f - L\hat{K}^{-1} L')^{-1} L\hat{K}^{-1} & (I_f - L\hat{K}^{-1} L')^{-1} \end{bmatrix}$$

Note: $\hat{\phi}_{\text{MLE}}$ is asymptotically independent of $\hat{\alpha}_{\text{p+1,MLE}}$ and $\hat{\beta}_{\text{MLE}}$

Identifiability in LAD case?

- Minimizer may not be unique.
- Gaussian case: $\{Z_t\}$ iid $\mathrm{N}(0,\sigma_0^2\phi_{0p}^{-2}) = \mathrm{N}(0,\sigma_1^2\phi_{1p}^{-2})$, so

$$E\,|\,z_1(\phi_1)\,| = E\left|\frac{Z_1\sigma_1}{\sigma_0\phi_{1p}}\right| = E\left|\frac{Z_1\sigma_0}{\sigma_0\phi_{0p}}\right| = E\,|\,z_1(\phi_0)\,|$$

- Consider $\{c_j\}$ with at least two non-zero elements and

$$\sum_{j=-\infty}^{\infty} |c_j| < \infty \text{ and } \sum_{j=-\infty}^{\infty} c_j^2 = 1$$

Jian and Pawitan (1998) show

$$E\,\left|\sum_{j=-\infty}^{\infty} c_j Z_j\right| > E\,|Z_1|$$

holds for Laplace, Student's t, contaminated normal, etc.

- Non-Gaussian case: $E\,|\,z_1(\phi_1)\,| = E\left|\frac{\phi_0(B^{-1})\phi_1(B)}{\phi_{0p}\phi_1(B^{-1})\phi_0(B)}Z_t\right| > E\,|\,z_1(\phi_0)\,|$

Central Limit Theorem (LAD case)

- Think of $\mathbf{u} = n^{1/2}(\phi - \phi_0)$ as an element of $\mathbb{R}^p$

- Define

$$S_n(\mathbf{u}) = \sum_{t=1}^{n-p} \left(| z_t(\phi_0 + n^{-1/2}\mathbf{u}) | - | z_t(\phi_0) | \right)$$

$$= m_n(\phi_0 + n^{-1/2}\mathbf{u}) - \sum_{t=1}^{n-p} | z_t(\phi_0) |$$

- Then $S_n(\mathbf{u}) \to S(\mathbf{u})$ in distribution on $C(\mathbb{R}^p)$, where

$$S(\mathbf{u}) = \frac{f_\sigma(0)}{|\phi_{0r}|} \mathbf{u}'\Gamma_p \mathbf{u} + \mathbf{u}'\mathbf{N}, \ \mathbf{N} \sim N(\mathbf{0}, \frac{2\mathrm{Var}(|Z_1|)}{\phi_{0r}^2 \sigma^2} \Gamma_p),$$

- Hence,

$$\arg\min S_n(\mathbf{u}) = n^{1/2}(\hat{\phi}_{LAD} - \phi_0)$$

$$\underset{D}{\to} \arg\min S(\mathbf{u})$$

$$= -\frac{|\phi_{0r}| \Gamma_p^{-1}}{2 f_\sigma(0)} \mathbf{N} \sim N(\mathbf{0}, \frac{\mathrm{Var}(|Z_1|)}{2\sigma^4 f_\sigma^4(0)} \sigma^2 \Gamma_p^{-1})$$

Asymptotic Results (LAD case):

Theorem 1. Let $\{Y_t\}$ be the linear process

$$Y_t = \sum_{j=-\infty}^{\infty} c_j z_{t-j},$$

where $c_0=0$, $\sum_{j=-\infty}^{\infty} |c_j| < \infty$, $\{z_t\} \sim \text{IID}(0,\sigma^2)$, $\text{median}(z_1)=0$, $g(0)>0$ (g density of z_1). Then

$$S_n = \sum_{t=1}^{n-p} \left(| z_t - n^{-1/2} Y_t | - | z_t | \right)$$

$$\to \text{Var}(Y_1) g(0) + N$$

where $N \sim \text{N}(0, \gamma^*(0) + 2\sum_{h \geq 1} \gamma^*(h))$ and $\gamma^*(h)$ is the covariance function for $Y_t \,\text{sgn}(z_t)$

Key idea:

$$S_n = \sum_{t=1}^{n-p} \left(| z_t - n^{-1/2} Y_t | - | z_t | \right)$$

$$= -n^{-1/2} \sum_{t=1}^{n-p} Y_t \operatorname{sgn}(z_t)$$

$$+ 2 \sum_{t=1}^{n-p} (n^{-1/2} Y_t - z_t) \left\{ 1_{\{0 < z_t < n^{-1/2} Y_t\}} - 1_{\{n^{-1/2} Y_t < z_t < 0\}} \right\}$$

$$\rightarrow N + Var(Y_1) g(0)$$

Theorem 2. On C(R^p),

$$S_n(\mathbf{u}) = \sum_{t=1}^{n-p}\left(|z_t(\phi_0 + n^{-1/2}\mathbf{u})| - |z_t(\phi_0)|\right)$$

$$\to S(\mathbf{u}),$$

where

$$S(\mathbf{u}) = \frac{f_\sigma(0)}{|\phi_{0r}|}\mathbf{u}'\Gamma_p\mathbf{u} + \mathbf{u}'\mathbf{N},$$

$$\mathbf{N} \sim N(\mathbf{0}, \frac{2Var(|Z_1|)}{\phi_{0r}^2\sigma^2}\Gamma_p),$$

and Γ_p is the covariance matrix of a causal AR(p).

Limit theory for LAD estimate. Note that

$$\hat{\phi}_{\text{LAD}} = \phi_0 + \hat{\mathbf{u}}_n / \sqrt{n}$$

so that $\hat{\mathbf{u}}_n = \sqrt{n}(\hat{\phi}_{\text{LAD}} - \phi_0) = \arg\min S_n(\mathbf{u})$

$$\to \hat{\mathbf{u}} = \arg\min S(\mathbf{u}).$$

Minimizing S, we find that the minimizer or limit random variable is

$$\hat{\mathbf{u}}_n = \sqrt{n}(\hat{\phi}_{\text{LAD}} - \phi_0) \to -\frac{|\phi_{0r}|\Gamma_p^{-1}}{2 f_\sigma(0)} \mathbf{N}$$

$$-\frac{|\phi_{0r}|\Gamma_p^{-1}}{2 f_\sigma(0)} \mathbf{N} \sim N(\mathbf{0}, \frac{Var(|Z_1|)}{2\sigma^4 f_\sigma^2(0)} \sigma^2 \Gamma_p^{-1})$$

Asymptotic Covariance Matrix

- For LS estimators of AR(p):

$$\sqrt{n}(\hat{\phi}_{\mathrm{LS}} - \phi_0) \xrightarrow{D} N(0, \sigma^2 \Gamma_p^{-1})$$

- For LAD estimators of AR(p):

$$\sqrt{n}(\hat{\phi}_{\mathrm{LAD}} - \phi_0) \xrightarrow{D} N(0, \frac{1}{4\sigma^2 f^2(0)} \sigma^2 \Gamma_p^{-1})$$

- For LAD estimators of AP(p):

$$\sqrt{n}(\hat{\phi}_{\mathrm{LAD}} - \phi_0) \xrightarrow{D} N(0, \frac{\mathrm{Var}(|Z_1|)}{2\sigma^4 f_\sigma^2(0)} \sigma^2 \Gamma_p^{-1})$$

- For MLE estimators of AP(p):

$$\sqrt{n}(\hat{\phi}_{\mathrm{MLE}} - \phi_0) \xrightarrow{D} N(0, \frac{1}{2(\sigma^2 \hat{I} - 1)} \sigma^2 \Gamma_p^{-1})$$

Laplace: (LAD=MLE)

$$\frac{\mathrm{Var}(|Z_1|)}{2\sigma^4 f_\sigma^2(0)} = \frac{1}{2} = \frac{1}{2(\sigma^2 \hat{I} - 1)}$$

Students t_ν, $\nu > 2$:

LAD: $$\frac{\mathrm{Var}(|Z_1|)}{2\sigma^4 f_\sigma^2(0)} = \frac{\Gamma^2(\nu/2)(\nu-2)\pi}{2\Gamma^2((\nu+1)/2)} - \frac{2(\nu-2)^2}{(\nu-1)^2}$$

MLE: $$\frac{1}{2(\sigma^2 \hat{I} - 1)} = \frac{(\nu-2)(\nu+3)}{12}$$

Student's t_3:

LAD: .7337

MLE: 0.5

ARE: .7337/.5=1.4674

Order Selection:

Partial ACF From the previous result, if true model is of order r and fitted model is of order p > r, then

$$n^{1/2}\hat{\phi}_{p,LAD} \to N\left(0, \frac{\mathrm{Var}(|Z|)}{2\sigma^4 f_\sigma^2(0)}\right)$$

where $\hat{\phi}_{p,LAD}$ is the pth element of $\hat{\phi}_{LAD}$.

Procedure:

1. Fit high order (P-th order), obtain residuals and estimate scalar,

$$\theta^2 = \frac{\mathrm{Var}(|Z_1|)}{2\sigma^4 f_\sigma^2(0)},$$

by empirical moments of residuals and density estimates.

2. Fit AP models of order p=1,2, . . . , P via LAD and obtain p-th coefficient $\hat{\phi}_{p,p}$ for each.

3. Choose model order r as the smallest order beyond which the estimated coefficients are statistically insignificant.

Note: Can replace $\hat{\phi}_{p,p}$ with $\hat{\phi}_{p,\text{MLE}}$ if using MLE. In this case for p > r

$$n^{1/2}\hat{\phi}_{p,MLE} \to N(0, \frac{1}{2(\sigma^2 \hat{I} - 1)}).$$

AIC: 2p or not 2p?

- An approximately unbiased estimate of the Kullback-Leiber index of fitted to true model:

$$AIC(p) := -2L_X(\hat{\phi},\hat{\kappa}) + \frac{\mathrm{Var}(|Z_1|)}{E|Z_1|\sigma^2 f_\sigma(0)}\,p$$

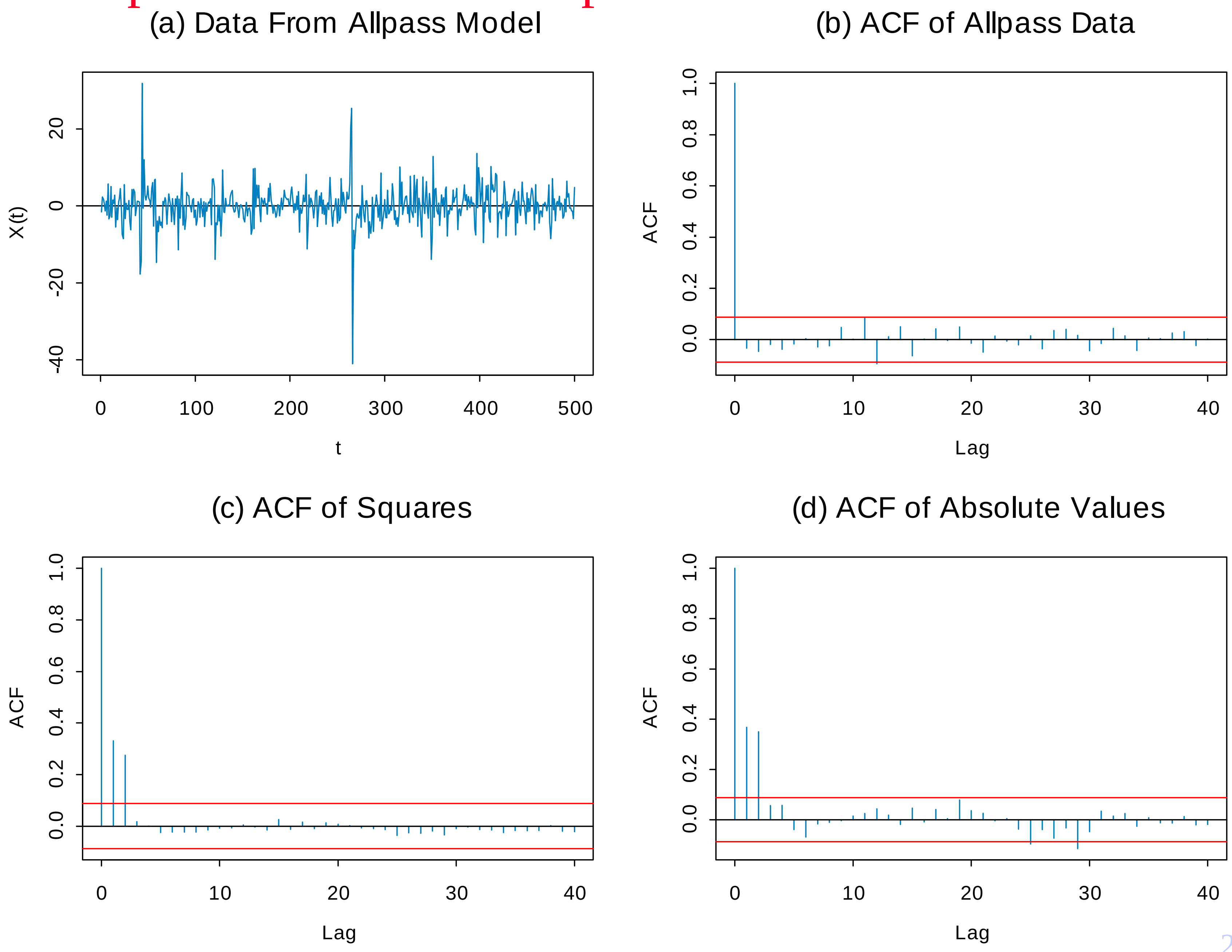
- Penalty term for Laplace case:

$$\frac{\mathrm{Var}(|Z_1|)}{E|Z_1|\sigma^2 f_\sigma(0)}\,p = \frac{\sigma^2/2}{(\sigma/\sqrt{2})\sigma^2(1/\sqrt{2}\sigma)}\,p = p$$

- Estimated penalty term:

$$\frac{\mathrm{var}(|z_t(\hat{\phi})|)}{ave\{|z_t(\hat{\phi})|\}\hat{f}_{z_t(\hat{\phi})}(0)}\,p \xrightarrow[\mathrm{P}]{} \frac{\mathrm{Var}(|Z_1|)}{E|Z_1|\sigma^2 f_\sigma(0)}\,p$$

Sample realization of all-pass of order 2



Estimates:

$$\hat{\phi}_1 = .297(.0381), \hat{\phi}_2 = .374(.0381)$$

Standard errors computed as $\hat{\theta}\, sqrt\{(1-\hat{\phi}_2^2)/500\}$

where $\hat{\theta} = .919$

Order selection:

- cut-off value for PACF is 1.96*.908/sqrt(500)=.0796
- $AIC(p) := -2L_X(\hat{\phi}, \hat{\kappa}) + 1.896p$

	1	2	3	4	5
phi_p	0.289	0.374	0.009	0.011	0.01
AIC(p)	2451	2346	2347	2348	2350

	6	7	8	9	10
phi_p	0.047	0.034	-0.05	0.083	0.021
AIC(p)	2348	2349	2345	2343	2345

Simulation results:

- 1000 replicates of all-pass models
- model order parameter value

model order	parameter value
1	$\phi_1 = .5$
2	$\phi_1 = .3$, $\phi_2 = .4$

- noise distribution is t with 3 d.f.
- sample sizes n=500, 5000
- estimation method is LAD

To guard against being trapped in local minima, we adopted the following strategy.

- 250 random starting values were chosen at *random*. For model of order p, k-th starting value was computed recursively as follows:

  1. Draw $\phi_{11}^{(k)}, \phi_{22}^{(k)}, \ldots, \phi_{pp}^{(k)}$ iid uniform (-1,1).
  2. For j=2, …, p, compute

$$\begin{bmatrix} \phi_{j1}^{(k)} \\ \vdots \\ \phi_{j,j-1}^{(k)} \end{bmatrix} = \begin{bmatrix} \phi_{j-1,1}^{(k)} \\ \vdots \\ \phi_{j-1,j-1}^{(k)} \end{bmatrix} - \phi_{jj}^{(k)} \begin{bmatrix} \phi_{j-1,j-1}^{(k)} \\ \vdots \\ \phi_{j-1,1}^{(k)} \end{bmatrix}$$

- Select top 10 based on minimum function evaluation.

- Run Hooke and Jeeves with each of the 10 starting values and choose best optimized value.

N	Asymptotic mean	Asymptotic std dev	Empirical mean	Empirical std dev	%coverage	rel eff*
500	ϕ_1=.5	**.0332**	.4979	**.0397**	94.2	11.8
5000	ϕ_1=.5	**.0105**	.4998	**.0109**	95.4	9.3

N	Asymptotic mean	Asymptotic std dev	Empirical mean	Empirical std dev	%coverage
500	ϕ_1=.3	**.0351**	.2990	**.0456**	92.5
	ϕ_2=.4	**.0351**	.3965	**.0447**	92.1
5000	ϕ_1=.3	**.0111**	.3003	**.0118**	95.5
	ϕ_2=.4	**.0111**	.3990	**.0117**	94.7

*Efficiency relative to maximum absolute residual kurtosis:

$$\left|\frac{1}{n-p}\sum_{t=1}^{n-p}\left(\frac{z_t(\phi)}{v_2^{1/2}}\right)^4 - 3\right|, \quad v_2 = \frac{1}{n-p}\sum_{t=1}^{n-p}(z_t(\phi)-\overline{z}(\phi))^2$$

MLE Simulations Results using t-distr(3.5)

	Asymptotic		Empirical		
N	mean	**std dev**	mean	**std dev**	%coverage
500	$\phi_1=.5$	**.0349**	.4983	**.0421**	91.7
	$\nu=3.5$	**.5853**	3.449	**.4527**	92.5
5000	$\phi_1=.5$	**.0110**	.4997	**.0088**	95.0
	$\nu=3.5$	**.1851**	3.449	**.1341**	96.7

	Asymptotic		Empirical		
N	mean	**std dev**	mean	**std dev**	%coverage
500	$\phi_1=.3$	**.0369**	.2969	**.0451**	89.6
	$\phi_2=.4$	**.0369**	.3973	**.0446**	90.6
	$\nu=3.5$	**.5853**	3.556	**.5685**	92.4
5000	$\phi_1=.3$	**.0117**	.3002	**.0099**	94.7
	$\phi_2=.4$	**.0117**	.4001	**.0106**	93.6
	$\nu=3.5$	**.1764**	3.510	**.1764**	94.7

Minimum Dispersion Estimator: Minimize the objective fcn

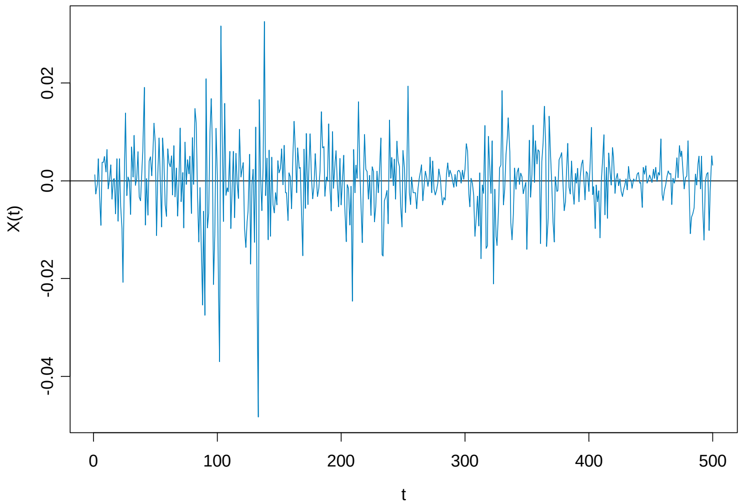
$$S(\phi) = \sum_{t=1}^{n-p} \left(\frac{t}{n-p+1} - \frac{1}{2} \right) z_{(t)}(\phi)$$

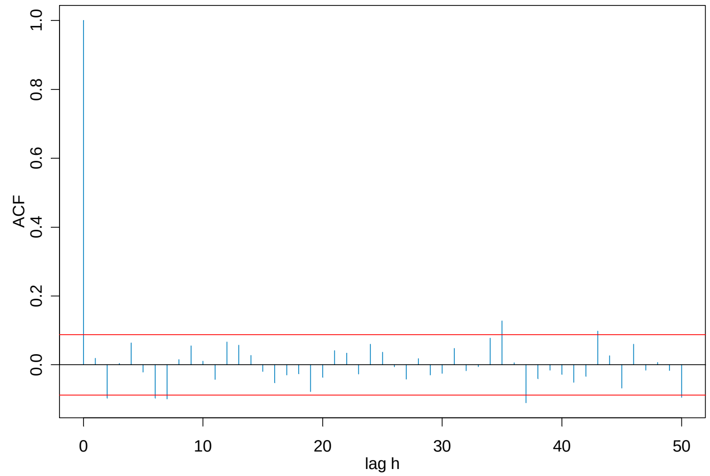
where $\{z_{(t)}(\phi)\}$ are the ordered $\{z_t(\phi)\}$.

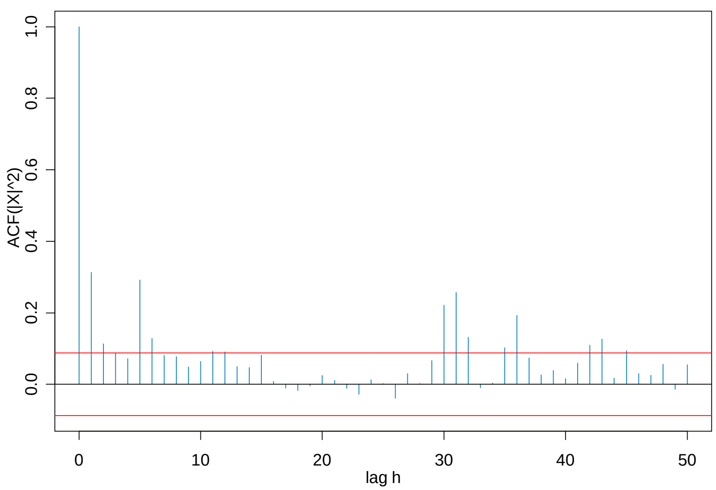
		Empirical		Empirical LAD	
N		mean	**std dev**	mean	**std dev**
500	ϕ_1=.5	.4978	**.0315**	.4979	**.0397**
5000	ϕ_1=.5	.4997	**.0094**	.4998	**.0109**
500	ϕ_1=.3	.2988	**.0374**	.2990	**.0456**
	ϕ_2=.4	.3957	**.0360**	.3965	**.0447**
5000	ϕ_1=.3	.3007	**.0101**	.3003	**.0118**
	ϕ_2=.4	.3993	**.0104**	.3990	**.0117**

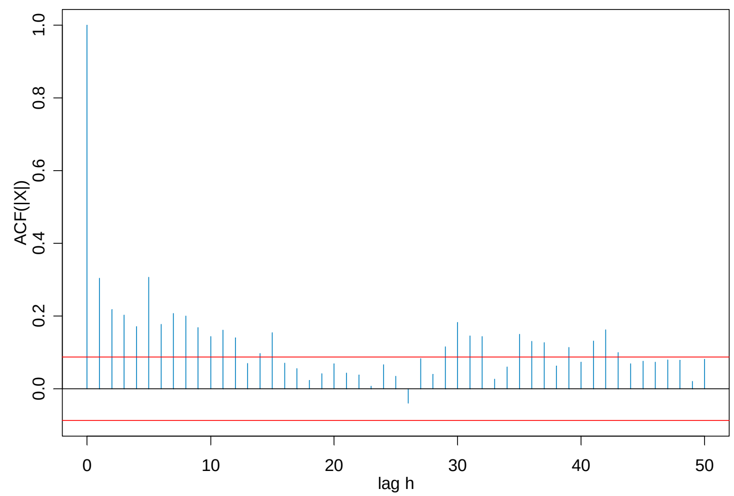
Application to financial data

500-daily log-returns of NZ/US exchange rate





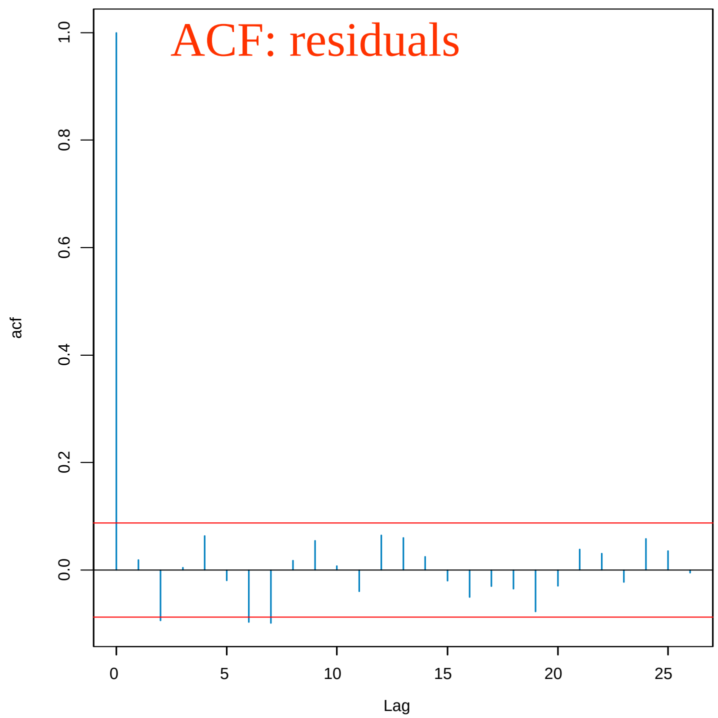


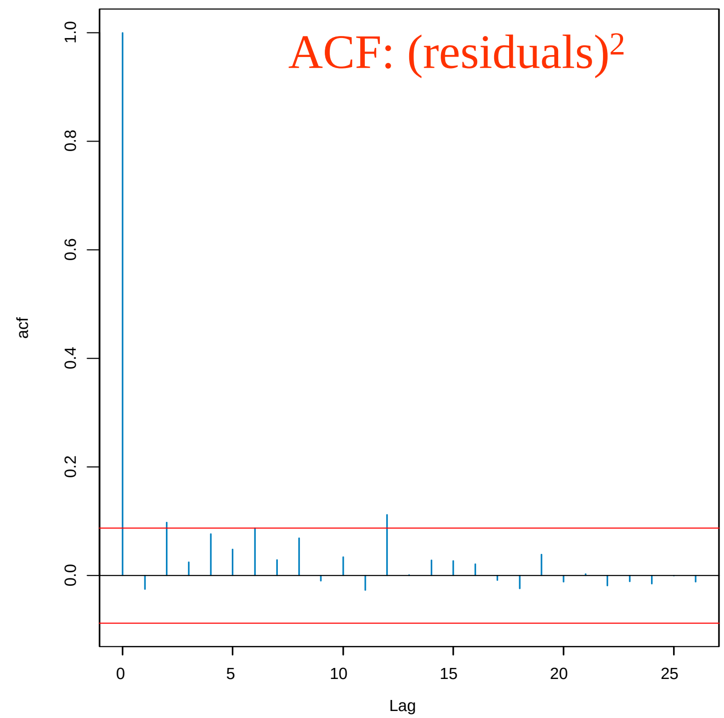


All-pass model fitted to NZ-USA exchange rates (using LAD):

Order = 6, ϕ_1=-.367, ϕ_2=-.750, ϕ_3=-.391, ϕ_4=.088, ϕ_5= -.193, ϕ_6=-.096

(AIC had local minima at p=6 and 10)





Noninvertible MA models with heavy tailed noise

$$X_t = Z_t + \theta_1 Z_{t-1} + \cdot\ \cdot\ \cdot\ + \theta_q Z_{t-q}\,,$$

a. $\{Z_t\} \sim \text{IID}(\alpha)$ with Pareto tails

b. $\theta(z) = 1 + \theta_1 z + \cdot\ \cdot\ \cdot + \theta_q z^q$

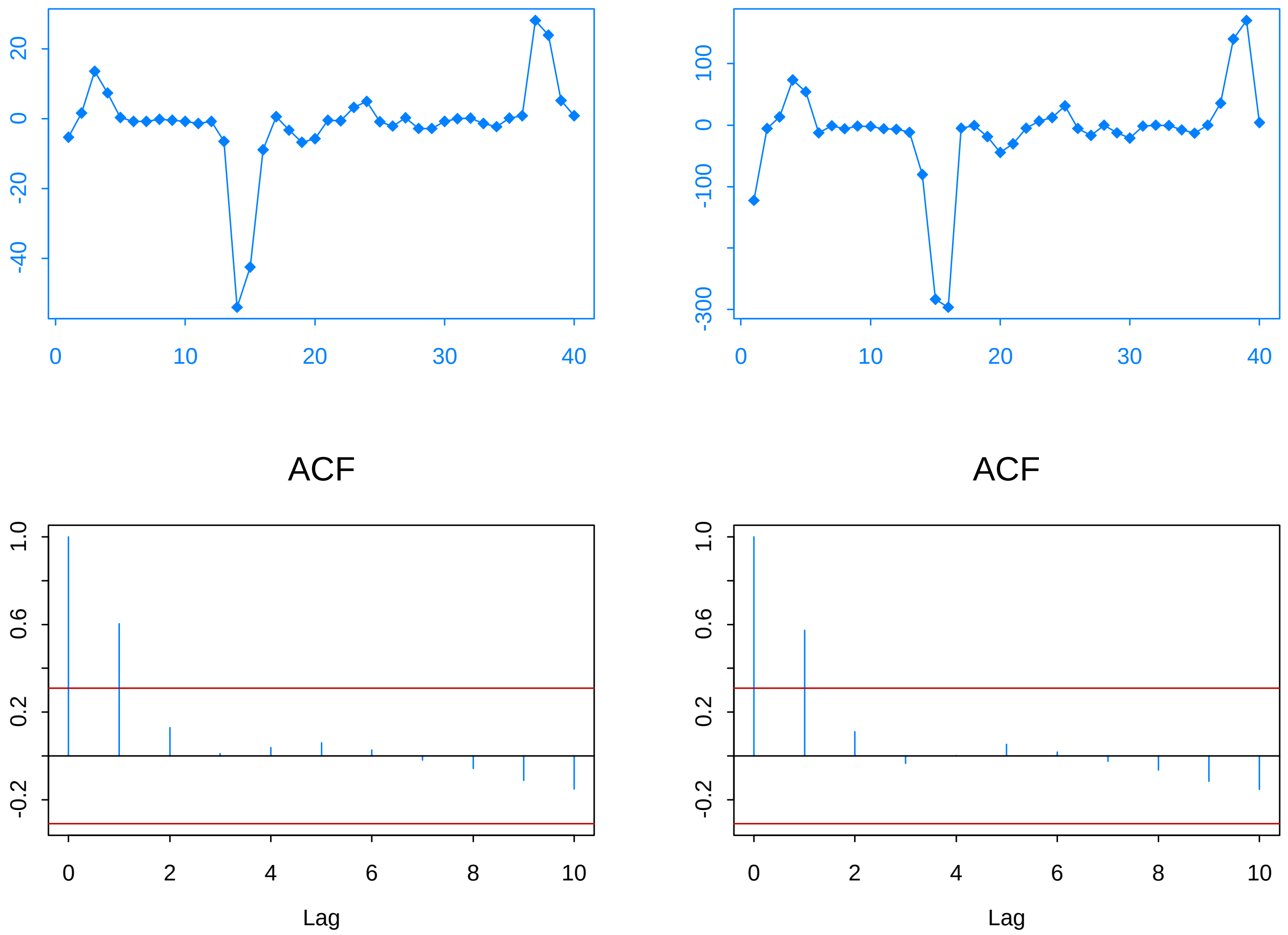
No zeros inside the unit circle $\Rightarrow$ invertible

Some zero(s) inside the unit circle $\Rightarrow$ noninvertible

Realizations of an invertible and noninvertible MA(2) processes

Model: $X_t = \theta_*(B)\, Z_t$, $\{Z_t\} \sim \text{IID}(\alpha = 1)$, where

$\theta_i(B) = (1 + 1/2B)(1 + 1/3B)$ and $\theta_{ni}(B) = (1 + 2B)(1 + 3B)$



Application of all-pass to noninvertible MA model fitting

Suppose $\{X_t\}$ follows the noninvertible MA model

$$X_t = \theta_i(B)\,\theta_{ni}(B)\,Z_t\,, \quad \{Z_t\} \sim \text{IID}.$$

Step 1: Let $\{U_t\}$ be the residuals obtained by fitting a purely invertible MA model, i.e.,

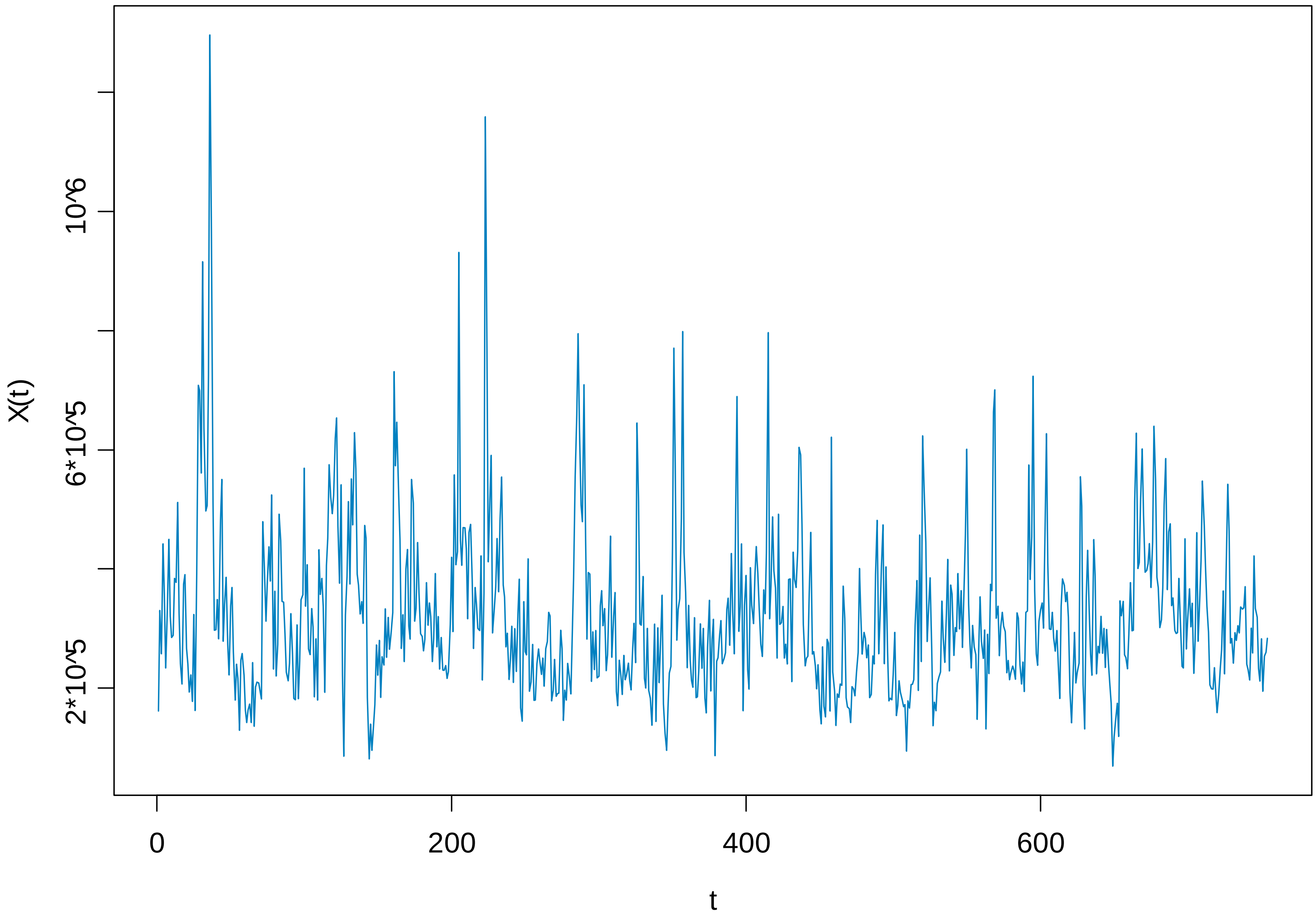
$$\begin{aligned} X_t &= \hat{\theta}(B)U_t \\ &\approx \theta_i(B)\tilde{\theta}_{ni}(B)U_t, \quad (\tilde{\theta}_{ni} \text{ is the invertible version of } \theta_{ni}). \end{aligned}$$

So $$U_t \approx \frac{\theta_{ni}(B)}{\tilde{\theta}_{ni}(B)} Z_t$$

Step 2: Fit a purely causal AP model to $\{U_t\}$

$$\tilde{\theta}_{ni}(B)U_t = \theta_{ni}(B)Z_t.$$

Volumes of Microsoft (MSFT) stock traded over 755 transaction days (6/3/96 to 5/28/99)



Analysis of MSFT:

Step 1: Log(volume) follows MA(4).

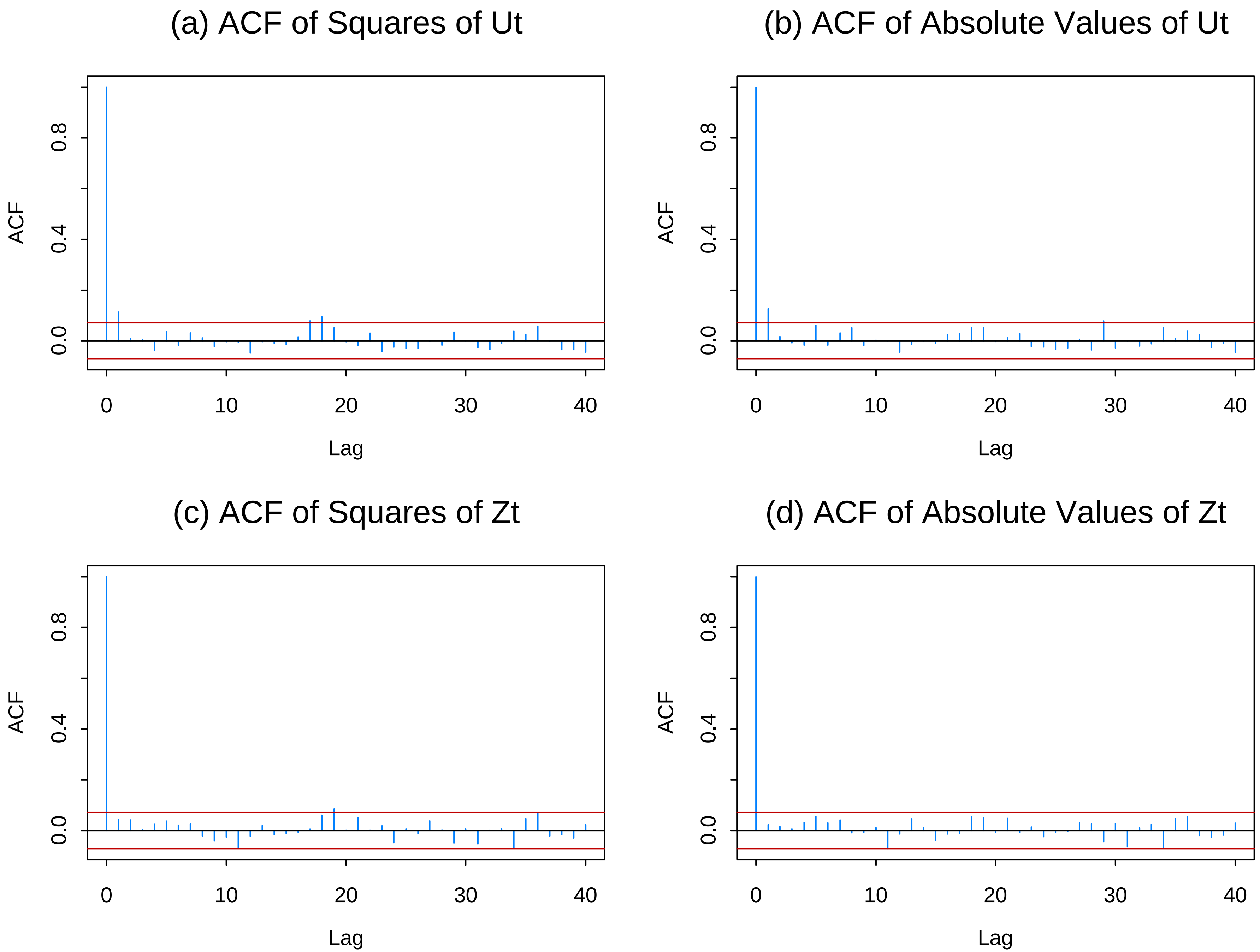
$X_t = (1+.513B+.277B^2+.270B^3+.202B^4)\, U_t$ (invertible MA(4))

Step 2: All-pass model of order 4 fitted to $\{U_t\}$ using MLE (t-dist):

$$(1+.184B+.132B^2-.833B^3-.314B^4)U_t$$
$$=(1+2.65B-.418B^2-.586B^3-3.18B^4)Z_t.$$

Conclude that $\{X_t\}$ follows a noninvertible MA(4) which after refitting has the form:

$X_t = (1+1.34B+1.374B^2+2.54B^3+4.96B^4)\, Z_t$, $\{Z_t\}$~IID t(6.3)


(a) ACF of Squares of Ut
(b) ACF of Absolute Values of Ut
(c) ACF of Squares of Zt
(d) ACF of Absolute Values of Zt
ACF
0.0
0.4
0.8
Lag
0
10
20
30
40

Summary: Microsoft Trading Volume

☞ Two-step fit of noninvertible MA(4):

- invertible MA(4): residuals not iid
- causal AP(4); residuals iid

☞ Direct fit of purely noninvertible MA(4):

$$(1+1.34B+1.374B^2+2.54B^3+4.96B^4)$$

☞ For MCHP, invertible MA(4) fits.

Summary

☞ All-pass models and their properties

- linear time series with “nonlinear” behavior

☞ Estimation

- likelihood approximation
- MLE and LAD
- order selection

☞ Emprirical results

- simulation study
- AP(6) for NZ/USA exchange rates

☞ Noninvertible moving average processes

- two-step estimation procedure using all-pass
- noninvertible MA(4) for Microsoft trading volume

Further Work

☞ Least absolute deviations

- further simulations
- order selection
- heavy-tailed case
- other smooth objective functions (e.g., min dispersion)

☞ Maximum likelihood

- Gaussian mixtures
- simulation studies
- applications

☞ Noninvertible moving average modeling

- initial estimates from two-step all-pass procedure
- adaptive procedures