Limit Theory for Some Non-Linear Time Series Models Including GARCH and SV

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Characteristics of Some Financial Time Series

Define \( X_t = 100^* (\ln(P_t) - \ln(P_{t-1})) \) (log returns)

- heavy tailed

\[
P(|X_1| > x) \sim C x^{-\alpha}, \quad 0 < \alpha < 4.
\]

- uncorrelated

\[
\hat{\rho}_x(h) \text{ near 0 for all lags } h > 0 \text{ (MGD sequence)}
\]

- \(|X_t|\) and \(X_t^2\) have slowly decaying autocorrelations

\[
\hat{\rho}_{|X|}(h) \text{ and } \hat{\rho}_{X^2}(h) \text{ converge to 0 slowly as } h \text{ increases.}
\]

- process exhibits ‘stochastic volatility’.
Sample ACF IBM (a) 1962-1981, (b) 1982-2000

(a) ACF of IBM (1st half)

(b) ACF of IBM (2nd half)
Sample ACF of abs values for IBM (a) 1961-1981, (b) 1982-2000

(a) ACF, Abs Values of IBM (1st half)

(b) ACF, Abs Values of IBM (2nd half)
Sample ACF of squares for IBM (a) 1961-1981, (b) 1982-2000

(a) ACF, Squares of IBM (1st half)

(b) ACF, Squares of IBM (2nd half)
Sample ACF of original data and squares for IBM 1962-2000
Plot of $M_t(4)/S_t(4)$ for IBM
500-daily log-returns of NZ/US exchange rate
ACF of log-returns of NZ/US exchange rate
Plot of $M_t(4)/S_t(4)$
Hill’s plot of tail index
Models for Log(returns)

Basic model

\[ X_t = 100*(\ln (P_t) - \ln (P_{t-1})) \]  \hspace{1cm} \text{(log returns)}

\[ = \sigma_t Z_t , \]

where

- \{Z_t\} is IID with mean 0, variance 1 (if exists). (e.g. N(0,1) or a \(t\)-distribution with \(\nu\) df.)
- \{\sigma_t\} is the volatility process
- \(\sigma_t\) and \(Z_t\) are independent.
Models for Log(returns)-cont

\[ X_t = \sigma_t Z_t \] (observation eqn in state-space formulation)

Examples of models for volatility:

(i) GARCH(p,q) process (observation-driven specification)

\[ \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_p X_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_q \sigma_{t-q}^2. \]

Special case: ARCH(1), \[ X_t^2 = (\alpha_0 + \alpha_1 X_{t-1}^2)Z_t^2. \]

\[ \rho_{X^2}(h) = \alpha_1^h, \text{ if } \alpha_1^2 < 1/3. \]

(ii) stochastic volatility process (parameter-driven specification)

\[ \log \sigma_t^2 = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}, \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty, \{ \varepsilon_t \} \sim \text{IID N}(0, \sigma^2) \]

\[ \rho_{X^2}(h) = \text{Cor}(\sigma_t^2, \sigma_{t+h}^2) / EZ_1^4 \]
Linear Processes

**Model:** \( X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j} \) \( \{Z_t\} \sim \text{IID}, \) \( P(|Z_t|>x) \sim C x^{-\alpha}, \) \( 0<\alpha<2. \)

**Properties:**

- \( P(|X_t|>x) \sim C_2 x^{-\alpha} \)
- Define \( \rho(h) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} / \sum_{j=-\infty}^{\infty} \psi_j^2. \)

**Case \( \alpha > 2: \)**
\[
n^{1/2}(\hat{\rho}(h) - \rho(h)) \xrightarrow{d} \sum_{j=1}^{\infty} (\rho(h+j)+\rho(h-j)-2\rho(j)\rho(h)) N_j, \quad \{N_t\} \sim \text{IIDN}
\]

**Case \( 0 < \alpha < 2: \)**
\[
(n / \ln n)^{1/\alpha} (\hat{\rho}(h) - \rho(h)) \xrightarrow{d} \sum_{j=1}^{\infty} (\rho(h+j)+\rho(h-j)-2\rho(j)\rho(h)) S_j / S_0,
\]
\( \{S_t\} \sim \text{IID stable (}\alpha\text{), } S_0 \text{ stable (}\alpha/2\text{)} \)
Multivariate regular variation of \( X=(X_1, \ldots, X_m) \): There exists a random vector \( \theta \in S^{m-1} \) such that

\[
P(|X|> tx, X/|X| \in \bullet)/P(|X|>t) \xrightarrow{v} x^{-\alpha} P( \theta \in \bullet)
\]

\( (\xrightarrow{v} \text{ vague convergence on } S^{m-1}) \).

- \( P( \theta \in \bullet) \) is called the spectral measure
- \( \alpha \) is the index of \( X \).

**Equivalence:** There exist positive constants \( a_n \) and a measure \( \mu \),

\[
nP(X/ a_n \in \bullet) \xrightarrow{v} \mu(\bullet)
\]

In this case, one can choose \( a_n \) and \( \mu \) such that

\[
\mu((x, \infty) \times B) = x^{-\alpha} P( \theta \in B)
\]
Another equivalence?

MRV ⇔ all linear combinations of $X$ are regularly varying

i.e., if and only if

$$
P(c^T X > t) / P(1^T X > t) \rightarrow w(c), \text{ exists for all real-valued } c,
$$

in which case,

$$
w(tc) = t^{-\alpha} w(c).
$$

$(\Rightarrow)$ true

$(\Leftarrow)$ established by Basrak, Davis and Mikosch (2000) for $\alpha$ not an even integer—case of even integer is unknown.
Background Results—point process convergence

**Theorem** (Davis & Hsing `95, Davis & Mikosch `97). Let \( \{X_t\} \) be a stationary sequence of random vectors. Suppose

(i) finite dimensional distributions are jointly regularly varying (let \((\theta_{-k}, \ldots, \theta_k)\) be the vector in \(S^{(2k+1)m-1}\) in the definition).

(ii) mixing condition \( A(a_n) \) or strong mixing.

(iii) \(\lim_{k \to \infty} \limsup_{n \to \infty} P(\sup_{k \leq |t| \leq r_n} |X_t| > a_n y | |X_0| > a_n y) = 0.\)

Then

\[
\gamma = \lim_{k \to \infty} E(\sum_{j=1}^{k} \theta_j^{(k)} | |X_0| |)^\alpha / E(\theta_0^{(k)} | |X_0| |)^\alpha
\]

exists. If \(\gamma > 0\), then

\[
N_n := \sum_{t=1}^{n} \varepsilon_{X_t/a_n} \xrightarrow{d} N := \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varepsilon_{P_iQ_{ij}},
\]
where

- \((P_i)\) are points of a Poisson process on \((0, \infty)\) with intensity function \(\nu(dy) = \gamma \alpha y^{-\alpha-1} dy\).
- \(\sum_{j=1}^{\infty} \epsilon_{Q_{ij}}, \ i \geq 1,\) are iid point process with distribution \(Q\), and \(Q\) is the weak limit of

\[
\lim_{k \to \infty} E(|\theta_0^{(k)}|^{\alpha} - \vee_{j=1}^{k} |\theta_j^{(k)}|) + I_* \left( \sum_{|l| \leq k} \epsilon_{\theta_l^{(k)}} \right) / E(|\theta_0^{(k)}|^{\alpha} - \vee_{j=1}^{k} |\theta_j^{(k)}|).\]
Set-up: Let \( \{X_t\} \) be a stationary sequence and set
\[
X_t = X_t(m) = (X_t, \ldots, X_{t+m}).
\]
Suppose \( X_t \) satisfies the conditions of previous theorem. Then
\[
N_n := \sum_{t=1}^{n} \varepsilon_{X_t/a_n} \xrightarrow{d} N := \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varepsilon_{P_iQ_{ij}},
\]
Sample ACVF and ACF:
\[
\hat{\gamma}_X(h) = n^{-1} \sum_{t=1}^{n-h} X_t X_{t+h}, \ h \geq 0, \quad \text{ACVF}
\]
\[
\hat{\rho}_X(h) = \frac{\hat{\gamma}_X(h)}{\hat{\gamma}_X(0)}, \ h \geq 1, \quad \text{ACF}
\]
If \( \text{E}X_0^2 < \infty \), then define \( \gamma_X(h) = \text{E}X_0 X_h \) and \( \rho_X(h) = \gamma_X(h)/\gamma_X(0) \).
Background Results—application to ACVF & ACF

(i) If $\alpha \in (0,2)$, then

$$(na_n^{-2}\hat{\gamma}_X(h))_{h=0,\ldots,m} \xrightarrow{d} (V_h)_{h=0,\ldots,m}$$

$$(\hat{\rho}_X(h))_{h=1,\ldots,m} \xrightarrow{d} (V_h / V_0)_{h=1,\ldots,m},$$

where

$$V_h = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P_i^2 Q_{ij}^{(0)} Q_{ij}^{(h)}, \quad h = 0,\ldots,m.$$

(ii) If $\alpha \in (2,4) +$ additional condition, then

$$(na_n^{-2}(\hat{\gamma}_X(h) - \gamma_X(h)))_{h=0,\ldots,m} \xrightarrow{d} (V_h)_{h=0,\ldots,m}$$

$$(na_n^{-2}(\hat{\rho}_X(h) - \rho_X(h)))_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1}(0)(V_h - \rho_X(h)V_0)_{h=1,\ldots,m}.$$
Applications—stochastic recurrence equations

\[ Y_t = A_t Y_{t-1} + B_t, \quad (A_t, B_t) \sim \text{IID}, \]

\[ A_t \text{ } d \times d \text{ random matrices, } B_t \text{ random } d \text{-vectors} \]

Examples

ARCH(1): \[ X_t = (\alpha_0 + \alpha_1 X_{t-1}^2)^{1/2} Z_t, \quad \{Z_t\} \sim \text{IID}. \] Then the squares follow an SRE with \[ Y_t = X_t^2, \quad A_t = \alpha_1 Z_t^2, \quad B_t = \alpha_0 Z_t^2. \]

GARCH(2,1): \[ X_t = \sigma_t Z_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \alpha_2 X_{t-2}^2 + \beta_1 \sigma_{t-1}^2. \] Then \[ Y_t = (X_t^2, X_{t-1}^2, \sigma_t^2)' \] follows the SRE given by

\[
\begin{bmatrix}
    X_t^2 \\
    X_{t-1}^2 \\
    \sigma_t^2
\end{bmatrix}
= \begin{bmatrix}
    \alpha_1 Z_t^2 & \alpha_2 Z_t^2 & \beta_1 Z_t^2 \\
    1 & 0 & 0 \\
    \alpha_1 & \alpha_2 & \beta_1
\end{bmatrix}
\begin{bmatrix}
    X_{t-1}^2 \\
    X_{t-2}^2 \\
    \sigma_{t-1}^2
\end{bmatrix}
+ \begin{bmatrix}
    \alpha_0 Z_t^2 \\
    0 \\
    0
\end{bmatrix}
\]
Examples (tricks)

GARCH(1,1): \[ X_t = \sigma_t Z_t, \quad \sigma^2_t = \alpha_0 + \alpha_1 X^2_{t-1} + \beta_1 \sigma^2_{t-1}. \]

Although this process does not have a 1-dimensional SRE representation, the process \( \sigma^2_t \) does. Iterating, we have

\[
\sigma^2_t = \alpha_0 + \alpha_1 X^2_{t-1} + \beta_1 \sigma^2_{t-1} = \alpha_0 + \alpha_1 \sigma^2_{t-1} Z^2_{t-1} + \beta_1 \sigma^2_{t-1}
\]
\[= (\alpha_1 Z^2_{t-1} + \beta_1) \sigma^2_{t-1} + \alpha_0.\]

Bilinear(1): \[ X_t = aX_{t-1} + bX_{t-1}Z_{t-1} + Z_t, \quad \{Z_t\} \sim \text{IID} \]
\[= Y_{t-1} + Z_t, \]
\[Y_t = A_t Y_{t-1} + B_t, \quad A_t = a + bZ_t, \quad B_t = A_t Z_t\]
Stochastic Recurrence Equations (cont)

\[ Y_t = A_t Y_{t-1} + B_t, \quad (A_t, B_t) \sim \text{IID} \]

Existence of stationary solution

- \( E \ln^+ || A_1 || < \infty \)
- \( E \ln^+ || B_1 || < \infty \)
- \( \inf n^{-1} E \ln || A_1 \ldots A_n || =: \gamma < 0 \) \( (\gamma \text{ -- top Lyapunov exponent}) \)

Ex. (d=1) \( E \ln |A_1| < 0 \).

Strong mixing

If \( E ||A_1||^\varepsilon < \infty, E |B_1|^\varepsilon < \infty \) for some \( \varepsilon > 0 \), then the SRE \( (Y_t) \) is geometrically ergodic \( \Rightarrow \) strong mixing with geometric rate (Meyn and Tweedie `93).
Regular variation of the marginal distribution (Kesten)

Assume $A$ and $B$ have non-negative entries and

- $E \|A_1\|^{\varepsilon} < 1$ for some $\varepsilon > 0$
- $A_1$ has no zero rows a.s.
- W.P. 1, $\{\ln \rho(A_1 \ldots A_n):$ is dense in $\mathbb{R}$ for some $n, A_1 \ldots A_n > 0\}$
- There exists a $\kappa_0 > 0$ such that $E\|A\|^{\kappa_0} \ln^+\|A\| < \infty$ and
  \[ E\left(\min_{i=1,\ldots,d} \sum_{j=1}^d A_{ij}\right)^{\kappa_0} \geq d^{\kappa_0/2} \]

Then there exists a $\kappa_1 \in (0, \kappa_0]$ such that all linear combinations of $Y$ are regularly varying with index $\kappa_1$. (Also need $E|B|^\kappa_i < \infty$.)
**Proposition:** Let \( (Y_t) \) be the soln to the SRE based on the squares of a GARCH model. Assume

- Top Lyapunov exponent \( \gamma < 0 \). (See Bougerol and Picard`92)
- \( Z \) has a positive density on \((-\infty, \infty)\) with all moments finite or \( E|Z|^h = \infty \), for all \( h \geq h_0 \) and \( E|Z|^h < \infty \) for all \( h < h_0 \).
- Not all the GARCH parameters vanish.

Then \( (Y_t) \) is *strongly mixing* with geometric rate and all finite dimensional distributions are *multivariate regularly varying* with index \( \kappa_1 \).

**Corollary:** The corresponding GARCH process is strongly mixing and has all finite dimensional distributions that are MRV with index \( \kappa = 2\kappa_1 \).
Application to GARCH (cont)

Remarks:
1. Kesten’s result applied to an iterate of $Y_t$, i.e., $Y_{tm} = \tilde{A}_t Y_{(t-1)m} + \tilde{B}_t$

2. Determination of $\kappa$ is difficult. Explicit expressions only known in two(?) cases.

- ARCH(1): $E|\alpha_1 Z^2|^\kappa/2 = 1$.

<table>
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<tr>
<th>$\alpha_1$</th>
<th>0.312</th>
<th>0.577</th>
<th>1.00</th>
<th>1.57</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa$</td>
<td>8.00</td>
<td>4.00</td>
<td>2.00</td>
<td>1.00</td>
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- GARCH(1,1): $E|\alpha_1 Z^2 + \beta_1|^\kappa/2 = 1$ (Mikosch and Starica)

- For IGARCH ($\alpha_1 + \beta_1 = 1$), then $\kappa = 2 \Rightarrow$ infinite variance.

- Can estimate $\kappa$ empirically by replacing expectations with sample moments.
Summary for GARCH(p,q)

κ∈ (0,2):

\[(\hat{\rho}_X (h))_{h=1,\ldots,m} \xrightarrow{d} (V_h / V_0)_{h=1,\ldots,m},\]

κ∈ (2,4):

\[(n^{1-2/\kappa}\hat{\rho}_X (h))_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1}(0)(V_h)_{h=1,\ldots,m}.\]

κ∈ (4,∞):

\[(n^{1/2}\hat{\rho}_X (h))_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1}(0)(G_h)_{h=1,\ldots,m}.\]

Remark: Similar results hold for the sample ACF based on |X_t| and X_t^2.
Realization of GARCH Process

Fitted GARCH(1,1) model for NZ-USA exchange:

\[ X_t = \sigma_t Z_t, \quad \sigma_t^2 = (6.70)10^{-7} + .1519X_{t-1}^2 + .772\sigma_{t-1}^2 \]

\( (Z_t) \sim \text{IID t-distr with 5 df.} \quad \kappa \text{ is approximately 3.8} \)
ACF of Fitted GARCH(1,1) Process

ACF of squares of realization 1

ACF of squares of realization 2
ACF of 2 realizations of an (ARCH)$^2$: $X_t = (.001 + .7 X_{t-1})^{1/2} Z_t$
ACF of 2 realizations of an ARCH: \( X_t = (0.001 + X_{t-1})^{1/2} Z_t \)
Stochastic Volatility Models

\[ X_t = \sigma_t Z_t \]

- \((Z_t) \sim \text{IID with mean 0 (if it exists)}\)
- \((\sigma_t)\) is a stationary process \((2 \log \sigma_t\) is a linear process) given by
  \[
  \log \sigma_t^2 = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}, \quad \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty, \quad (\varepsilon_t) \sim \text{IID N}(0, \sigma^2)\]

Heavy tails: Assume \(Z_t\) has Pareto tails with index \(\alpha\), i.e.,

\[
P(\left| Z_t \right| > z) \sim C z^{-\alpha} \Rightarrow P(\left| X_t \right| > z) \sim C E\sigma^{\alpha} z^{-\alpha}.
\]

Then if \(\alpha \in (0,2)\),

\[
\left( n / \ln n \right)^{1/\alpha} \hat{\rho}_X(h) \xrightarrow{d} \frac{\left\| \sigma_1 \sigma_{h+1} \right\|_\alpha}{\left\| \sigma_1 \right\|_\alpha^2} S_h. \quad \frac{S_h}{S_0}.
\]
Stochastic Volatility Models (cont)

Other powers:

1. Absolute values: \( \alpha \in (1,2) \),

\[
E|X_t| = E|\sigma_t|E|Z_t|, \quad E|X_t X_{t+h}| = (E|\sigma_t \sigma_{t+h}|)(E|Z_t|E|Z_{t+h}|)
\]

\[
Cov(|X_t|, |X_{t+h}|) = Cov(\sigma_t, \sigma_{t+h})(E|Z|)^2
\]

\[
Cor(|X_t|, |X_{t+h}|) = Cor(\sigma_t, \sigma_{t+h})(E|Z|)^2 / EZ^2 = 0 ?.
\]

We obtain

\[
n(n \ln n)^{-1/\alpha}(\hat{\gamma}_{|X|}(h) - \gamma_{|X|}(h)) \xrightarrow{d} \|\sigma_1 \sigma_{h+1}\|_{\alpha} S_h
\]

and

\[
(n / \ln n)^{1/\alpha} \hat{\rho}_{|X|}(h) \xrightarrow{d} \frac{\|\sigma_1 \sigma_{h+1}\|_{\alpha}}{\|\sigma_1\|^2_{\alpha}} \frac{S_h}{S_0}.
\]
2. Higher order: $\alpha \in (0,2)$

The squares are again a SV process and the results of the previous proposition apply. Namely,

\[
\left( \frac{n}{\ln n} \right)^{2/\alpha} \hat{\rho}_{X^2}(h) \xrightarrow{d} \frac{\|\sigma^2_1\sigma^2_{h+1}\|_{\alpha/2}}{\|\sigma^2_1\|_{\alpha/2}} \frac{S_h}{S_0}.
\]

In particular,

\[
\hat{\rho}_{X^2}(h) \xrightarrow{p} 0.
\]
(log $X^2$) - mean for NZ-USA exchange rates
Stochastic Volatility Models (cont)

ACF/PACF for (log $X^2$) suggests ARMA (1,1) model:

$\mu = -11.5403, \ Y_t = .9646Y_{t-1} + \varepsilon_t - .8709 \varepsilon_{t-1}, (\varepsilon_t) \sim WN(0,4.6653)$
The ARMA (1,1) model for log $X^2$ leads to the SV model

$$X_t = \sigma_t Z_t$$

with

$$2 \ln \sigma_t = -11.5403 + v_t + \epsilon_t$$

$$v_t = .9646 v_{t-1} + \gamma_t, \quad (\gamma_t) \sim WN(0,.07253)$$

$$(\epsilon_t) \sim WN(0,4.2432).$$
Simulation of SVM model: Took $\varepsilon_t$ to be distributed according to log of a $t$ random variable with 3 df (suitable normalized).

ACF: abs(realization)
Stochastic Volatility Models (cont)

ACF: realization^2

ACF: realization^4
Sample ACF for GARCH and SV Models (1000 reps)

(a) GARCH(1,1) Model, n=10000

(b) SV Model, n=10000
Sample ACF for Squares of GARCH and SV (1000 reps)

(a) GARCH(1,1) Model, n=10000

(b) SV Model, n=10000
Sample ACF for Squares of GARCH and SV (1000 reps)

(c) GARCH(1,1) Model, n=100000

(d) SV Model, n=100000