Linear Time Series With Nonlinear Behavior

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☞ Introduction

- properties of financial time series
- motivating example
- all-pass models and their properties

☞ Estimation

- likelihood approximation
- MLE and LAD
- asymptotic results
- order selection

☞ Empirical results

- simulation
- NZ/USA exchange rates

☞ Non-causal AR and non-invertible MA processes

- preliminaries
- a two-step estimation procedure
- Microsoft trading volume

☞ Summary

Financial Time Series

☞ Log returns, $X_t = 100*(\ln(P_t) - \ln(P_{t-1}))$, of financial assets often exhibit:

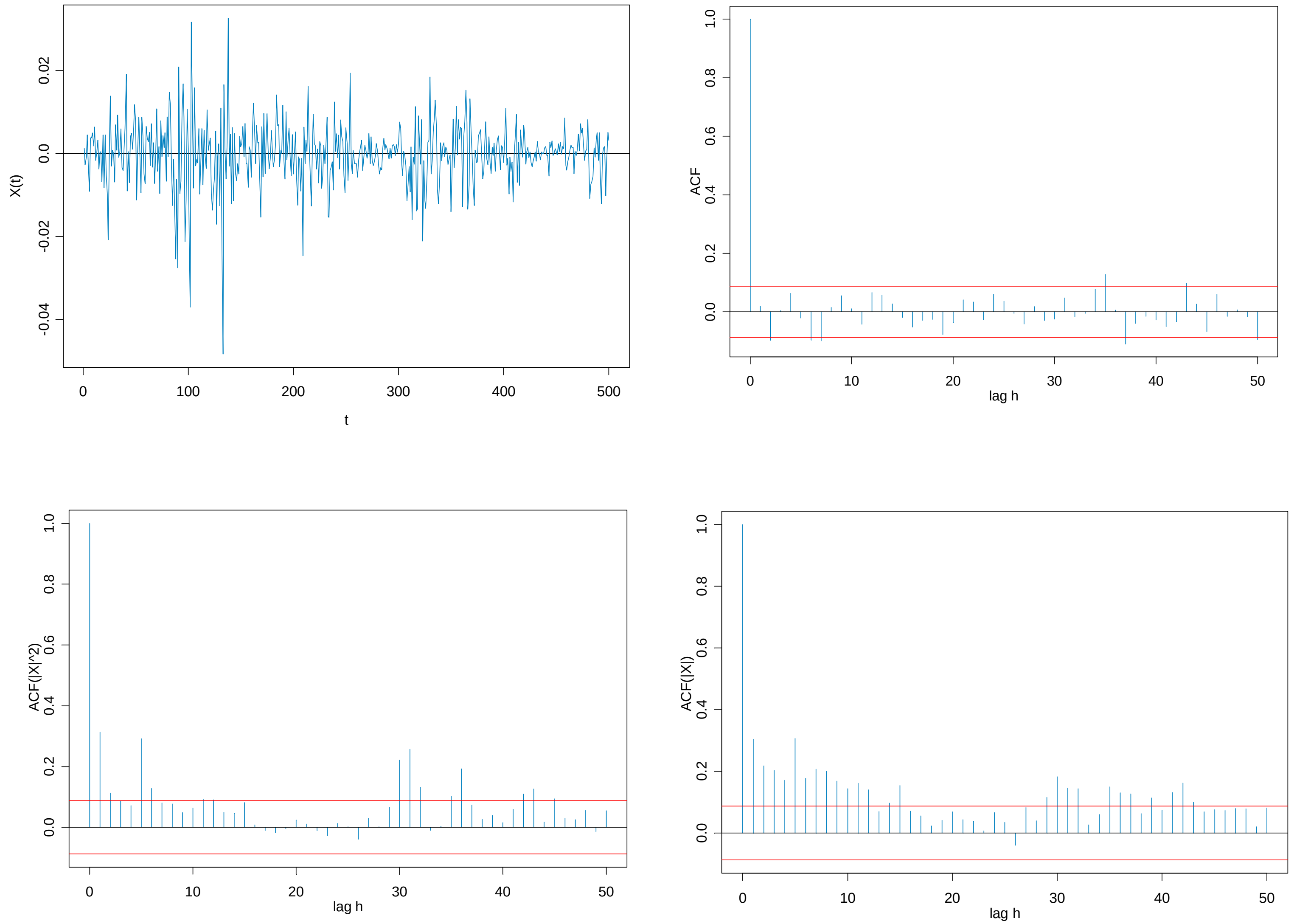
- heavy-tailed marginal distributions
  $$P(|X_1| > x) \sim C\, x^{-\alpha}, \quad 0 < \alpha < 4.$$
- lack of serial correlation
  $\hat{\rho}_X(h)$ near 0 for all lags $h > 0$ (MGD sequence)
- $|X_t|$ and X_t^2 have slowly decaying autocorrelations
  $\hat{\rho}_{|X|}(h)$ and $\hat{\rho}_{X^2}(h)$ converge to 0 *slowly* as $h \to \infty$
- process exhibits 'stochastic volatility'

☞ Nonlinear models $X_t = \sigma_t Z_t$, $\{Z_t\} \sim \mathrm{IID}(0,1)$

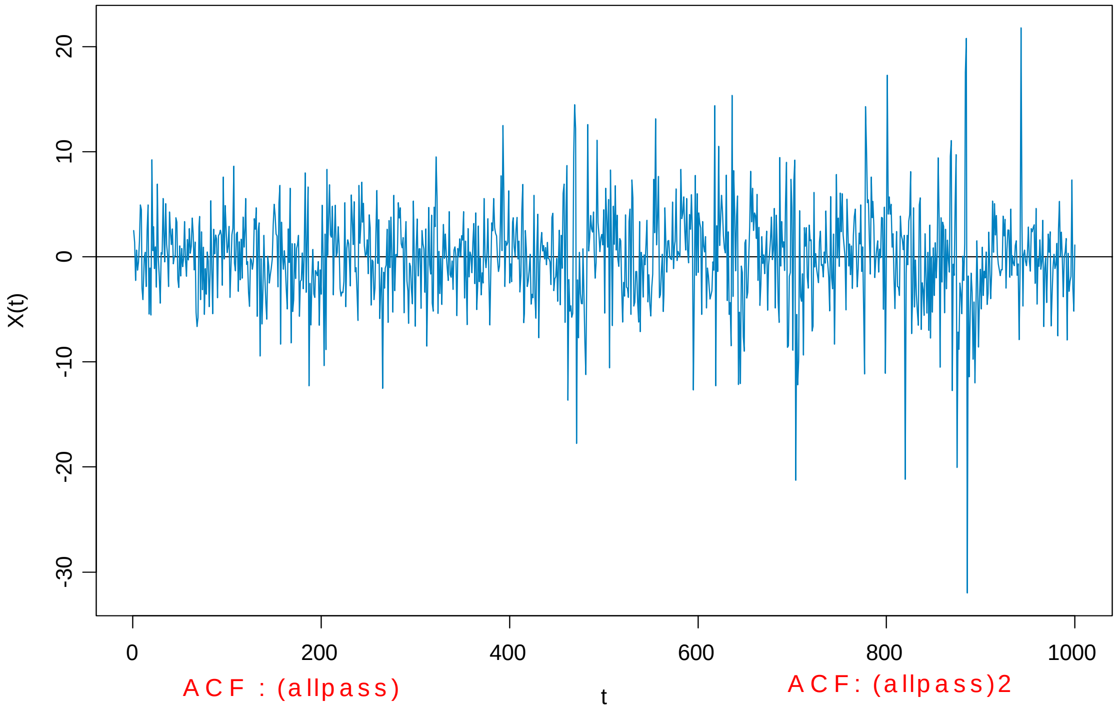
- ARCH and its variants (Engle `82; Bollerslev, Chou, and Kroner 1992)
- Stochastic volatility (Clark 1973; Taylor 1986)

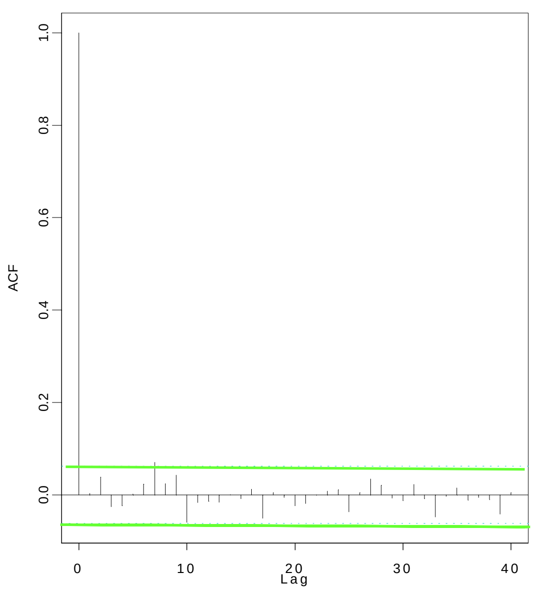
Motivating example

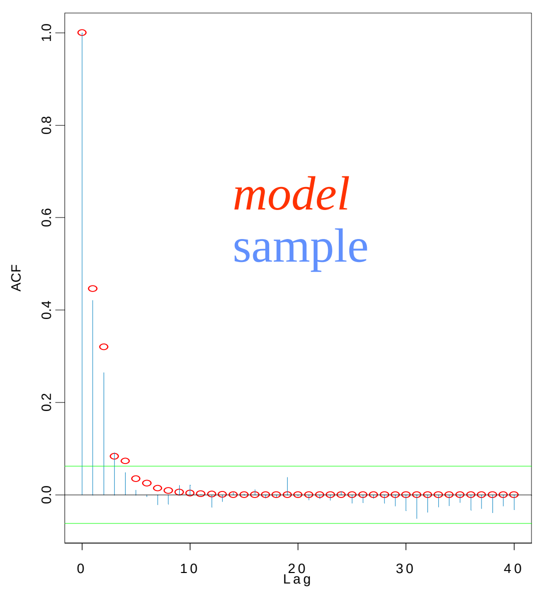
500-daily log-returns of NZ/US exchange rate



All-pass model of order 2 (t3 noise)







All-pass Models

Causal AR polynomial: $\phi(z)=1-\phi_1 z - \cdots - \phi_p z^p$, $\phi(z) \neq 0$ for $|z|\leq 1$.

Define MA polynomial:

$$\theta(z) = -z^p \phi(z^{-1})/\phi_p = -(z^p - \phi_1 z^{p-1} - \cdots - \phi_p)/\phi_p$$

$\neq 0$ for $|z|\geq 1$ (MA polynomial is non-invertible).

Model for data $\{X_t\}$: $\phi(B)X_t = \theta(B) Z_t$, $\{Z_t\} \sim$ IID (non-Gaussian)

$$B^k X_t = X_{t-k}$$

Examples:

All-pass(1): $X_t - \phi X_{t-1} = Z_t - \phi^{-1} Z_{t-1}$, $|\phi| < 1$.

All-pass(2): $X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t + \phi_1/\phi_2 Z_{t-1} - 1/\phi_2 Z_{t-2}$

Properties:

- causal, non-invertible ARMA with MA representation

$$X_t = \frac{B^p \phi(B^{-1})}{-\phi_p \phi(B^{-1})} Z_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

- uncorrelated (flat spectrum)

$$f_X(\omega) = \frac{\left|e^{-ip\omega}\right|^2 \left|\phi(e^{i\omega})\right|^2}{\phi_p^2 \left|\phi(e^{-i\omega})\right|^2} \frac{\sigma^2}{2\pi} = \frac{\sigma^2}{\phi_p^2 2\pi}$$

- zero mean
- data are dependent if noise is non-Gaussian (e.g. Breidt & Davis 1991).
- squares and absolute values are correlated.
- X_t is heavy-tailed if noise is heavy-tailed.

Estimation for All-Pass Models

☞ Second-order moment techniques do not work

- least squares
- Gaussian likelihood

☞ Higher-order cumulant methods

- Giannakis and Swami (1990)
- Chi and Kung (1995)

☞ Non-Gaussian likelihood methods

- likelihood approximation
- quasi-likelihood
- least absolute deviations
- minimum dispersion

Approximating the likelihood

Data: $(X_1, \ldots, X_n)$

Model: $$X_t = \phi_{01} X_{t-1} + \cdots + \phi_{0p} X_{t-p} + (Z_{t-p} - \phi_{01} Z_{t-p+1} - \cdots - \phi_{0p} Z_t) / \phi_{0r}$$

where ϕ_{0r} is the last non-zero coefficient among the ϕ_{0j}'s.

Noise: $z_{t-p} = \phi_{01} z_{t-p+1} + \cdots + \phi_{0p} z_t - (X_t - \phi_{01} X_{t-1} - \cdots - \phi_{0p} X_{t-p}),$

where $z_t = Z_t / \phi_{0r}$.

More generally define,

$$z_{t-p}(\phi) = \begin{cases} 0, & \text{if } t = n+p, \ldots, n+1, \\ \phi_1 z_{t-p+1}(\phi) + \cdots + \phi_p z_t(\phi) - \phi(B) X_t, & \text{if } t = n, \ldots, p+1. \end{cases}$$

Note: $z_t(\phi_0)$ is a close approximation to z_t (initialization error)

Assume that Z_t has density function f_σ and consider the vector

$$\mathbf{z} = (\underbrace{X_{1-p},\ldots,X_0, z_{1-p}(\phi),\ldots,z_0(\phi)}, \underbrace{z_1(\phi),\ldots,z_{n-p+1}(\phi),\ldots,z_n(\phi)})'$$

independent pieces

Joint density of $\mathbf{z}$:

$$h(\mathbf{z}) = h_1(X_{1-p},\ldots,X_0, z_{1-p}(\phi),\ldots,z_0(\phi))$$

$$\bullet\left(\prod_{t=1}^{n-p} f_\sigma(\phi_q z_t(\phi))\,|\,\phi_q\,|\right) h_2(z_{n-p+1}(\phi),\ldots,z_n(\phi)),$$

and hence the joint density of the data can be approximated by

$$h(\mathbf{x}) = \left(\prod_{t=1}^{n-p} f_\sigma(\phi_q z_t(\phi))\,|\,\phi_q\,|\right)$$

where $q=\max\{0 \le j \le p: \phi_j \neq 0\}$.

Assumptions

- ☞ Assume $\{Z_t\}$ iid $f_\sigma(z)=\sigma^{-1}f(\sigma^{-1}z)$ with
    - σ a scale parameter
    - mean 0, variance σ^2
- ☞ For f known, use maximum likelihood
    - further assumptions on f
    - Fisher information: $\tilde{I} = \sigma^{-2}\int (f'(z))^2 / f(z)dz$
- ☞ For f unknown, use quasi-likelihood
- ☞ Least absolute deviations
    - assume f has median 0
    - assume f continuous in neighborhood of 0
    - act as if f = Laplace to get criterion function

Results

- ☞ Let $\gamma(h)$ = ACVF of AR model with AR poly $\phi_0(.)$ and

$$\Gamma_p = [\gamma(j-k)]_{j,k=1}^{p}$$

- ☞ Maximum likelihood:

$$\sqrt{n}(\hat{\phi}_{\text{MLE}} - \phi_0) \xrightarrow{D} N(0, \frac{\sigma^{-2}}{2(\tilde{I} - \sigma^{-2})}\sigma^2\Gamma_p^{-1})$$

- ☞ Least absolute deviations:

$$\sqrt{n}(\hat{\phi}_{\text{LAD}} - \phi_0) \xrightarrow{D} N(0, \frac{\text{Var}(|Z_1|)}{2\sigma^4 f_\sigma^2(0)}\sigma^2\Gamma_p^{-1})$$

Log-likelihood:

$$L(\phi,\sigma) = -(n-p)\ln(\sigma/|\phi_q|) + \sum_{t=1}^{n-p} \ln f(\sigma^{-1}\phi_q z_t(\phi))$$

where $f_\sigma(z) = \sigma^{-1} f(z/\sigma)$.

Least absolute deviations: choose Laplace density

$$f(z) = \frac{1}{\sqrt{2}} \exp(-\sqrt{2}|z|)$$

and log-likelihood becomes

$$\text{constant} - (n-p)\ln\kappa - \sum_{t=1}^{n-p} \sqrt{2}|z_t(\phi)|/\kappa$$

Concentrated Laplacian likelihood

$$l(\phi) = \text{constant} - (n-p)\ln\sum_{t=1}^{n-p} |z_t(\phi)|$$

Maximizing $l(\phi)$ is equivalent to minimizing the absolute deviations

$$m_n(\phi) = \sum_{t=1}^{n-p} |z_t(\phi)|.$$

Identifiability?

- Minimizer may not be unique.

- Gaussian case: $\{Z_t\}$ iid $\mathrm{N}(0, \sigma_0^2 \phi_{0p}^{-2}) = \mathrm{N}(0, \sigma_1^2 \phi_{1p}^{-2})$, so

$$E\,|\,z_1(\phi_1)\,| = E\left|\frac{Z_1\sigma_1}{\sigma_0\phi_{1p}}\right| = E\left|\frac{Z_1\sigma_0}{\sigma_0\phi_{0p}}\right| = E\,|\,z_1(\phi_0)\,|$$

- Consider $\{c_j\}$ with at least two non-zero elements and

$$\sum_{j=-\infty}^{\infty} |\,c_j\,| < \infty \quad \text{and} \quad \sum_{j=-\infty}^{\infty} c_j^2 = 1$$

Jian and Pawitan (1998) show

$$E\,\Big|\sum_{j=-\infty}^{\infty} c_j Z_j\,\Big| > E\,|\,Z_1\,|$$

holds for Laplace, Student's t, contaminated normal, etc.

- Non-Gaussian case: $E\,|\,z_1(\phi_1)\,| = E\left|\frac{\phi_0(B^{-1})\phi_1(B)}{\phi_{0p}\phi_1(B^{-1})\phi_0(B)} Z_t\right| > E\,|\,z_1(\phi_0)\,|$

Central Limit Theorem

- Think of $\mathbf{u} = n^{1/2}(\phi - \phi_0)$ as an element of $\mathbb{R}^p$

- Define

$$S_n(\mathbf{u}) = \sum_{t=1}^{n-p} \left(| z_t(\phi_0 + n^{-1/2}\mathbf{u}) | - | z_t(\phi_0) | \right)$$

$$= m_n(\phi_0 + n^{-1/2}\mathbf{u}) - \sum_{t=1}^{n-p} | z_t(\phi_0) |$$

- Then $S_n(\mathbf{u}) \to S(\mathbf{u})$ in distribution on $C(\mathbb{R}^p)$, where

$$S(\mathbf{u}) = \frac{f_\sigma(0)}{|\phi_{0p}|}\mathbf{u}'\Gamma_p\mathbf{u} + \mathbf{u}'\mathbf{N}, \ \mathbf{N} \sim N(\mathbf{0}, \frac{2\mathrm{Var}(|Z_1|)}{\phi_{0p}^2\sigma^2}\Gamma_p),$$

- Hence,

$$\arg\min S_n(\mathbf{u}) = n^{1/2}(\hat{\phi}_{LAD} - \phi_0)$$

$$\underset{D}{\to} \arg\min S(\mathbf{u})$$

$$= -\frac{|\phi_p|\Gamma_p^{-1}}{2 f_\sigma(0)}\mathbf{N} \sim N(\mathbf{0}, \frac{\mathrm{Var}(|Z_1|)}{2\sigma^4 f_\sigma^4(0)}\sigma^2\Gamma_p^{-1})$$

Asymptotic Results:

Theorem 1. Let $\{Y_t\}$ be the linear process

$$Y_t = \sum_{j=-\infty}^{\infty} c_j z_{t-j},$$

where $c_0=0$, $\sum_{j=-\infty}^{\infty} |c_j| < \infty$, $\{z_t\} \sim \text{IID}(0,\sigma^2)$, $\text{median}(z_1)=0$, $g(0)>0$ (g density of z_1). Then

$$S_n = \sum_{t=1}^{n-p} \left(| z_t - n^{-1/2} Y_t | - | z_t | \right)$$

$$\to \text{Var}(Y_1) g(0) + N$$

where $N \sim \text{N}(0, \gamma^*(0) + 2\sum_{h\geq 1} \gamma^*(h))$ and $\gamma^*(h)$ is the covariance function for $Y_t \,\text{sgn}(z_t)$

Key idea:

$$S_n = \sum_{t=1}^{n-p} \left(| z_t \text{ -} n^{\text{-}1/2} Y_t | \text{ - } | z_t | \right)$$

$$= -n^{-1/2} \sum_{t=1}^{n-p} Y_t \, \mathrm{sgn}(z_t)$$

$$+ 2 \sum_{t=1}^{n-p} (n^{-1/2} Y_t - z_t) \left\{ 1_{\{0 < z_t < n^{-1/2} Y_t\}} - 1_{\{n^{-1/2} Y_t < z_t < 0\}} \right\}$$

$$\to N + Var(Y_1) g(0)$$

Theorem 2. On C(R^p),

$$S_n(\mathbf{u}) = \sum_{t=1}^{n-p} \left(| z_t(\phi_0 + n^{-1/2}\mathbf{u}) | - | z_t(\phi_0) | \right)$$

$$\to S(\mathbf{u}),$$

where

$$S(\mathbf{u}) = \frac{f_\sigma(0)}{|\phi_{0r}|} \mathbf{u}'\Gamma_p \mathbf{u} + \mathbf{u}'\mathbf{N},$$

$$\mathbf{N} \sim N(\mathbf{0}, \frac{2Var(|Z_1|)}{\phi_{0r}^2 \sigma^2} \Gamma_p),$$

and Γ_p is the covariance matrix of a causal AR(p).

Limit theory for LAD estimate. Note that

$$\hat{\phi}_{\text{LAD}} = \phi_0 + \hat{\mathbf{u}}_n / \sqrt{n}$$

so that $\hat{\mathbf{u}}_n = \sqrt{n}(\hat{\phi}_{\text{LAD}} - \phi_0) = \arg\min S_n(\mathbf{u})$

$$\to \hat{\mathbf{u}} = \arg\min S(\mathbf{u}).$$

Minimizing S, we find that the minimizer or limit random variable is

$$\hat{\mathbf{u}}_n = \sqrt{n}(\hat{\phi}_{\text{LAD}} - \phi_0) \to -\frac{|\phi_{0r}|\Gamma_p^{-1}}{2f_\sigma(0)}\mathbf{N}$$

$$-\frac{|\phi_{0r}|\Gamma_p^{-1}}{2f_\sigma(0)}\mathbf{N} \sim N(\mathbf{0}, \frac{Var(|Z_1|)}{2\sigma^4 f_\sigma^2(0)}\sigma^2\Gamma_p^{-1})$$

Asymptotic Covariance Matrix

- For LS estimators of AR(p):

$$\sqrt{n}(\hat{\phi}_{\mathrm{LS}} - \phi_0) \xrightarrow{D} N(0, \sigma^2 \Gamma_p^{-1})$$

- For LAD estimators of AR(p):

$$\sqrt{n}(\hat{\phi}_{\mathrm{LAD}} - \phi_0) \xrightarrow{D} N(0, \frac{1}{4\sigma^2 f^2(0)} \sigma^2 \Gamma_p^{-1})$$

- For LAD estimators of AP(p)

$$\sqrt{n}(\hat{\phi}_{\mathrm{LAD}} - \phi_0) \xrightarrow{D} N(0, \frac{\mathrm{Var}(|Z_1|)}{2\sigma^4 f_\sigma^2(0)} \sigma^2 \Gamma_p^{-1})$$

Laplace: $\frac{\mathrm{Var}(|Z_1|)}{2\sigma^4 f_\sigma^2(0)} = \frac{1}{2}$

Students t_ν, $\nu > 2$: $\frac{\mathrm{Var}(|Z_1|)}{2\sigma^4 f_\sigma^2(0)} = \frac{\Gamma^2(\nu/2)(\nu-2)\pi}{2\Gamma^2((\nu+1)/2)} - \frac{2(\nu-2)^2}{(\nu-1)^2}$

Student's t_3: 0.7337

Order Selection:

Partial ACF From the previous result, if true model is of order r and fitted model is of order p > r, then

$$n^{1/2}\hat{\phi}_{p,LAD} \to N\left(0, \frac{\mathrm{Var}(|Z|)}{2\sigma^4 f_\sigma^2(0)}\right)$$

where $\hat{\phi}_{p,LAD}$ is the pth element of $\hat{\phi}_{LAD}$.

Procedure:

1. Fit high order (P-th order), obtain residuals and estimate scalar,

$$\theta^2 = \frac{\mathrm{Var}(|Z_1|)}{2\sigma^4 f_\sigma^2(0)},$$

by empirical moments of residuals and density estimates.

2. Fit AP models of order p=1,2, . . . , P via LAD and obtain p-th coefficient $\hat{\phi}_{p,p}$ for each.

3. Choose model order r as the smallest order beyond which the estimated coefficients are statistically insignificant.

AIC: 2*p* or not 2*p*?

- An approximately unbiased estimate of the Kullback-Leiber index of fitted to true model:

$$AIC(p) := -2L_X(\hat{\phi}, \hat{\kappa}) + \frac{\mathrm{Var}(|Z_1|)}{E|Z_1|\sigma^2 f_\sigma(0)}\, p$$

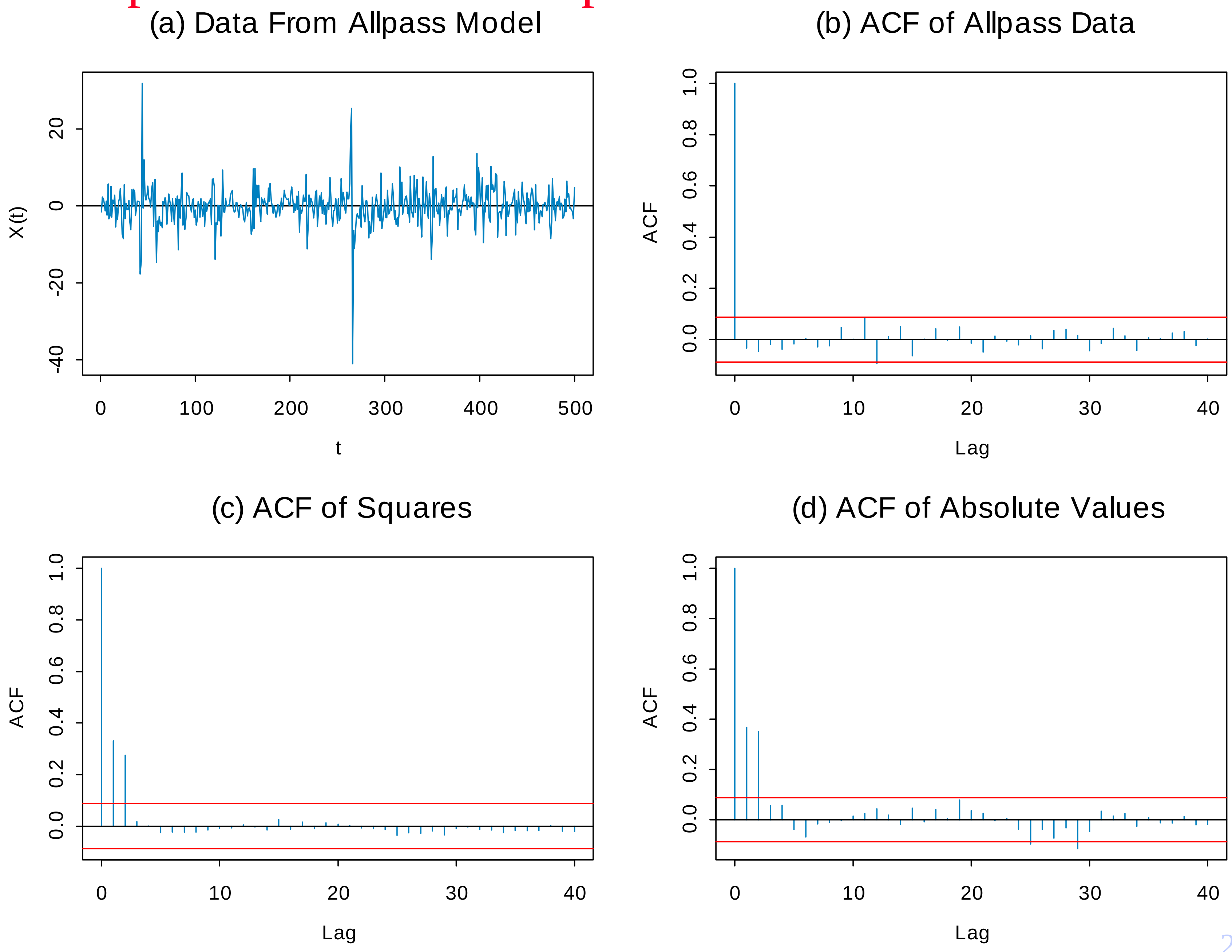
- Penalty term for Laplace case:

$$\frac{\mathrm{Var}(|Z_1|)}{E|Z_1|\sigma^2 f_\sigma(0)}\, p = \frac{\sigma^2/2}{(\sigma/\sqrt{2})/\sigma^2(1/\sqrt{2}\sigma)}\, p = p$$

- Estimated penalty term:

$$\frac{\mathrm{var}(|z_t(\hat{\phi})|)}{ave\{|z_t(\hat{\phi})|\}\mathrm{var}\{|z_t(\hat{\phi})|\}\hat{f}_{z_t(\hat{\phi})}(0)}\, p \xrightarrow[\mathrm{P}]{} \frac{\mathrm{Var}(|Z_1|)}{E|Z_1|\sigma^2 f_\sigma(0)}\, p$$

Sample realization of all-pass of order 2



Estimates:

$$\hat{\phi}_1 = .297(.0381), \hat{\phi}_2 = .374(.0381)$$

Standard errors computed as $\hat{\theta}\, sqrt\{(1-\hat{\phi}_2^2)/500\}$

where $\hat{\theta} = .919$

Order selection:

- cut-off value for PACF is 1.96*.908/sqrt(500)=.0796
- $AIC(p) := -2L_X(\hat{\phi}, \hat{\kappa}) + 1.896\,p$

	1	2	3	4	5
phi_p	0.289	0.374	0.009	0.011	0.01
AIC(p)	2451	2346	2347	2348	2350

	6	7	8	9	10
phi_p	0.047	0.034	-0.05	0.083	0.021
AIC(p)	2348	2349	2345	2343	2345

Simulation results:

- 1000 replicates of all-pass models
- model order parameter value

model order	parameter value
1	$\phi_1 = .4$
2	$\phi_1 = .3,\ \phi_2 = .4$

- noise distribution is t with 3 d.f.
- sample sizes n=500, 5000
- estimation method is LAD

To guard against being trapped in local minima, we adopted the following strategy.

- 250 random starting values were chosen at *random*. For model of order p, k-th starting value was computed recursively as follows:

 1. Draw $\phi_{11}^{(k)}, \phi_{22}^{(k)}, \ldots, \phi_{pp}^{(k)}$ iid uniform (-1,1).
 2. For j=2, …, p, compute

$$\begin{bmatrix} \phi_{j1}^{(k)} \\ \vdots \\ \phi_{j,j-1}^{(k)} \end{bmatrix} = \begin{bmatrix} \phi_{j-1,1}^{(k)} \\ \vdots \\ \phi_{j-1,j-1}^{(k)} \end{bmatrix} - \phi_{jj}^{(k)} \begin{bmatrix} \phi_{j-1,j-1}^{(k)} \\ \vdots \\ \phi_{j-1,1}^{(k)} \end{bmatrix}$$

- Select top 10 based on minimum function evaluation.

- Run Hooke and Jeeves with each of the 10 starting values and choose best optimized value.

N	Asymptotic mean	Asymptotic std dev	Empirical mean	Empirical std dev	Empirical %coverage	rel eff*
500	ϕ_1=.5	.0332	.4979	.0397	94.2	11.4
5000	ϕ_1=.5	.0105	.4998	.0109	95.4	9.3

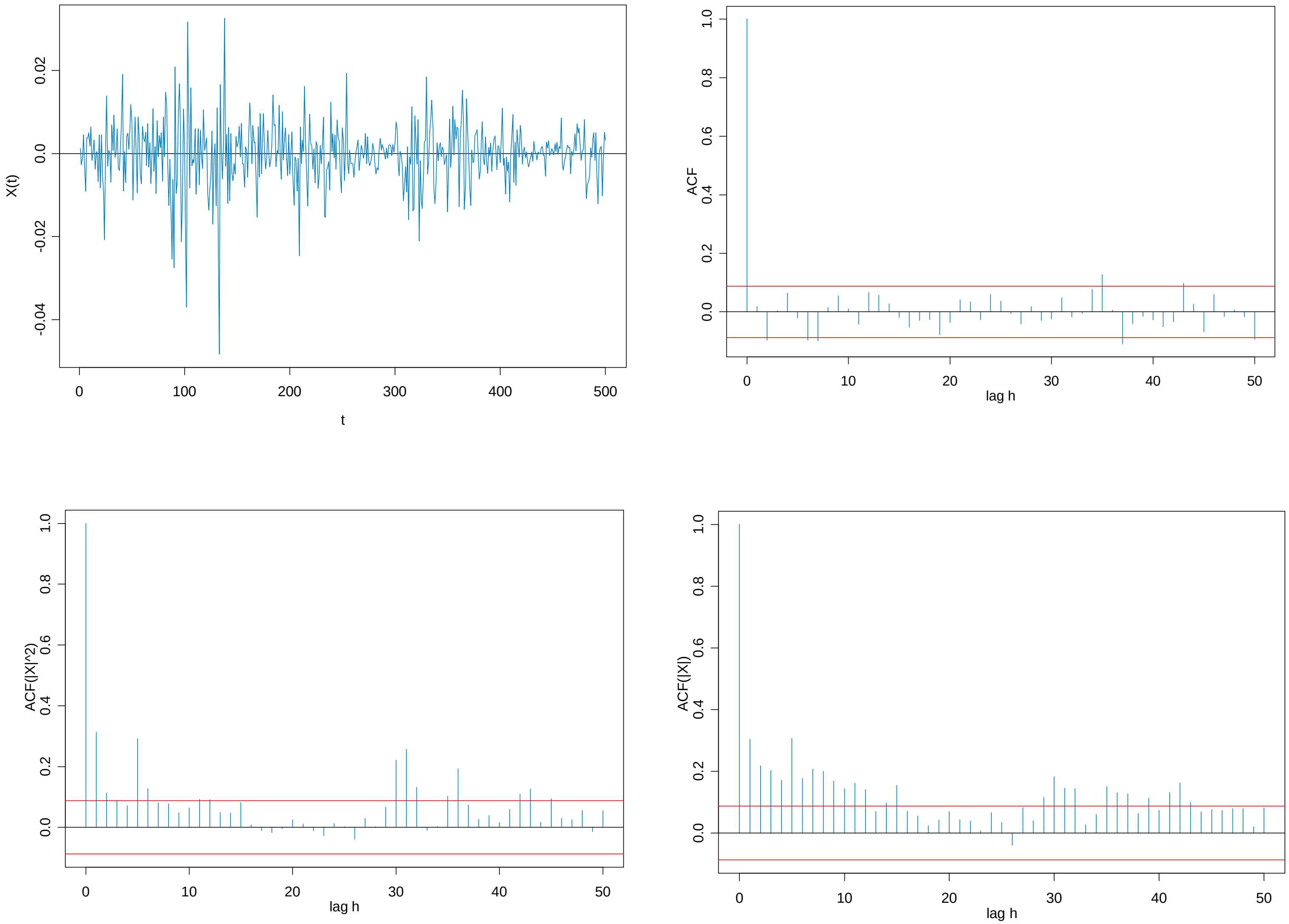
N	Asymptotic mean	Asymptotic std dev	Empirical mean	Empirical std dev	Empirical %coverage
500	ϕ_1=.3	.0351	.2990	.0456	92.5
	ϕ_2=.4	.0351	.3965	.0447	92.1
5000	ϕ_1=.3	.0111	.3003	.0118	95.5
	ϕ_2=.4	.0111	.3990	.0117	94.7

*Efficiency relative to maximum absolute residual kurtosis:

$$\left| \frac{1}{n-p} \sum_{t=1}^{n-p} \left(\frac{z_t(\phi)}{v_2^{1/2}} \right)^4 - 3 \right|$$

Application to financial data

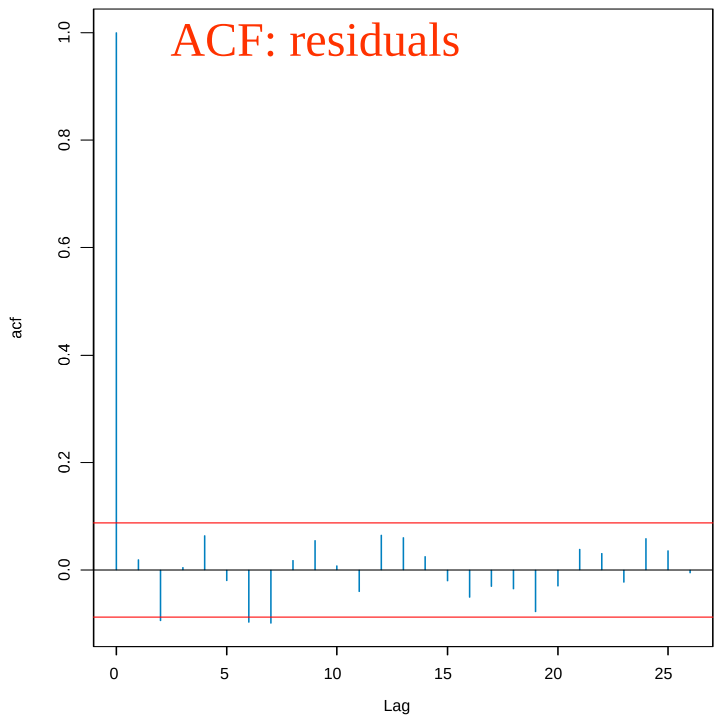
500-daily log-returns of NZ/US exchange rate

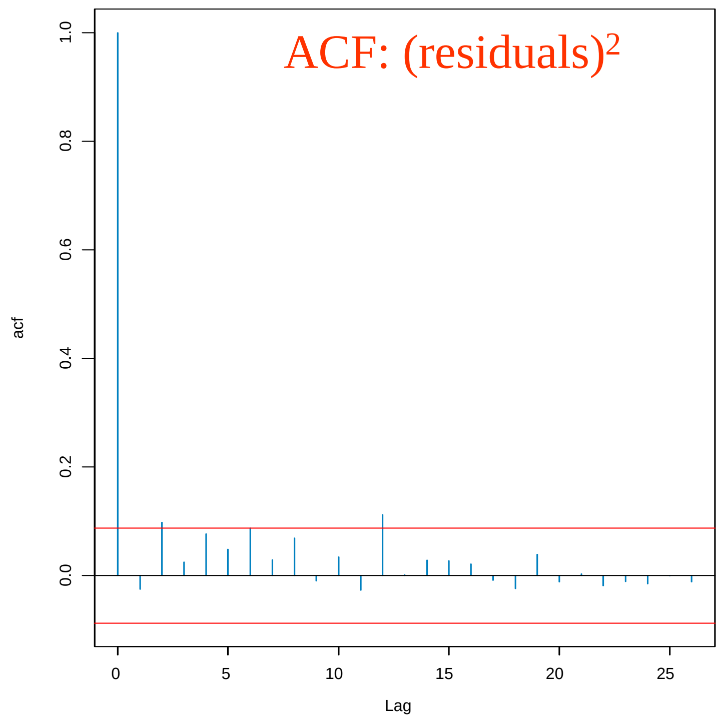


All-pass model fitted to NZ-USA exchange rates :

Order = 6, ϕ_1=-.367, ϕ_2=-.750, ϕ_3=-.391, ϕ_4=.088, ϕ_5= -.193, ϕ_6=-.096

(AIC had local minima at p=6 and 10)





Non-causal AR and non-invertible MA models with heavy tailed noise

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t ,$$

a. $\{Z_t\} \sim \text{IID}(\alpha)$ with Pareto tails

b. $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$

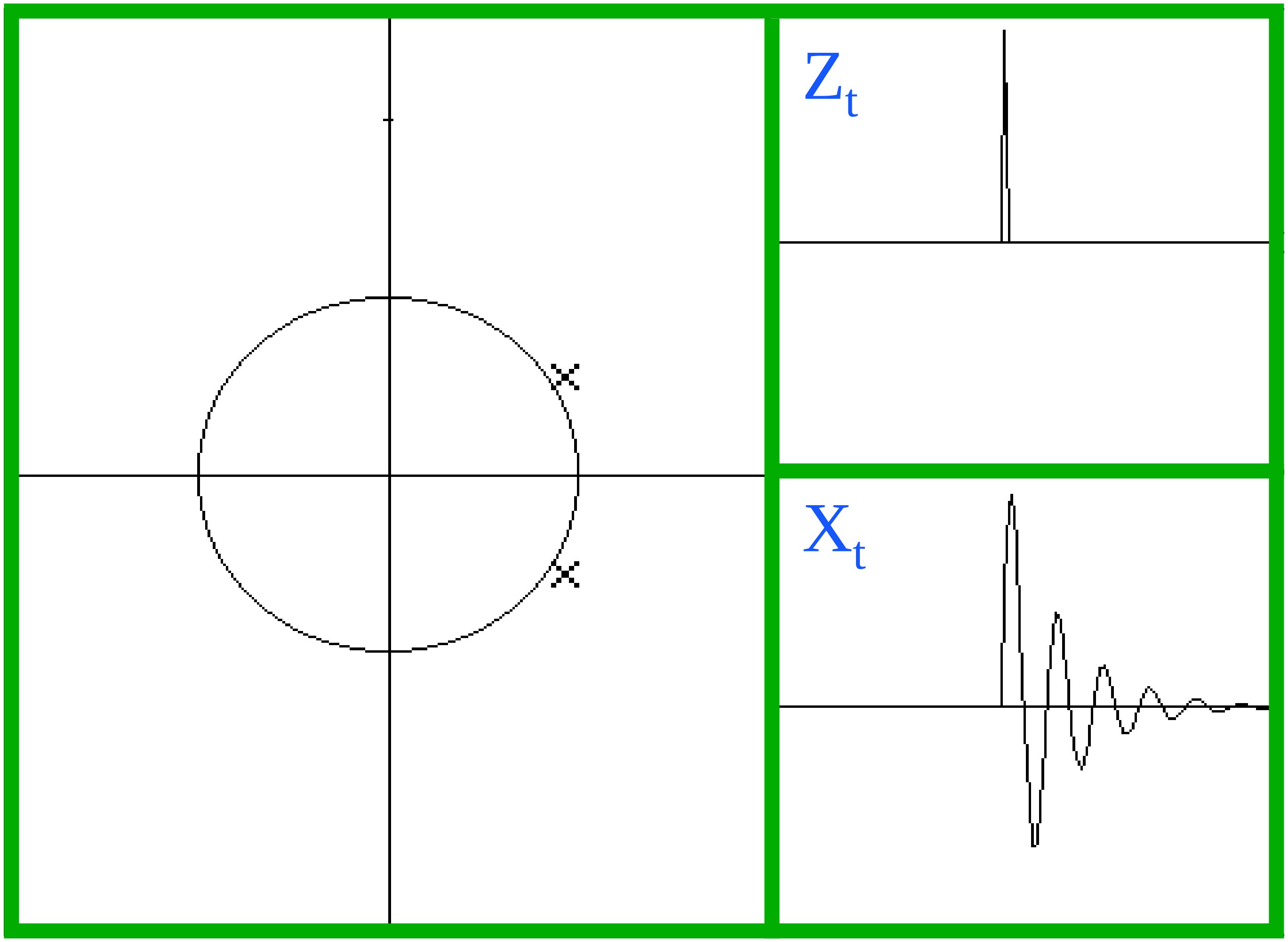
No zeros on the unit circle $\Rightarrow$ stationary

No zeros inside the unit circle $\Rightarrow$ causal

Some zero(s) inside the unit circle $\Rightarrow$ non-causal

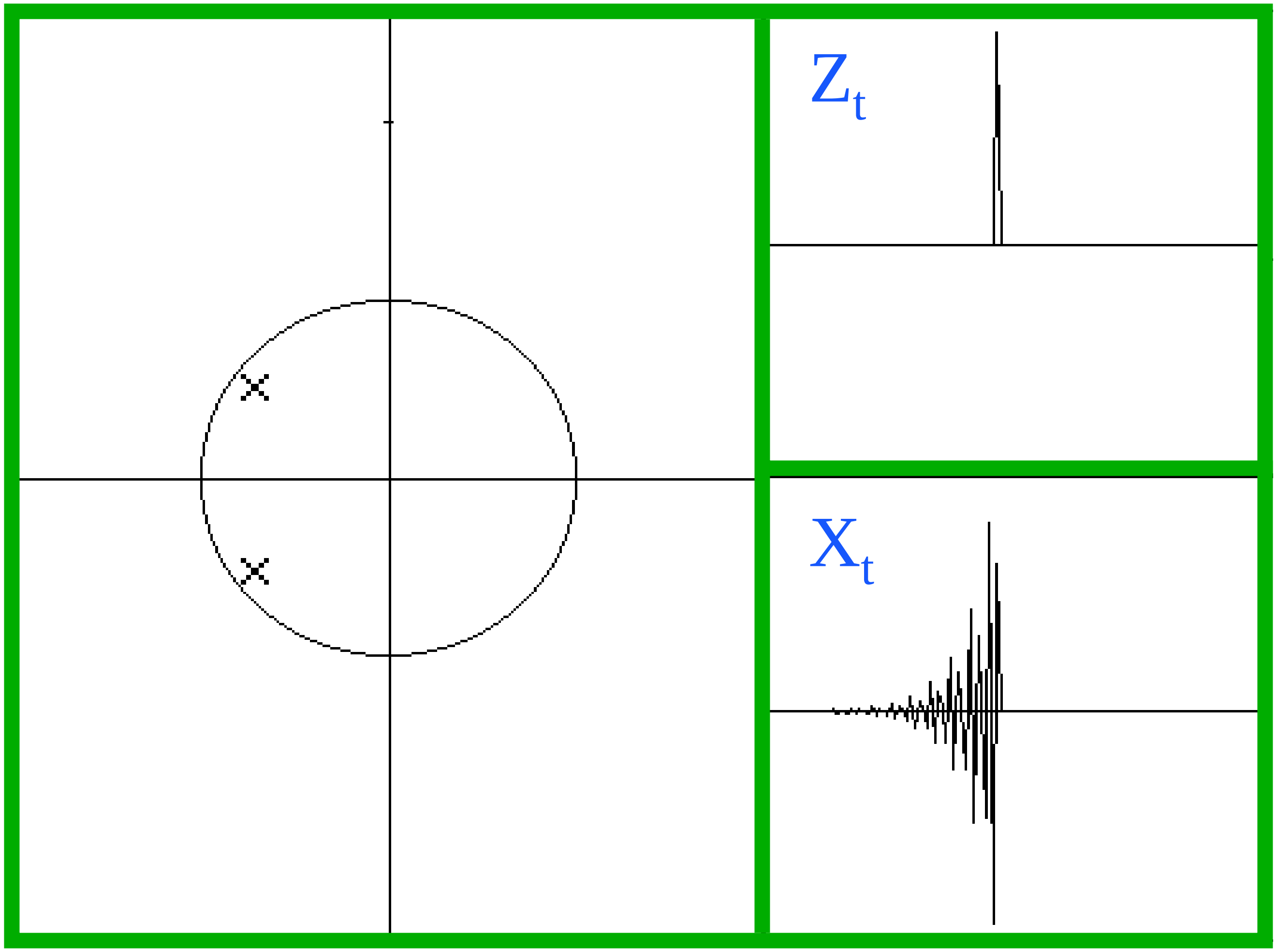
Impulse Response

Causal - Low frequency



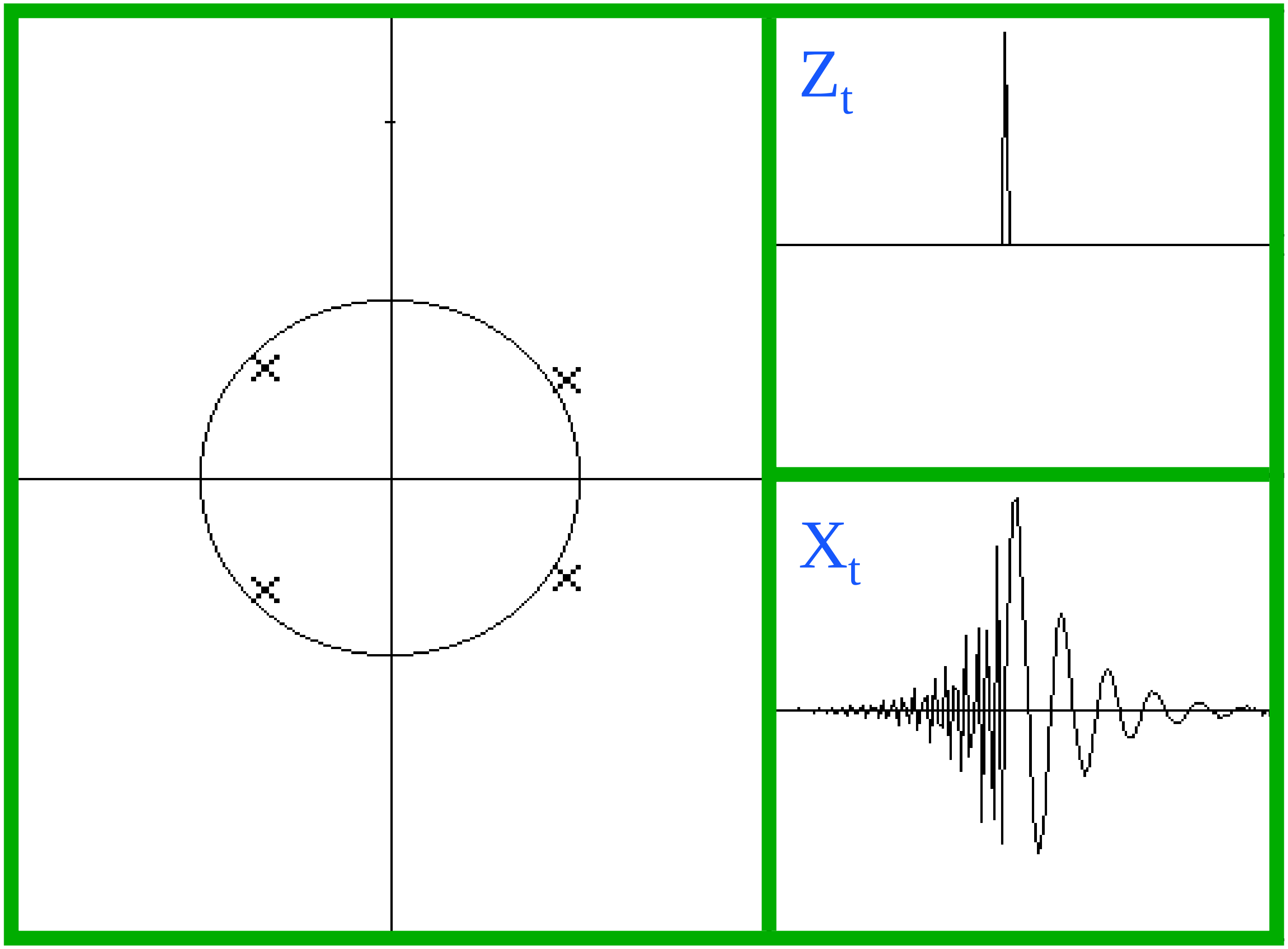
Impulse Response

Noncausal - High frequency



Impulse Response

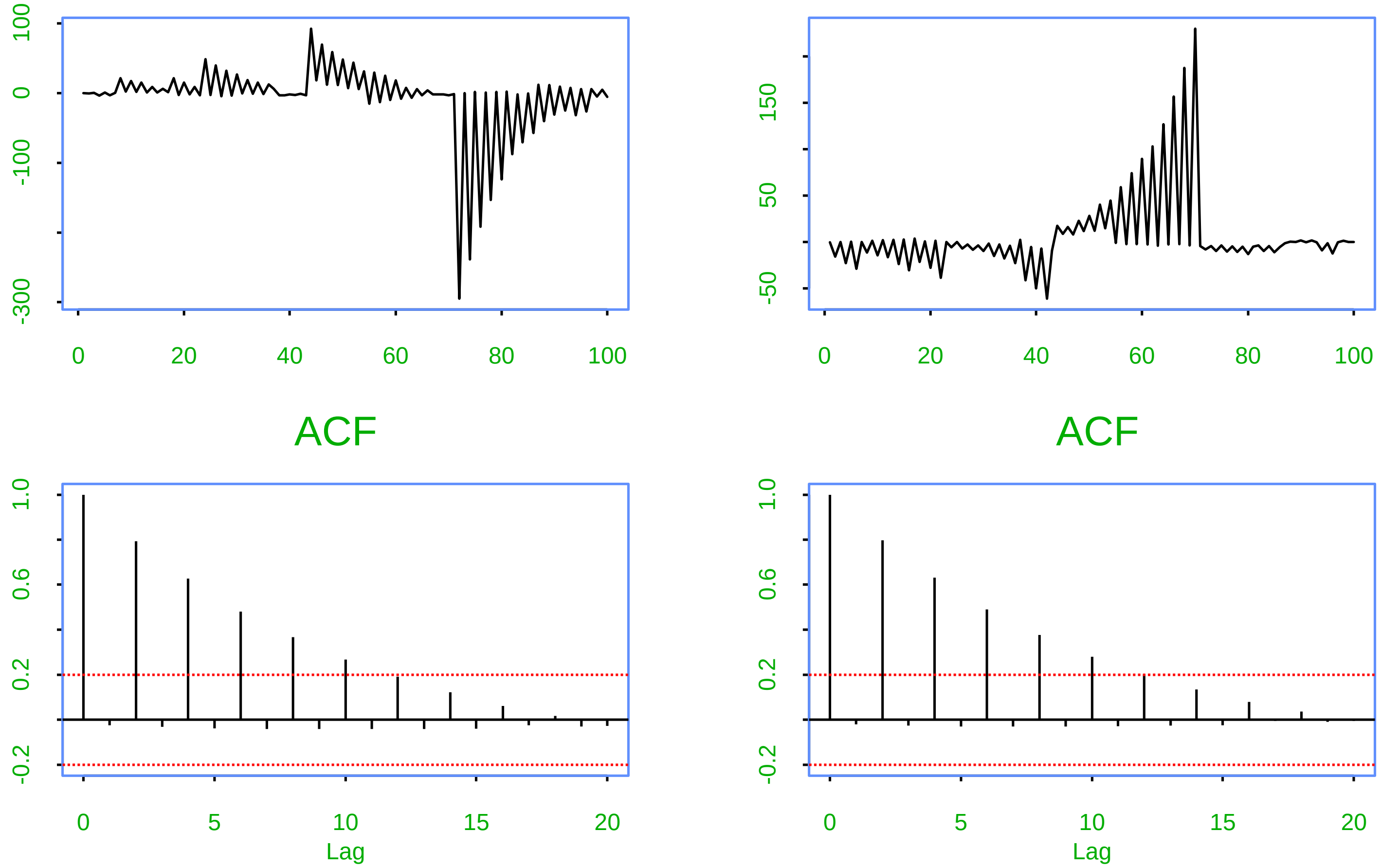
Mixed: High (non-causal) & Low (causal) frequency



Realization of a causal AR(2), and a non-causal AR(2)

Model: $\phi_*(B)X_t = Z_t$, $\{Z_t\} \sim \text{IID}(\alpha = 1)$, **where**

$\phi_c(B) = (1 - 0.9B)(1 + 0.9B)$ **and** $\phi_{nc}(B) = (1 - 1.1B)(1 + 1.1B)$



Application of all-pass to non-causal AR model fitting

Suppose $\{X_t\}$ follows the non-causal AR model

$$\phi_c(B)\,\phi_{nc}(B)\,X_t = Z_t\,, \quad \{Z_t\} \sim \text{IID}.$$

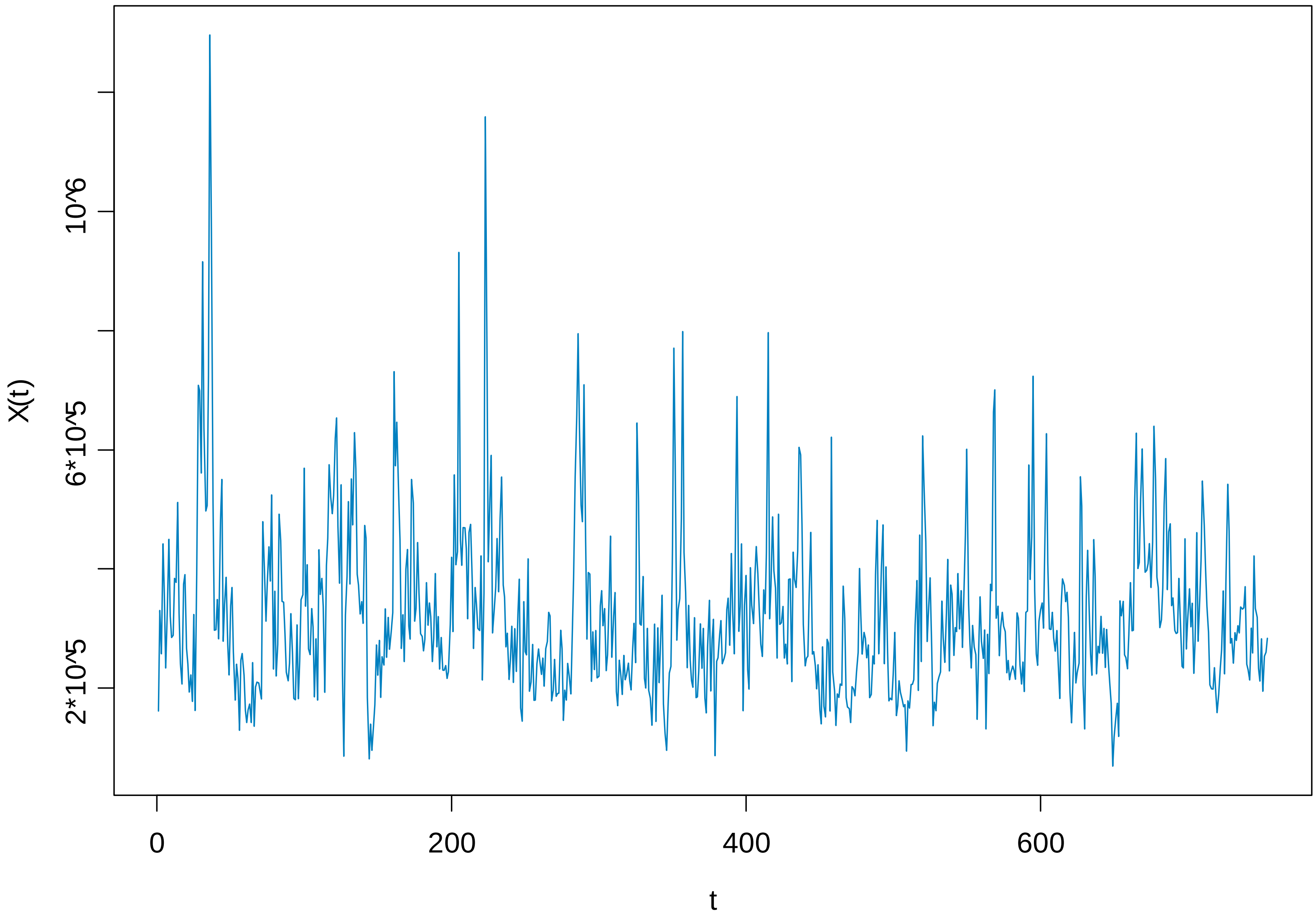
Step 1: Let $\{U_t\}$ be the residuals obtained by fitting a purely causal AR model, i.e.,

$$\begin{aligned} U_t &= \hat{\phi}(B)X_t \\ &\approx \phi_c(B)\tilde{\phi}_{nc}(B)X_t, \quad (\tilde{\phi}_{nc} \text{ is the causal version of } \phi_{nc}) \\ &= \frac{\tilde{\phi}_{nc}(B)}{\phi_{nc}(B)} Z_t \end{aligned}$$

Step 2: Fit a purely non-causal AP model to $\{U_t\}$

$$\phi_{nc}(B)U_t = \tilde{\phi}_{nc}(B)Z_t.$$

Volumes of Microsoft (MSFT) stock traded over 754 transaction days (6/3/96 to 5/27/99)



Analysis of MSFT:

Step 1: Log(volume) follows AR(1) or AR(3).

$$U_t = (1 - .5834\,B)\,X_t \qquad \text{(causal AR(1))}$$

Step 2: All-pass model of order 1 fitted to $\{U_t\}$:

$$(1 - 1.752\,B)U_t = (1 - .5708\,B)Z_t.$$

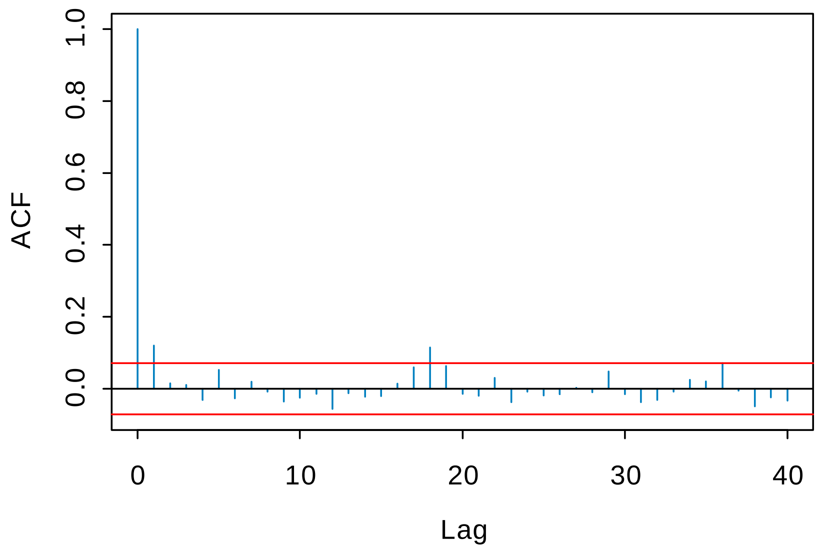
Combining the two models, we obtain the approximate non-causal model for $\{X_t\}$:

$$(1 - 1.752\,B)X_t = \frac{(1 - .5708\,B)}{(1 - .5834\,B)} Z_t \approx Z_t.$$

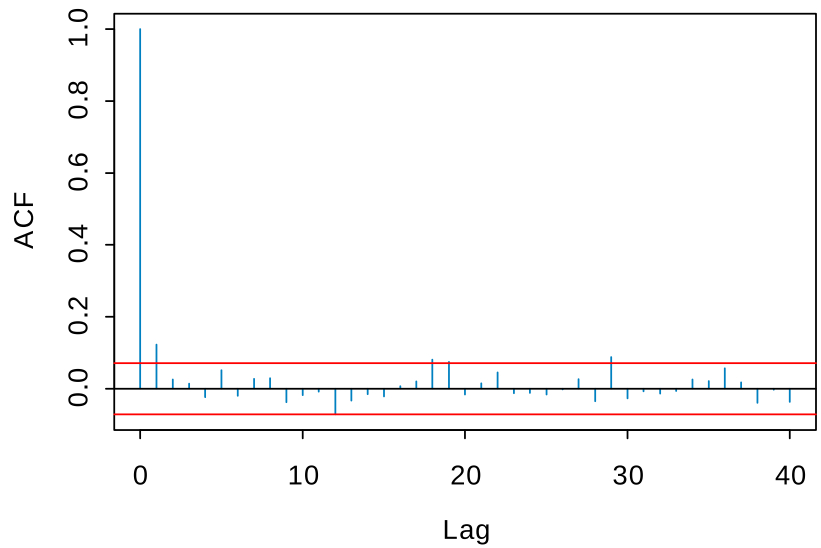
Estimated residuals from all-pass model fit:

$$\tilde{Z}_t = \frac{(1 - 1.752B)(1 - .5834\,B)}{(1 - .5708\,B)} X_t$$

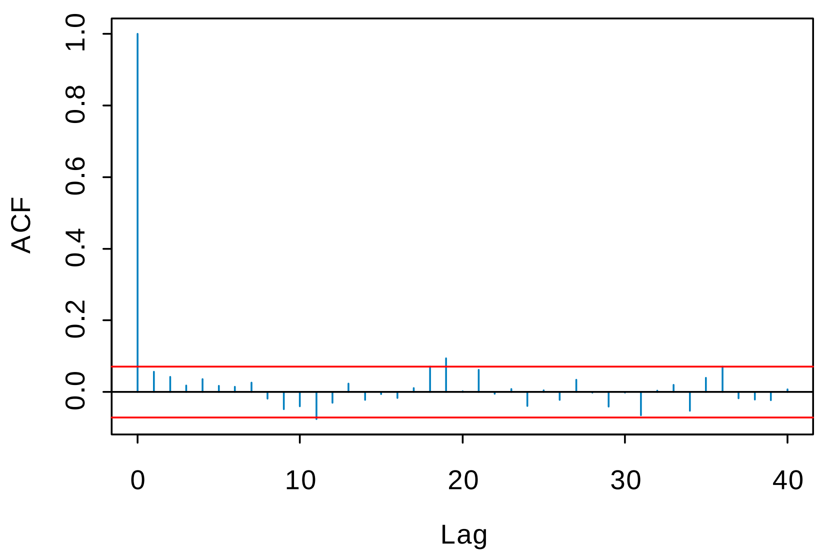
(a) ACF of Squares of Ut



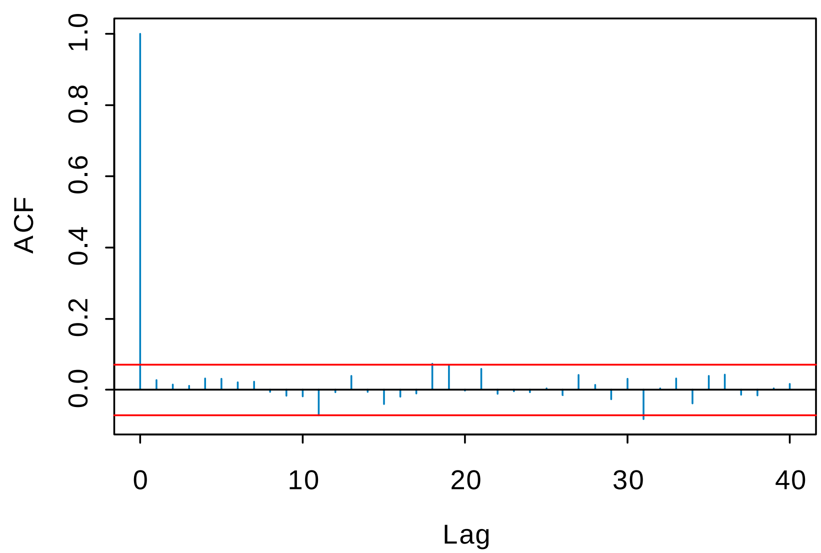
(b) ACF of Absolute Values of Ut



(c) ACF of Squares of Zt



(d) ACF of Absolute Values of Zt



Summary: Microsoft Trading Volume

☞ Two-step fit of noncausal AR(1): 1-1.7522B

- causal AR(1): residuals not iid
- purely noncausal AP(1); residuals iid

☞ Direct fit of noncausal AR(1): 1-1.7141B

☞ For ATML and MCHP, causal AR models fit

Summary

☞ All-pass models and their properties

- linear time series with "nonlinear" behavior

☞ Estimation

- likelihood approximation
- MLE and LAD
- order selection

☞ Emprirical results

- simulation study
- AP(6) for NZ/USA exchange rates

☞ Noncausal autoregressive processes

- two-step estimation procedure using all-pass
- noncausal AR(1) for Microsoft trading volume

Further Work

☞ Least absolute deviations

- further simulations
- order selection
- heavy-tailed case
- other smooth objective functions (e.g., min dispersion)

☞ Maximum likelihood

- Gaussian mixtures
- simulation studies
- applications

☞ Noninvertible moving average modeling

- initial estimates from two-step all-pass procedure
- adaptive procedures