Limit Theory for Some Non-Linear Time Series Models Including GARCH and SV

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Outline

- Characteristics of some financial time series
    - IBM returns
    - NZ-USA exchange rate
- Linear processes
- Background results
    - Multivariate regular variation
    - Point process convergence
    - Application to sample ACVF and ACF
- Applications
    - Stochastic recurrence equations (GARCH)
    - Stochastic volatility models

Characteristics of Some Financial Time Series

Define $X_t = 100*(\ln(P_t) - \ln(P_{t-1}))$ (log returns)

- heavy tailed

$$P(|X_1| > x) \sim C\,x^{-\alpha}, \quad 0 < \alpha < 4.$$

- uncorrelated

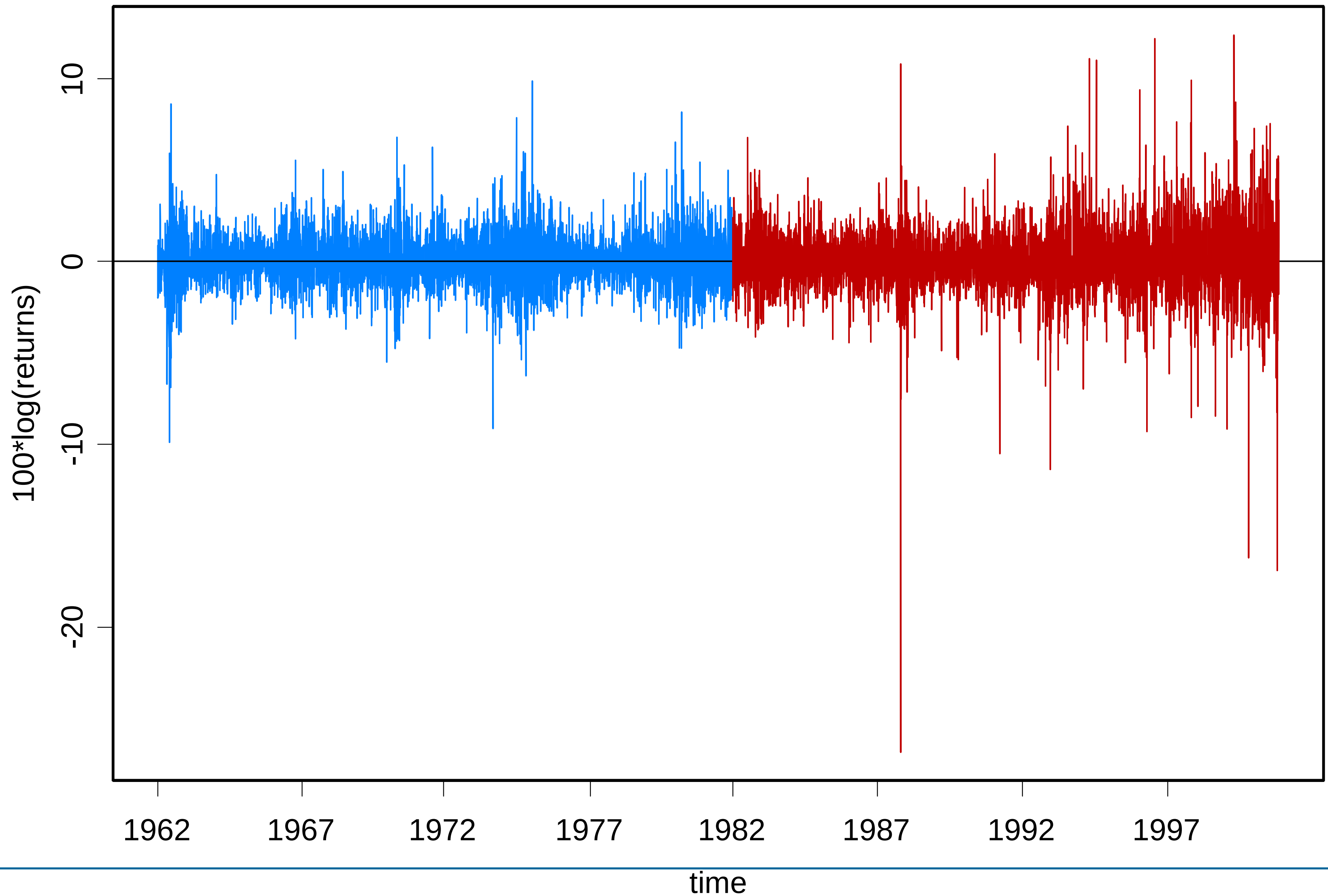
$\hat{\rho}_X(h)$ near 0 for all lags $h > 0$ (MGD sequence)

- $|X_t|$ and X_t^2 have slowly decaying autocorrelations

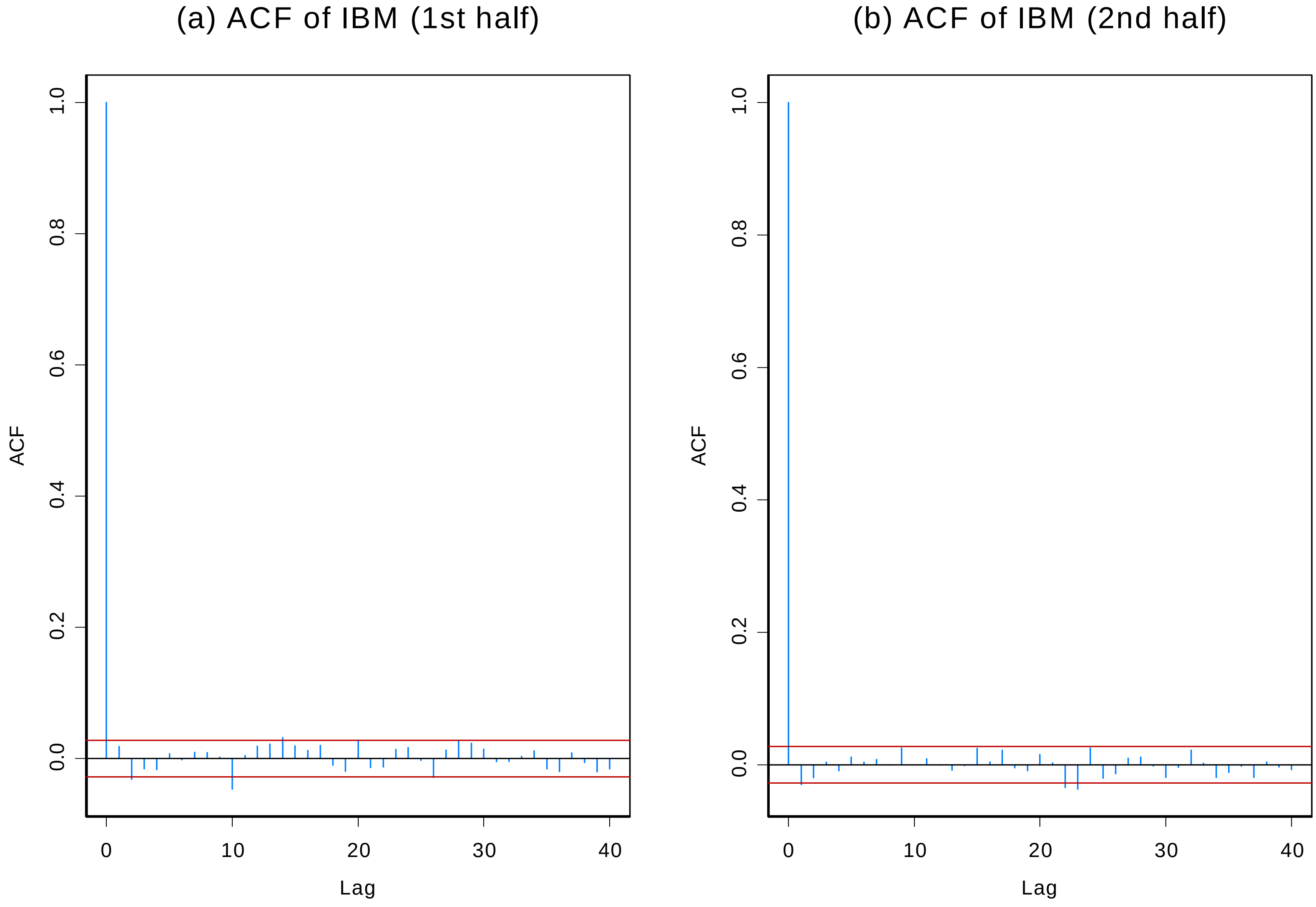
$\hat{\rho}_{|X|}(h)$ and $\hat{\rho}_{X^2}(h)$ converge to 0 *slowly* as h increases.

- process exhibits ‘stochastic volatility’.

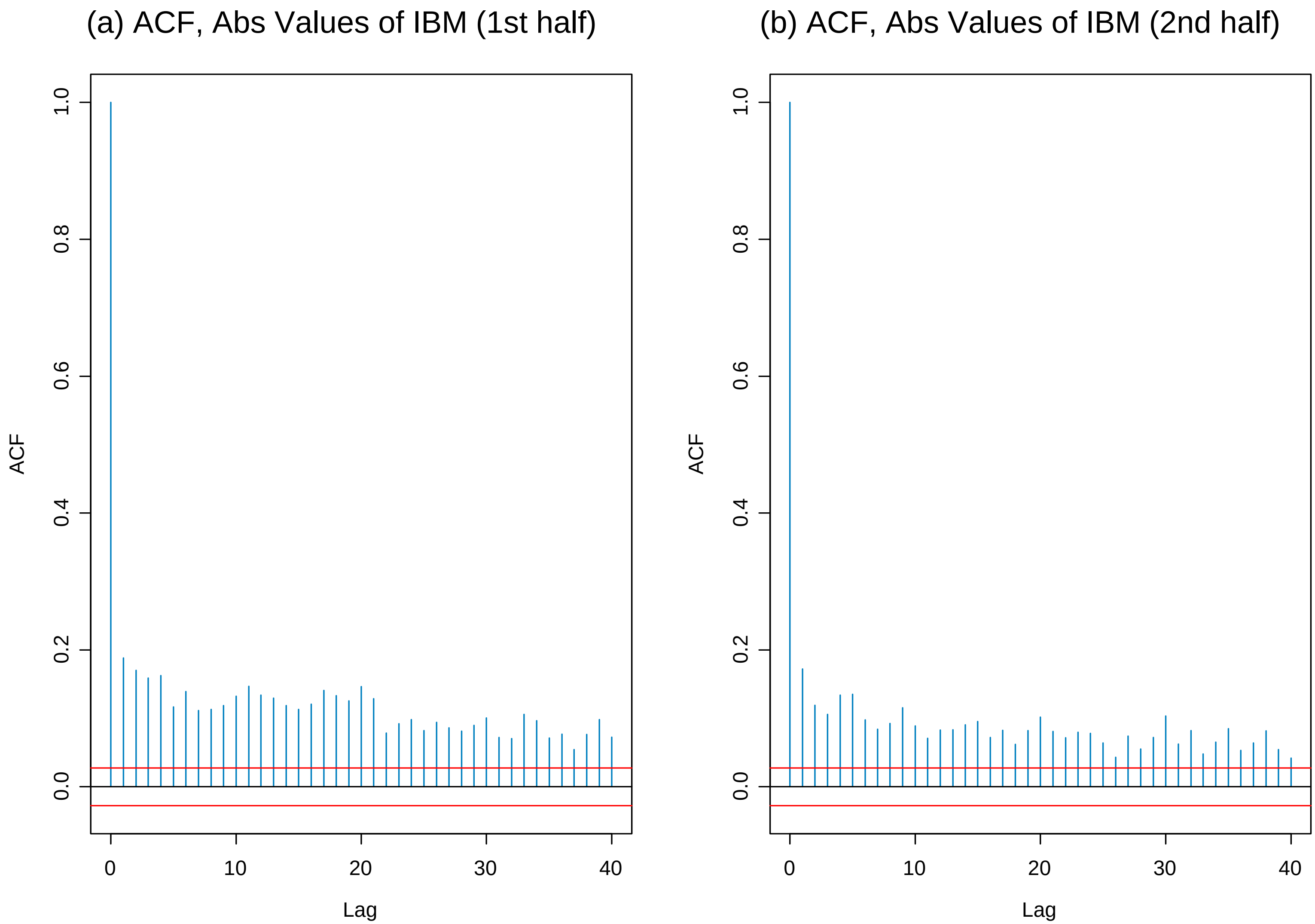
Log returns for IBM 1/3/62-11/3/00 (blue=1961-1981)



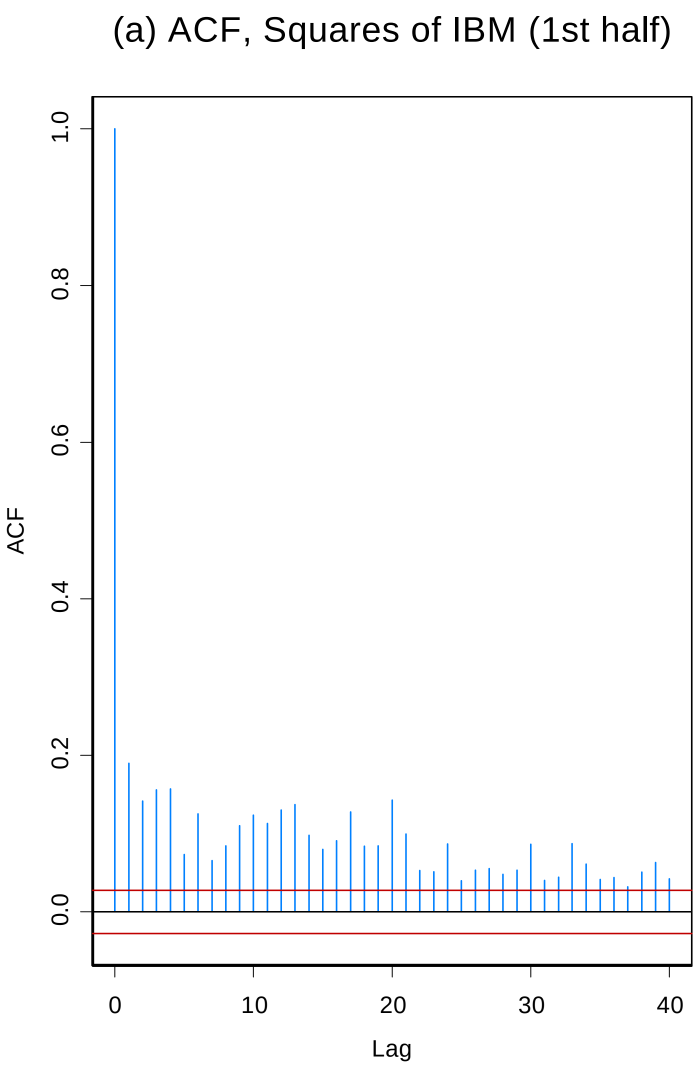
Sample ACF IBM (a) 1962-1981, (b) 1982-2000

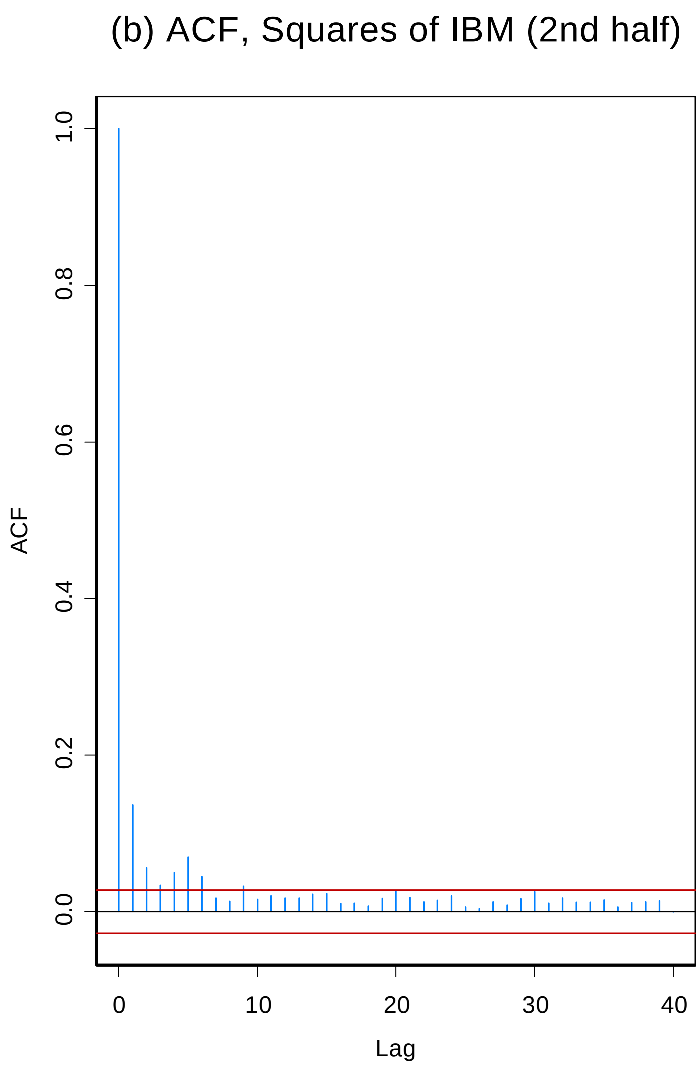


Sample ACF of abs values for IBM (a) 1961-1981, (b) 1982-2000

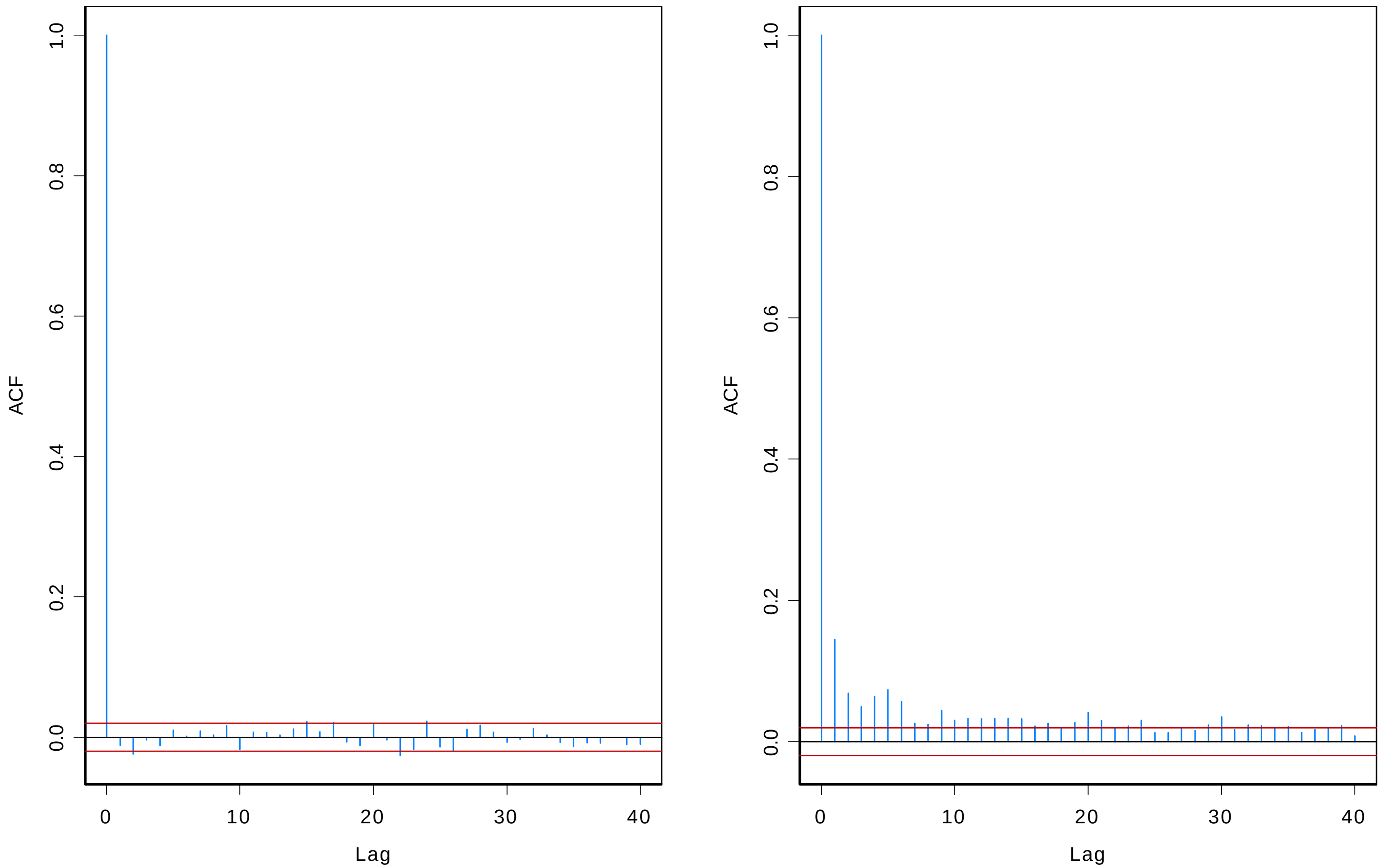


Sample ACF of squares for IBM (a) 1961-1981, (b) 1982-2000

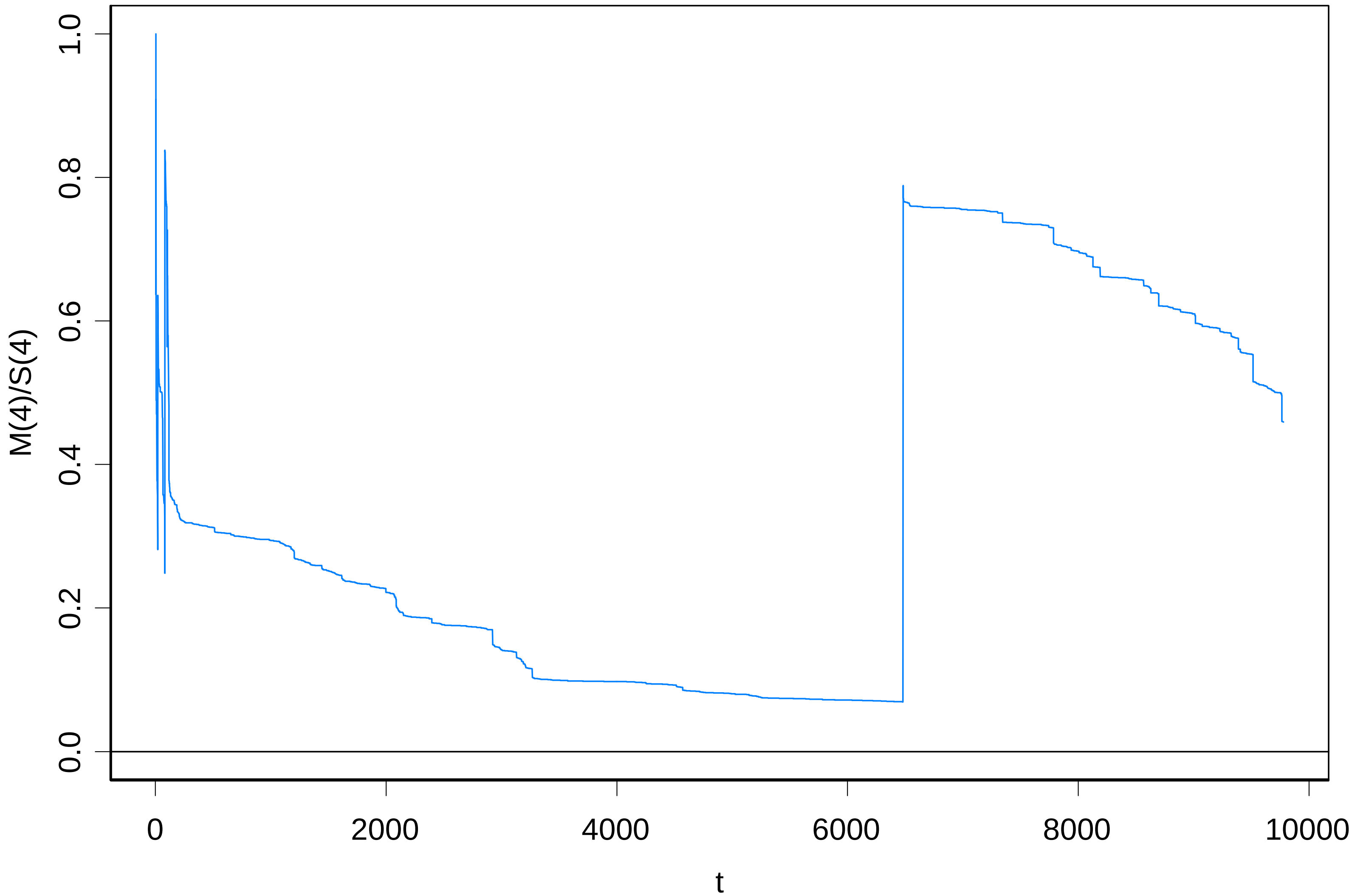




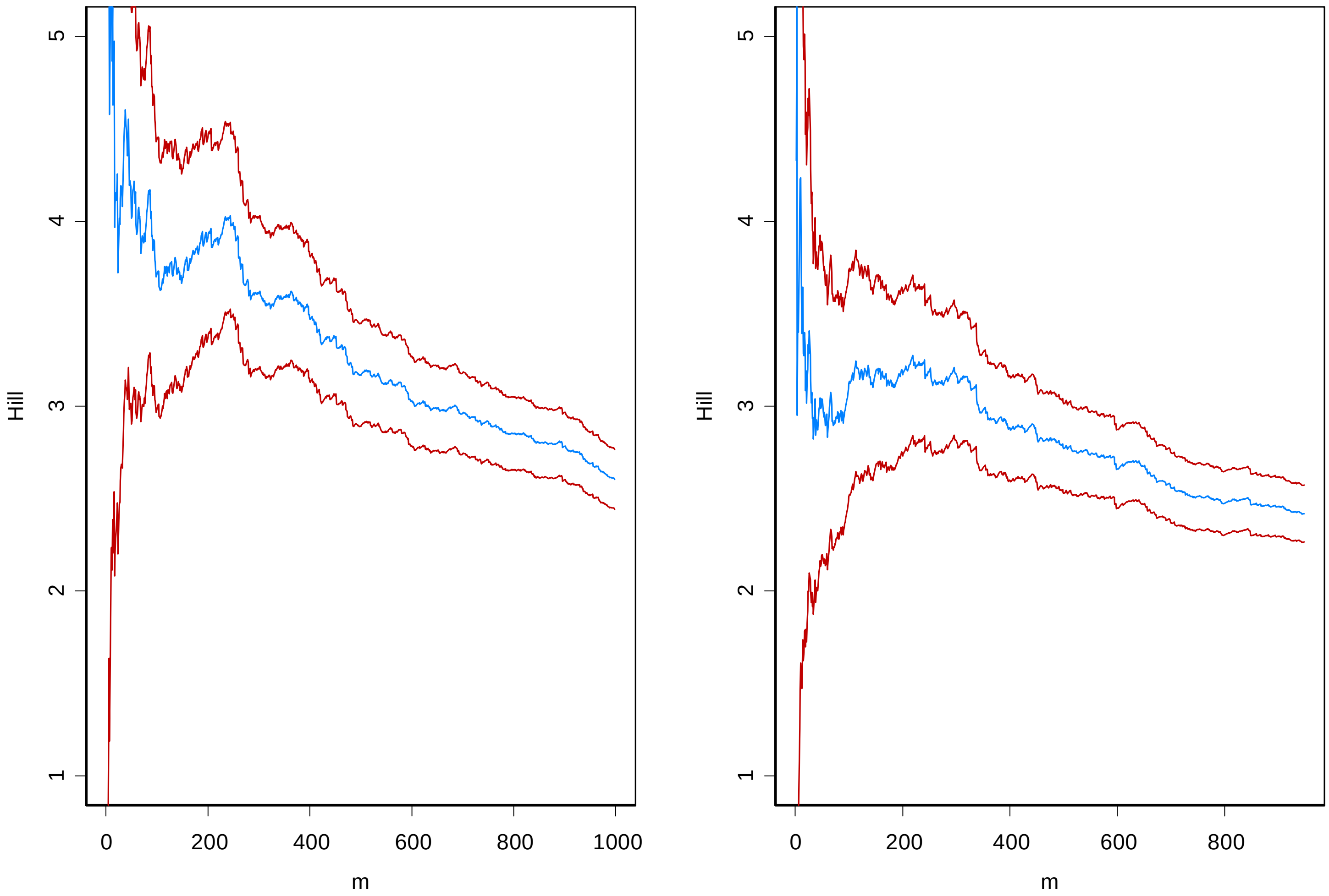
Sample ACF of original data and squares for IBM 1962-2000



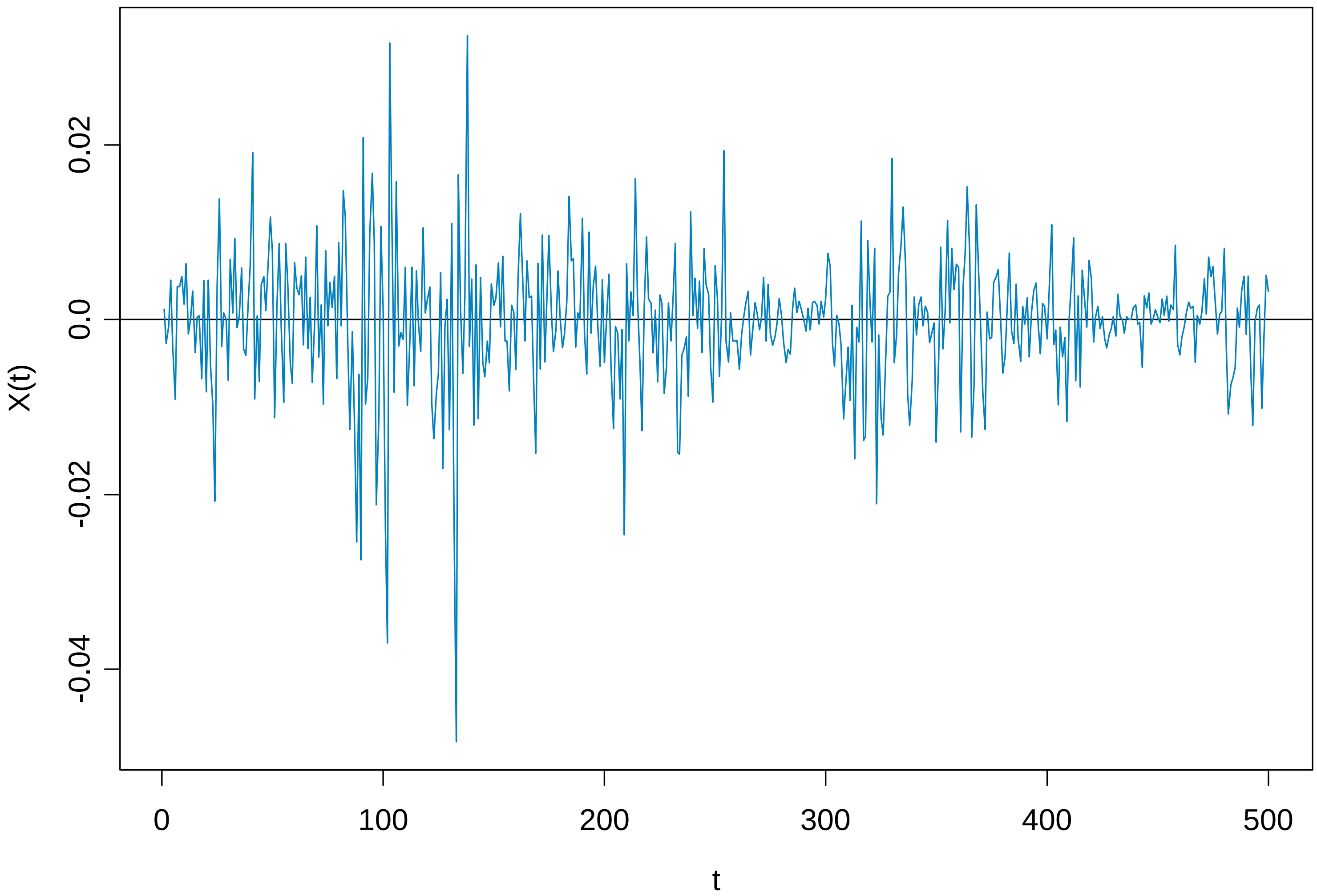
Plot of $M_t(4)/S_t(4)$ for IBM



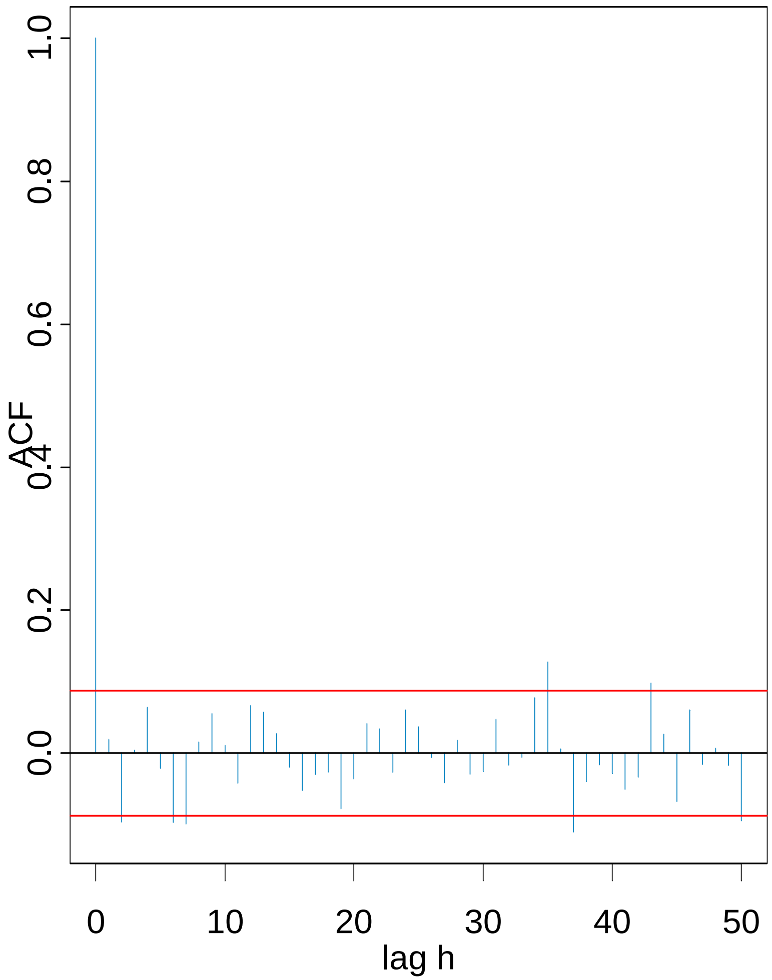
Hill's plot of tail index for IBM (1962-1981, 1982-2000)

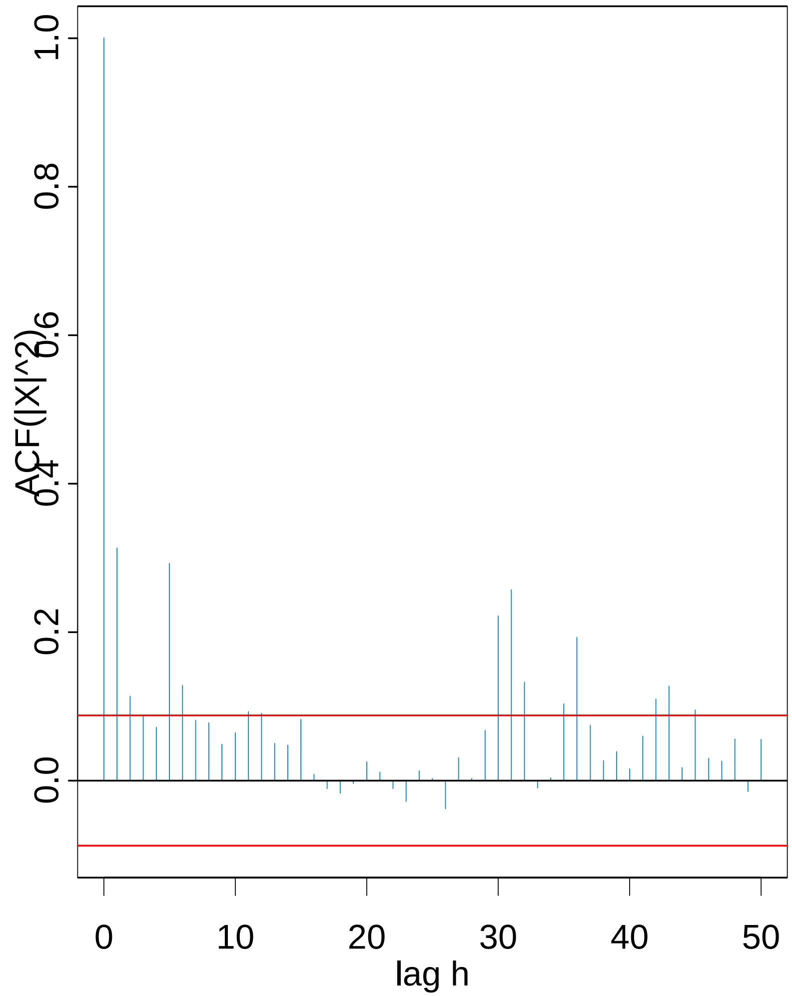


500-daily log-returns of NZ/US exchange rate

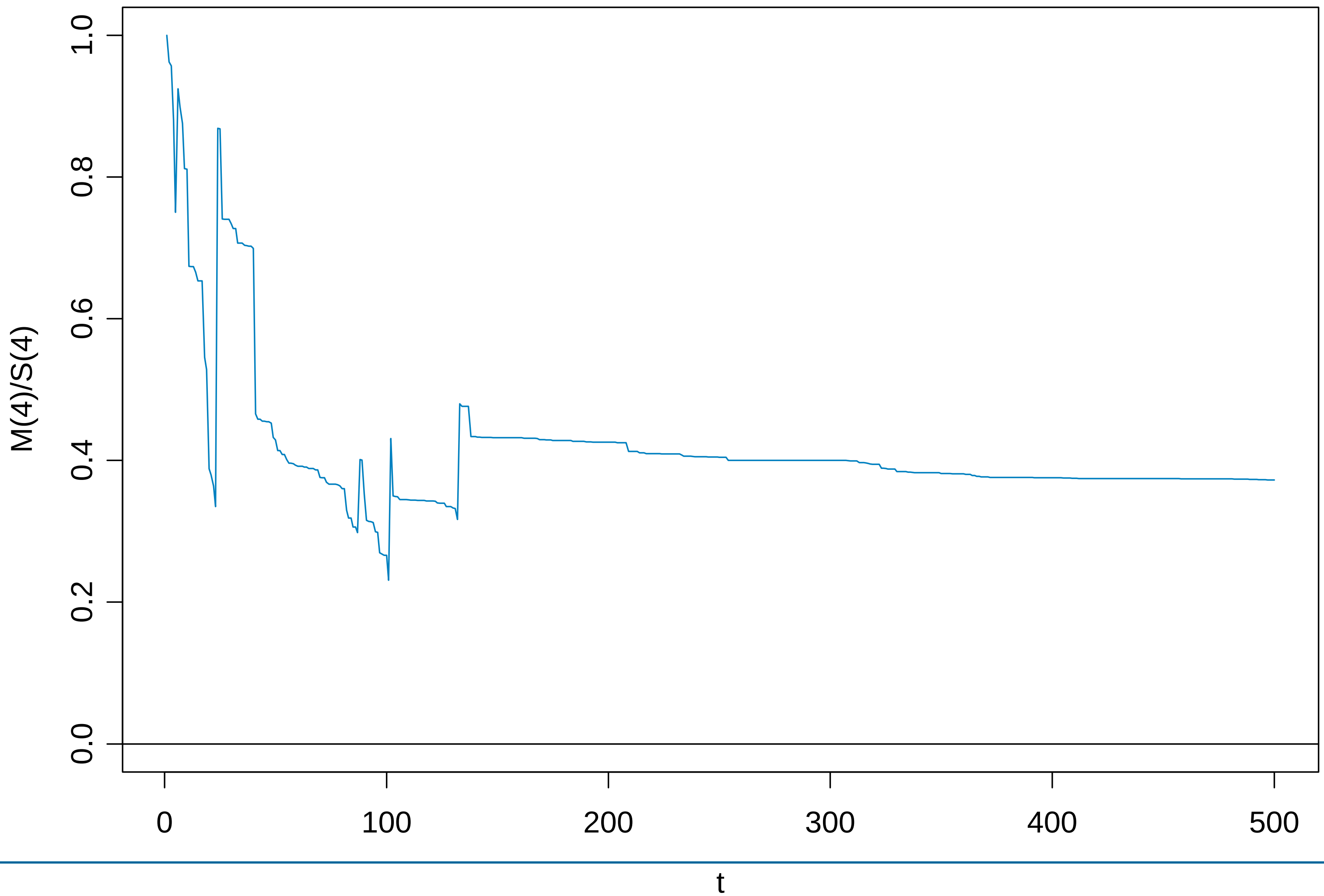


ACF of log-returns of NZ/US exchange rate

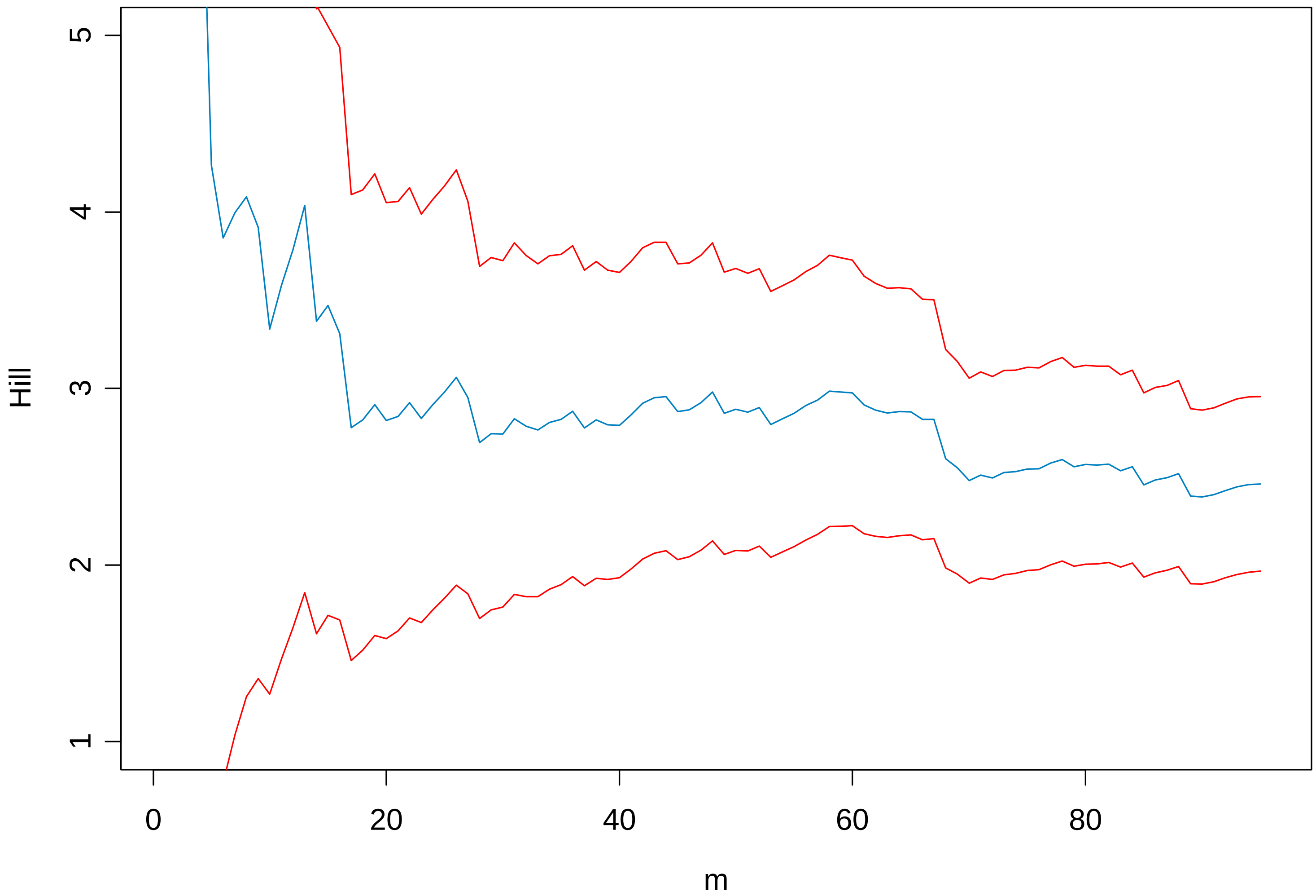




Plot of $M_t(4)/S_t(4)$



Hill's plot of tail index



Models for Log(returns)

Basic model

$$X_t = 100*(\ln(P_t) - \ln(P_{t-1})) \quad \text{(log returns)}$$

$$= \sigma_t Z_t ,$$

where

- $\{Z_t\}$ is IID with mean 0, variance 1 (if exists). (e.g. N(0,1) or a t-distribution with ν df.)
- $\{\sigma_t\}$ is the volatility process
- σ_t and Z_t are independent.

Models for Log(returns)-cont

$X_t = \sigma_t Z_t$ (observation eqn in state-space formulation)

Examples of models for volatility:

(i) GARCH(p,q) process (observation-driven specification)

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_p X_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_q \sigma_{t-q}^2 \ .$$

Special case: ARCH(1), $X_t^2 = (\alpha_0 + \alpha_1 X_{t-1}^2) Z_t^2$.

$$\rho_{X^2}(h) = \alpha_1^h, \text{ if } \alpha_1^2 < 1/3.$$

(ii) stochastic volatility process (parameter-driven specification)

$$\log \sigma_t^2 = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}, \ \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty, \{\varepsilon_t\} \sim \text{IID N}(0, \sigma^2)$$

$$\rho_{X^2}(h) = Cor(\sigma_t^2, \sigma_{t+h}^2) / EZ_1^4$$

Linear Processes

Model: $X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$ $\{Z_t\}$~IID, $P(|Z_t|>x) \sim Cx^{-\alpha}$, $0<\alpha<2$.

Properties:

- $P(|X_t|>x) \sim C_2\, x^{-\alpha}$
- Define $\rho(h) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} / \sum_{j=-\infty}^{\infty} \psi_j^2$.

Case $\alpha > 2$:

$$n^{1/2}(\hat{\rho}(h) - \rho(h)) \xrightarrow{d} \sum_{j=1}^{\infty} (\rho(h+j)+\rho(h-j)-2\rho(j)\rho(h))\, N_j,$$ $\{N_t\}$~IIDN

Case $0 < \alpha < 2$:

$$(n / \ln n)^{1/\alpha} (\hat{\rho}(h) - \rho(h)) \xrightarrow{d} \sum_{j=1}^{\infty} (\rho(h+j)+\rho(h-j)-2\rho(j)\rho(h))\, S_j / S_0,$$

$\{S_t\}$~IID stable (α), S_0 stable ($\alpha/2$)

Background Results—multivariate regular variation

Multivariate regular variation of $\mathbf{X}=(X_1, \ldots, X_m)$: There exists a random vector $\theta \in S^{m-1}$ such that

$$P(|\mathbf{X}|> t\, x,\ \mathbf{X}/|\mathbf{X}| \in \bullet\ )/P(|\mathbf{X}|>t) \to_v x^{-\alpha} P(\ \theta \in \bullet\ )$$

($\to_v$ vague convergence on S^{m-1}) .

- $P(\ \theta \in \bullet)$ is called the spectral measure
- α is the index of $\mathbf{X}$.

Equivalence: There exist positive constants a_n and a measure μ,

$$nP(\mathbf{X}/\ a_n \in \bullet\ ) \to_v \mu(\bullet\ )$$

In this case, one can choose a_n and μ such that

$$\mu((x, \infty) \times B) = x^{-\alpha} P(\ \theta \in B\ )$$

Background Results—multivariate regular variation

Another equivalence?

MRV $\Leftrightarrow$ all linear combinations of $\mathbf{X}$ are regularly varying

i.e., if and only if

$$P(\mathbf{c}^T\mathbf{X} > t)/P(\mathbf{1}^T\mathbf{X} > t) \to w(\mathbf{c}),$$ exists for all real-valued $\mathbf{c}$,

in which case,

$$w(t\mathbf{c}) = t^{-\alpha} w(\mathbf{c}).$$

($\Rightarrow$) true

($\Leftarrow$) established by Basrak, Davis and Mikosch (2000) for α **not** an even integer—case of even integer is unknown.

Background Results—point process convergence

Theorem (Davis & Hsing `95, Davis & Mikosch `97). Let $\{\mathbf{X}_t\}$ be a stationary sequence of random vectors. Suppose

(i) finite dimensional distributions are jointly regularly varying (let $(\theta_{-k}, \ldots, \theta_k)$ be the vector in $S^{(2k+1)m-1}$ in the definition).

(ii) mixing condition $\mathcal{A}$ (a_n) or strong mixing.

(iii) $\lim_{k\to\infty} \limsup_{n\to\infty} P\left(\bigvee_{k\le|t|\le r_n} |\mathbf{X}_t| > a_n y \,\middle|\, |\mathbf{X}_0| > a_n y\right) = 0.$

Then

$$\gamma = \lim_{k\to\infty} E\left(|\theta_0^{(k)}|^\alpha - \bigvee_{j=1}^{k} |\theta_j^{(k)}|\right)_+ / E|\theta_0^{(k)}|^\alpha$$

exists. If $\gamma > 0$, then

$$N_n := \sum_{t=1}^{n} \varepsilon_{\mathbf{X}_t/a_n} \xrightarrow{d} N := \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varepsilon_{P_i \mathbf{Q}_{ij}},$$

Background Results—point process convergence(cont)

where

- (P_i) are points of a Poisson process on $(0,\infty)$ with intensity function $\nu(dy)=\gamma\alpha y^{-\alpha-1}dy$.
- $\sum_{j=1}^{\infty}\varepsilon_{\mathbf{Q}_{ij}}$, $i \geq 1$, are iid point process with distribution Q, and Q is the weak limit of

$$\lim_{k\to\infty} E\left(|\theta_0^{(k)}|^\alpha - \bigvee_{j=1}^{k}|\theta_j^{(k)}|\right)_+ I_\bullet\left(\sum_{|t|\le k}\varepsilon_{\theta_t^{(k)}}\right) / E\left(|\theta_0^{(k)}|^\alpha - \bigvee_{j=1}^{k}|\theta_j^{(k)}|\right)_+$$

Background Results—application to ACVF & ACF

Set-up: Let $\{X_t\}$ be a stationary sequence and set

$$\mathbf{X}_t=\mathbf{X}_t(m) = (X_t, \ldots, X_{t+m}).$$

Suppose $\mathbf{X}_t$ satisfies the conditions of previous theorem. Then

$$N_n := \sum_{t=1}^{n} \varepsilon_{\mathbf{X}_t/a_n} \xrightarrow{d} N := \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varepsilon_{P_i \mathbf{Q}_{ij}},$$

Sample ACVF and ACF:

$$\hat{\gamma}_X(h) = n^{-1} \sum_{t=1}^{n-h} X_t X_{t+h}, \; h \geq 0, \qquad \text{ACVF}$$

$$\hat{\rho}_X(h) = \hat{\gamma}_X(h) / \hat{\gamma}_X(h), \; h \geq 1, \qquad \text{ACF}$$

If $EX_0^2 < \infty$, then define $\gamma_X(h) = EX_0X_h$ and $\rho_X(h)=\gamma_X(h)/\gamma_X(0)$.

Background Results—application to ACVF & ACF

(i) If $\alpha \in (0,2)$, then

$$(n a_n^{-2} \hat{\gamma}_X(h))_{h=0,\ldots,m} \xrightarrow{d} (V_h)_{h=0,\ldots,m}$$

$$(\hat{\rho}_X(h))_{h=1,\ldots,m} \xrightarrow{d} (V_h / V_0)_{h=1,\ldots,m},$$

where

$$V_h = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P_i^2 Q_{ij}^{(0)} Q_{ij}^{(h)}, \quad h = 0, \ldots, m.$$

(ii) If $\alpha \in (2,4)$ + additional condition, then

$$\left(n a_n^{-2} (\hat{\gamma}_X(h) - \gamma_X(h))\right)_{h=0,\ldots,m} \xrightarrow{d} \left(V_h\right)_{h=0,\ldots,m}$$

$$\left(n a_n^{-2} (\hat{\rho}_X(h) - \rho_X(h))\right)_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1}(0) \left(V_h - \rho_X(h) V_0\right)_{h=1,\ldots,m}.$$

Applications—stochastic recurrence equations

$$\mathbf{Y}_t = \mathbf{A}_t \mathbf{Y}_{t-1} + \mathbf{B}_t, \quad (\mathbf{A}_t, \mathbf{B}_t) \sim \text{IID},$$

$\mathbf{A}_t$ $d \times d$ random matrices, $\mathbf{B}_t$ random d-vectors

<u>Examples</u>

ARCH(1): $X_t = (\alpha_0 + \alpha_1 X^2_{t-1})^{1/2} Z_t$, $\{Z_t\} \sim$ IID. Then the squares follow an SRE with $Y_t = X_t^2$, $A_t = \alpha_1 Z_t^2$, $B_t = \alpha_0 Z_t^2$.

GARCH(2,1): $X_t = \sigma_t Z_t$, $\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \alpha_2 X_{t-2}^2 + \beta_1 \sigma_{t-1}^2$.

Then $\mathbf{Y}_t = (X_t^2, X_{t-1}^2, \sigma_t^2)'$ follows the SRE given by

$$\begin{bmatrix} X_t^2 \\ X_{t-1}^2 \\ \sigma_t^2 \end{bmatrix} = \begin{bmatrix} \alpha_1 Z_t^2 & \alpha_2 Z_t^2 & \beta_1 Z_t^2 \\ 1 & 0 & 0 \\ \alpha_1 & \alpha_2 & \beta_1 \end{bmatrix} \begin{bmatrix} X_{t-1}^2 \\ X_{t-2}^2 \\ \sigma_{t-1}^2 \end{bmatrix} + \begin{bmatrix} \alpha_0 Z_t^2 \\ 0 \\ 0 \end{bmatrix}$$

Stochastic Recurrence Equations (cont)

Examples (tricks)

GARCH(1,1): $X_t = \sigma_t Z_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2$.

Although this process does not have a 1-dimensional SRE representation, the process σ_t^2 does. Iterating, we have

$$\begin{aligned}\sigma_t^2 &= \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 Z_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \\ &= (\alpha_1 Z_{t-1}^2 + \beta_1)\sigma_{t-1}^2 + \alpha_0.\end{aligned}$$

Bilinear(1): $X_t = aX_{t-1} + bX_{t-1}Z_{t-1} + Z_t, \quad \{Z_t\}\sim\text{IID}$

$$= Y_{t-1} + Z_t\,,$$

$$Y_t = A_t Y_{t-1} + B_t \quad A_t = a + bZ_t, \;\; B_t = A_t Z_t$$

Stochastic Recurrence Equations (cont)

$$\mathbf{Y}_t = \mathbf{A}_t \mathbf{Y}_{t-1} + \mathbf{B}_t, \qquad (\mathbf{A}_t, \mathbf{B}_t) \sim \text{IID}$$

Existence of stationary solution

- $E \ln^+ \| \mathbf{A}_1 \| < \infty$
- $E \ln^+ \| \mathbf{B}_1 \| < \infty$
- $\inf n^{-1} E \ln \| \mathbf{A}_1 \dots \mathbf{A}_n \| =: \gamma < 0$ (γ –top Lyapunov exponent)

Ex. (d=1) $E \ln |\mathbf{A}_1| < 0.$

Strong mixing

If $E \|\mathbf{A}_1\|^\varepsilon < \infty$, $E |\mathbf{B}_1|^\varepsilon < \infty$ for some $\boldsymbol{\varepsilon} > 0$, then the SRE $(\mathbf{Y}_t)$ is *geometrically ergodic* $\Rightarrow$ *strong mixing* with geometric rate (Meyn and Tweedie `93).

Stochastic Recurrence Equations (cont)

Regular variation of the marginal distribution (Kesten)

Assume **A** and **B** have non-negative entries and

- $E\,\|\mathbf{A}_1\|^{\varepsilon} < 1$ for some $\varepsilon > 0$
- $\mathbf{A}_1$ has no zero rows a.s.
- W.P. 1, $\{\ln \rho(\mathbf{A}_1 \ldots \mathbf{A}_n)$: is dense in $\mathbf{R}$ for some n, $\mathbf{A}_1 \ldots \mathbf{A}_n > 0\}$
- There exists a $\kappa_0 > 0$ such that $E\|A\|^{\kappa_0} \ln^+\|A\| < \infty$ and

$$E\left(\min_{i=1,\ldots,d} \sum_{j=1}^{d} A_{ij}\right)^{\kappa_0} \geq d^{\kappa_0/2}$$

Then there exists a $\kappa_1 \in (0, \kappa_0]$ such that **all** linear combinations of **Y** are regularly varying with index κ_1. (Also need $E\,|B|^{\kappa_1} < \infty$.)

Application to GARCH

Proposition: Let $(\mathbf{Y}_t)$ be the soln to the SRE based on the *squares* of a GARCH model. Assume

- Top Lyapunov exponent $\gamma < 0$. (See Bougerol and Picard`92)
- Z has a positive density on $(-\infty, \infty)$ with all moments finite or $E|Z|^h = \infty$, for all $h \geq h_0$ and $E|Z|^h < \infty$ for all $h < h_0$.
- Not all the GARCH parameters vanish.

Then $(\mathbf{Y}_t)$ is *strongly mixing* with geometric rate and all finite dimensional distributions are *multivariate regularly varying* with index κ_1.

Corollary: The corresponding GARCH process is strongly mixing and has all finite dimensional distributions that are MRV with index $\kappa = 2\kappa_1$.

Application to GARCH (cont)

Remarks:

1. Kesten's result applied to an iterate of $\mathbf{Y}_t$, i.e., $\mathbf{Y}_{tm} = \tilde{\mathbf{A}}_t \mathbf{Y}_{(t-1)m} + \tilde{\mathbf{B}}_t$

2. Determination of κ is difficult. Explicit expressions only known in two(?) cases.

- ARCH(1): $E|\alpha_1 Z^2|^{\kappa/2} = 1$.

α_1	.312	.577	1.00	1.57
κ	8.00	4.00	2.00	1.00

- GARCH(1,1): $E|\alpha_1 Z^2 + \beta_1|^{\kappa/2} = 1$ (Mikosch and Starica)
- For IGARCH ($\alpha_1 + \beta_1 = 1$), then $\kappa = 2 \Rightarrow$ infinite variance.
- Can estimate κ empirically by replacing expectations with sample moments.

Summary for GARCH(p,q)

$\kappa \in (0,2)$:

$$(\hat{\rho}_X(h))_{h=1,\ldots,m} \xrightarrow{d} (V_h / V_0)_{h=1,\ldots,m},$$

$\kappa \in (2,4)$:

$$\left(n^{1-2/\kappa}\hat{\rho}_X(h)\right)_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1}(0)\left(V_h\right)_{h=1,\ldots,m}.$$

$\kappa \in (4,\infty)$:

$$\left(n^{1/2}\hat{\rho}_X(h)\right)_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1}(0)\left(G_h\right)_{h=1,\ldots,m}.$$

Remark: Similar results hold for the sample ACF based on $|X_t|$ and X_t^2.

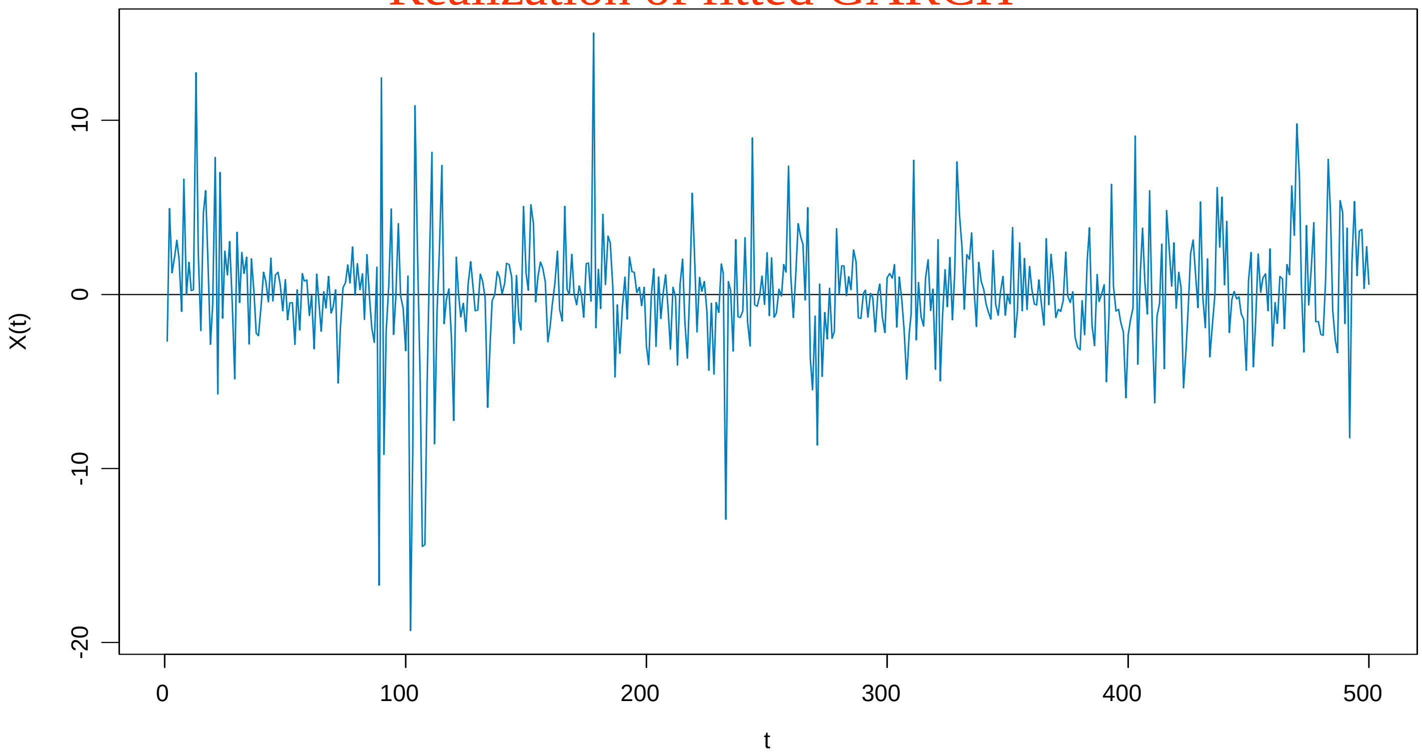
Realization of GARCH Process

Fitted GARCH(1,1) model for NZ-USA exchange:

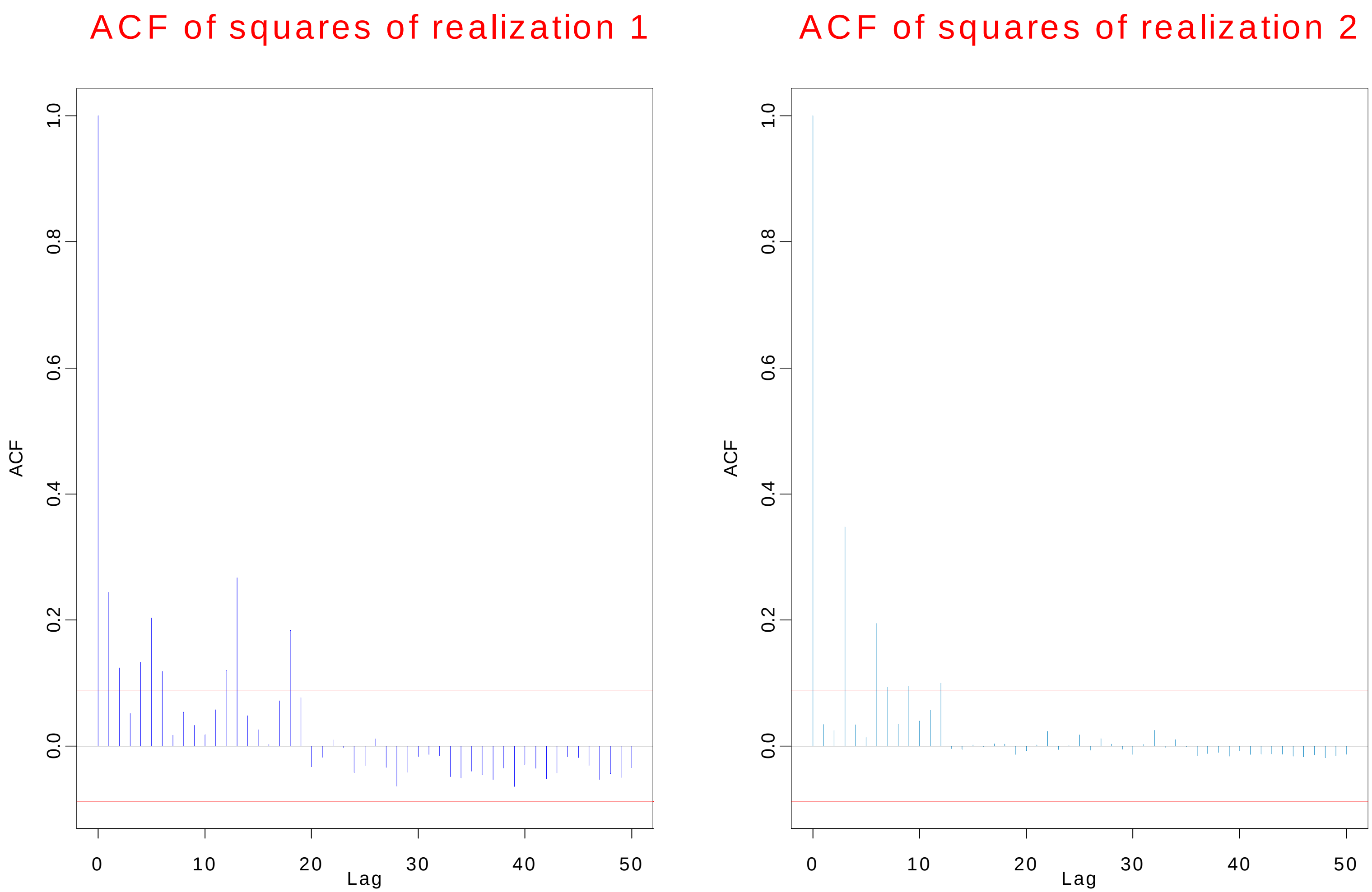
$$X_t = \sigma_t Z_t, \quad \sigma_t^2 = (6.70)10^{-7} + .1519X_{t-1}^2 + .772\sigma_{t-1}^2$$

(Z_t) ~ IID t-distr with 5 df. κ is approximately 3.8

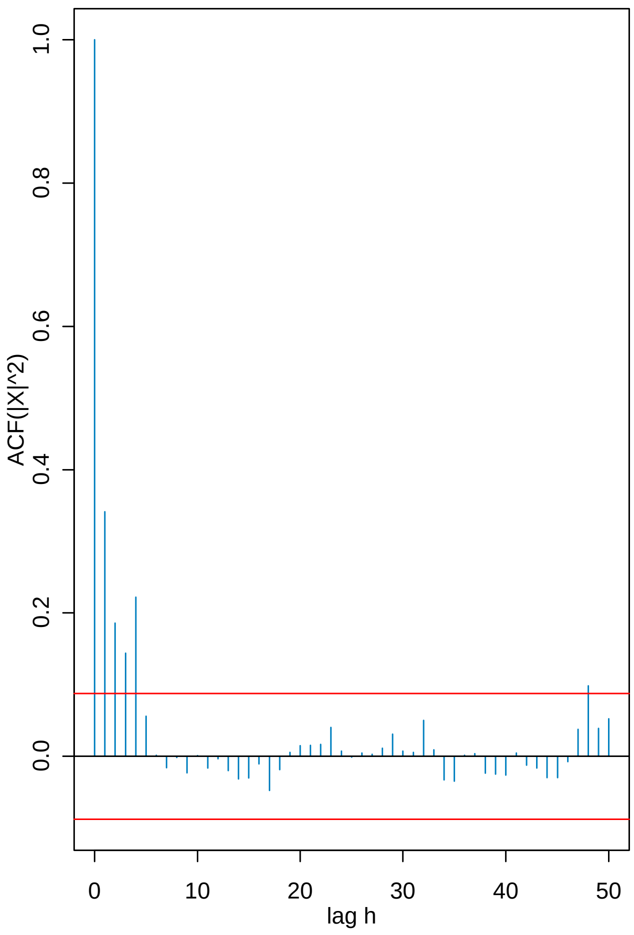
Realization of fitted GARCH

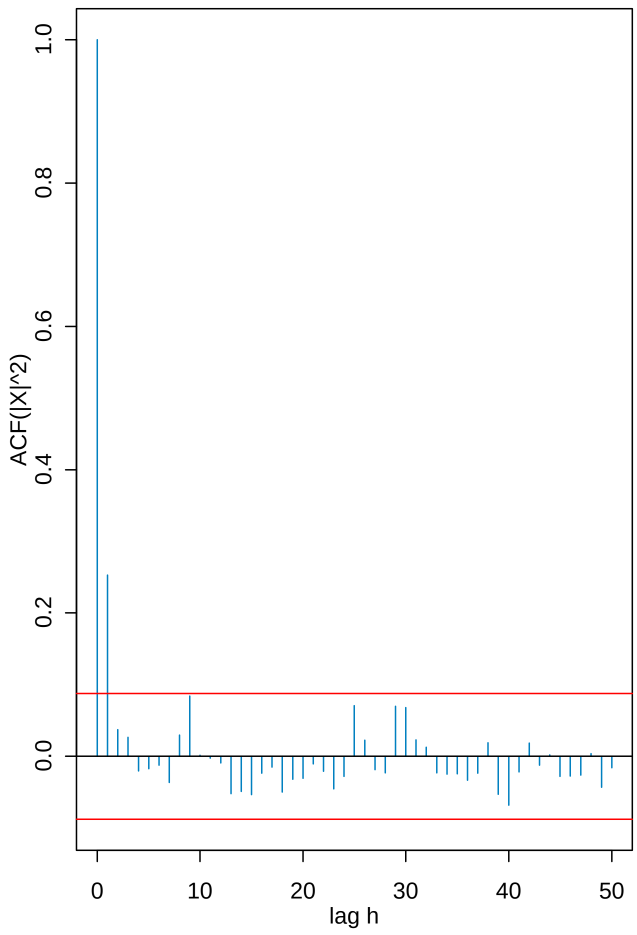


ACF of Fitted GARCH(1,1) Process

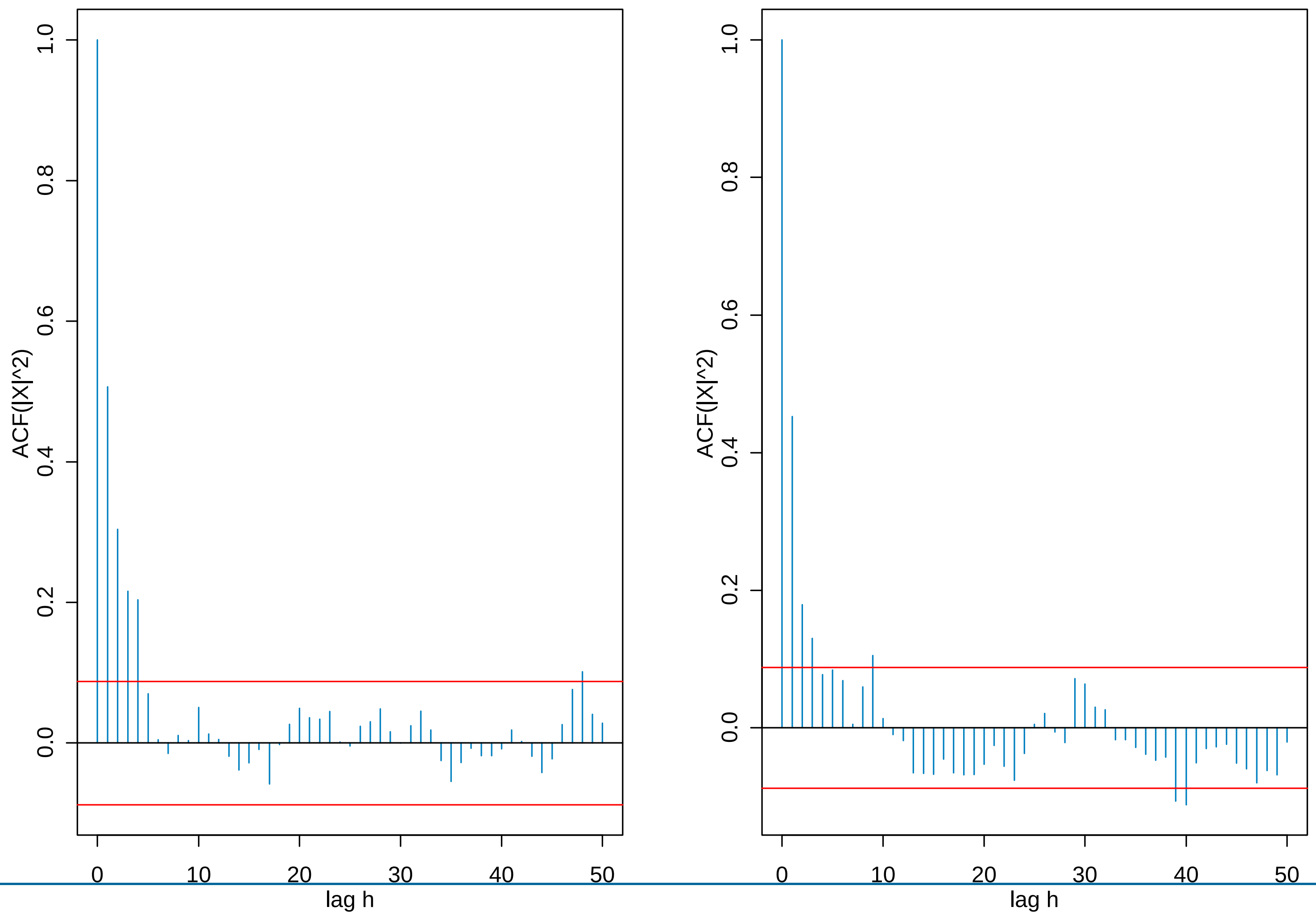


ACF of 2 realizations of an (ARCH)^2: $X_t=(.001+.7\,X_{t-1})^{1/2}\,Z_t$





ACF of 2 realizations of an |ARCH|: $X_t=(.001+ X_{t-1})^{1/2} Z_t$



Stochastic Volatility Models

SVM: $X_t = \sigma_t Z_t$

- $(Z_t) \sim$ IID with mean 0 (if it exists)
- (σ_t) is a stationary process (2 log σ_t is a linear process) given by

$$\boxed{\log \sigma_t^2 = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}, \quad \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty, (\varepsilon_t) \sim \text{IID N}(0, \sigma^2)}$$

Heavy tails: Assume Z_t has Pareto tails with index α, i.e.,

$$P(|Z_t| > z) \sim C z^{-\alpha} \Rightarrow P(|X_t| > z) \sim C E\sigma^{\alpha} z^{-\alpha}.$$

Then if $\alpha \in (0,2)$,

$$(n/\ln n)^{1/\alpha} \hat{\rho}_X(h) \xrightarrow{d} \frac{\|\sigma_1 \sigma_{h+1}\|_\alpha}{\|\sigma_1\|_\alpha^2} \frac{S_h}{S_0}.$$

Stochastic Volatility Models (cont)

Other powers:

1. Absolute values: $\alpha \in (1,2)$,

$$E|X_t| = E|\sigma_t|E|Z_t|, \;\; E|X_t X_{t+h}| = (E|\sigma_t \sigma_{t+h}|)(E|Z_t|E|Z_{t+h}|)$$

$$\mathrm{Cov}(|X_t|, |X_{t+h}|) = \mathrm{Cov}(\sigma_t, \sigma_{t+h})(E|Z|)^2$$

$$\mathrm{Cor}(|X_t|, |X_{t+h}|) = \mathrm{Cor}(\sigma_t, \sigma_{t+h})(E|Z|)^2 / EZ^2$$

$$= 0 \;(?).$$

We obtain

$$n(n \ln n)^{-1/\alpha}\left(\hat{\gamma}_{|X|}(h) - \gamma_{|X|}(h)\right) \xrightarrow{d} \|\sigma_1 \sigma_{h+1}\|_\alpha S_h$$

and

$$(n/\ln n)^{1/\alpha} \hat{\rho}_{|X|}(h) \xrightarrow{d} \frac{\|\sigma_1 \sigma_{h+1}\|_\alpha}{\|\sigma_1\|_\alpha^2} \frac{S_h}{S_0}.$$

Stochastic Volatility Models (cont)

2. Higher order: $\alpha \in (0,2)$

The squares are again a SV process and the results of the previous proposition apply. Namely,

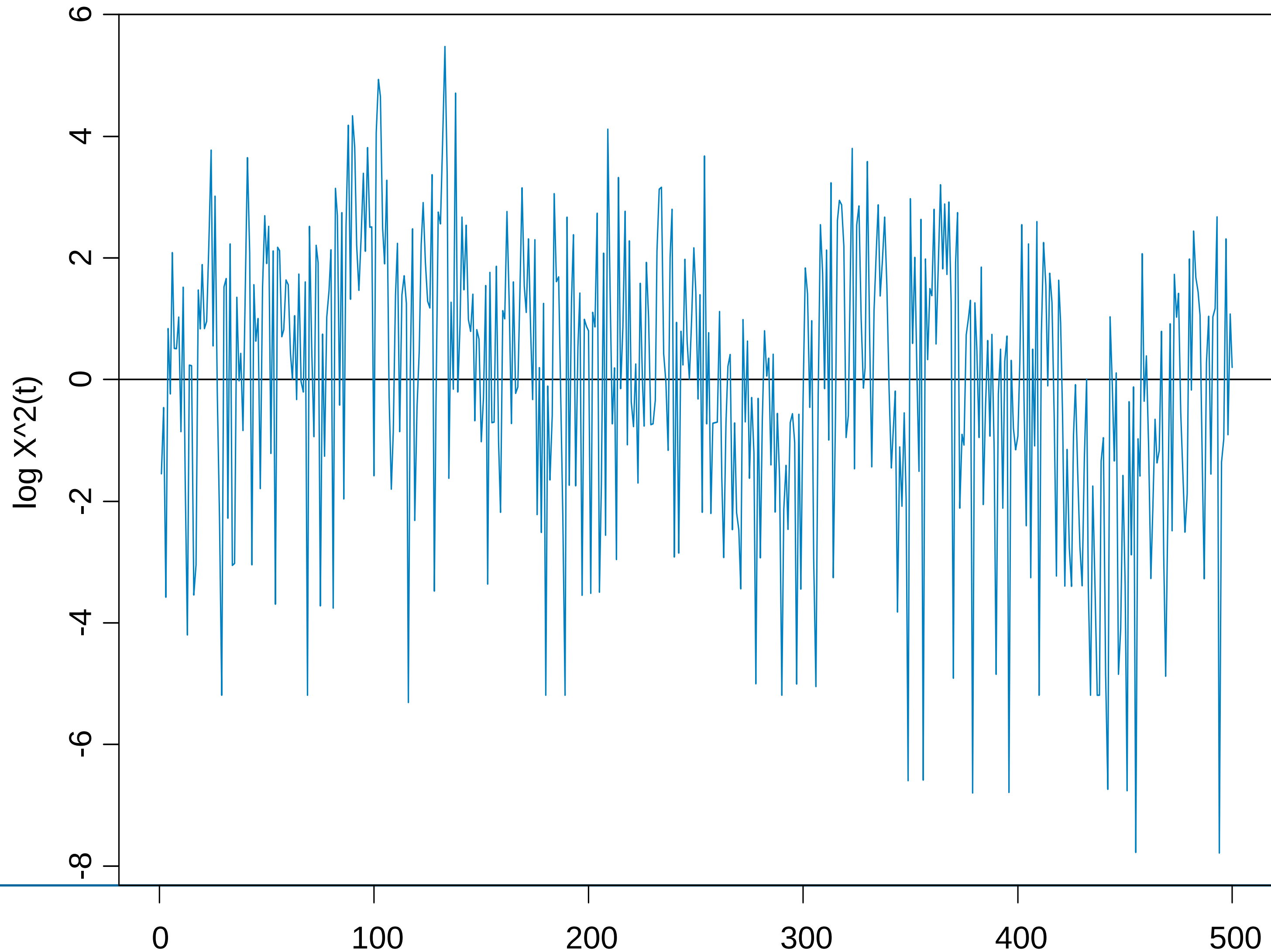
$$(n/\ln n)^{2/\alpha}\hat{\rho}_{X^2}(h) \xrightarrow{d} \frac{\left\|\sigma_1^2\sigma_{h+1}^2\right\|_{\alpha/2}}{\left\|\sigma_1^2\right\|_{\alpha/2}^2}\frac{S_h}{S_0}.$$

In particular,

$$\hat{\rho}_{X^2}(h) \xrightarrow{P} 0.$$

Stochastic Volatility Models (cont)

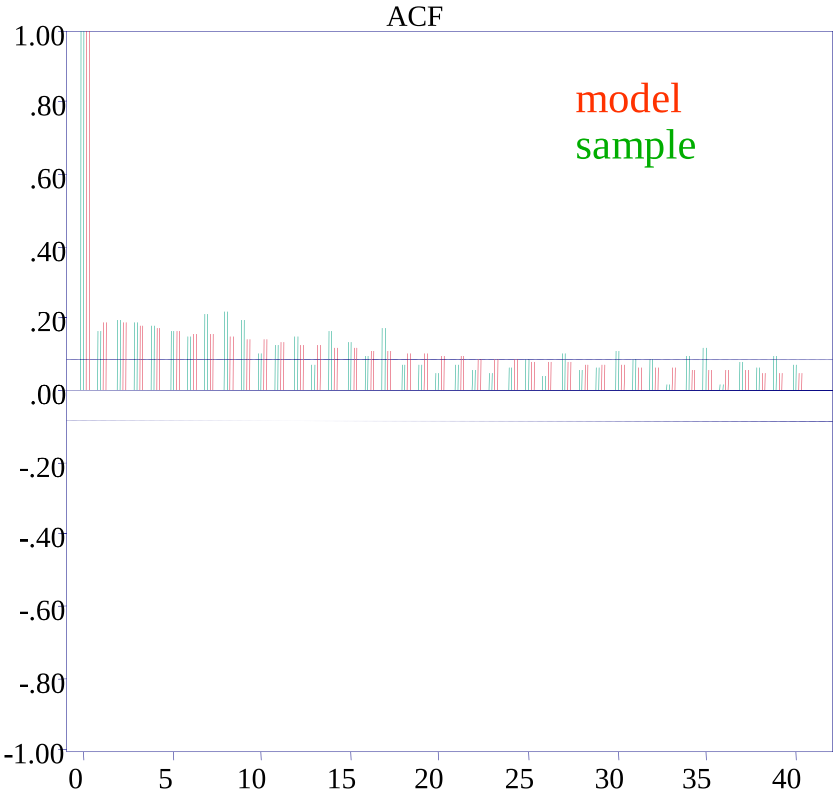
$(\log X^2)$ - mean for NZ-USA exchange rates

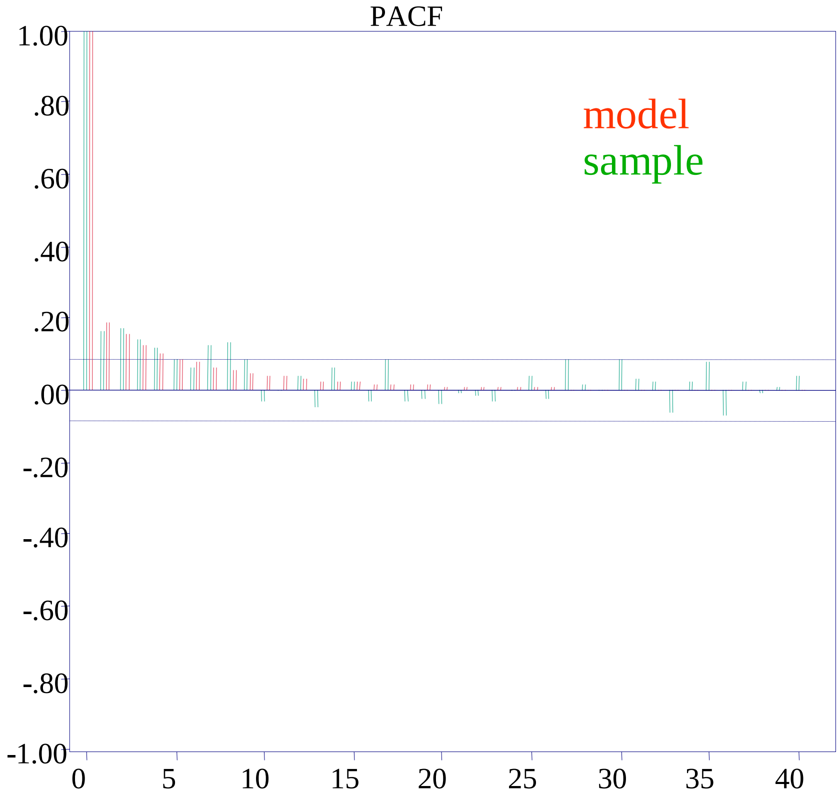


Stochastic Volatility Models (cont)

ACF/PACF for (log X^2) suggests ARMA (1,1) model:

$$\mu = -11.5403, \quad Y_t = .9646 Y_{t-1} + \varepsilon_t - .8709\, \varepsilon_{t-1}, \quad (\varepsilon_t) \sim WN(0, 4.6653)$$





Stochastic Volatility Models (cont)

The ARMA (1,1) model for log X^2 leads to the SV model

$$X_t = \sigma_t Z_t$$

with

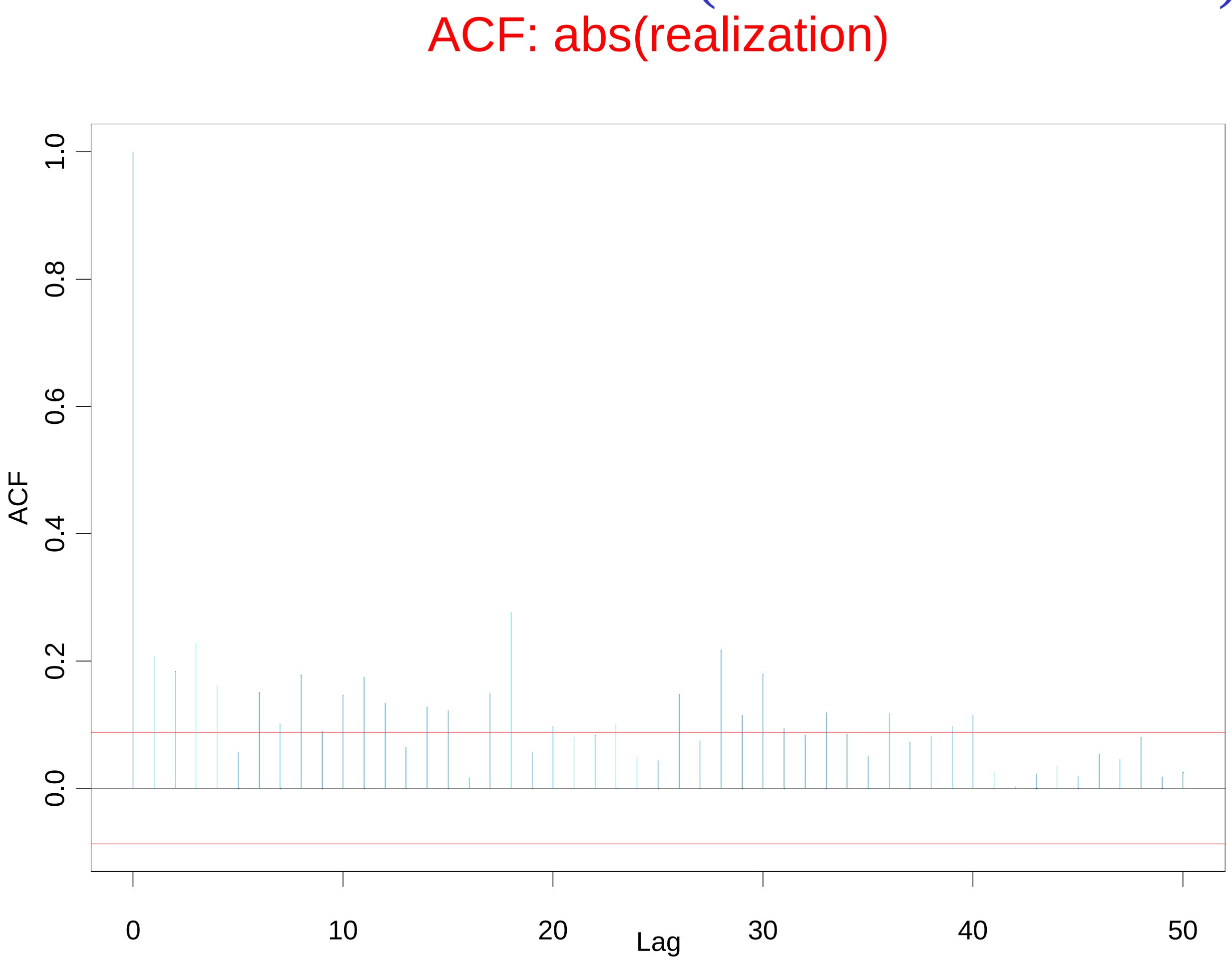
$$2 \ln \sigma_t = -11.5403 + v_t + \varepsilon_t$$

$$v_t = .9646\, v_{t-1} + \gamma_t\,, \quad (\gamma_t) \sim WN(0, .07253)$$

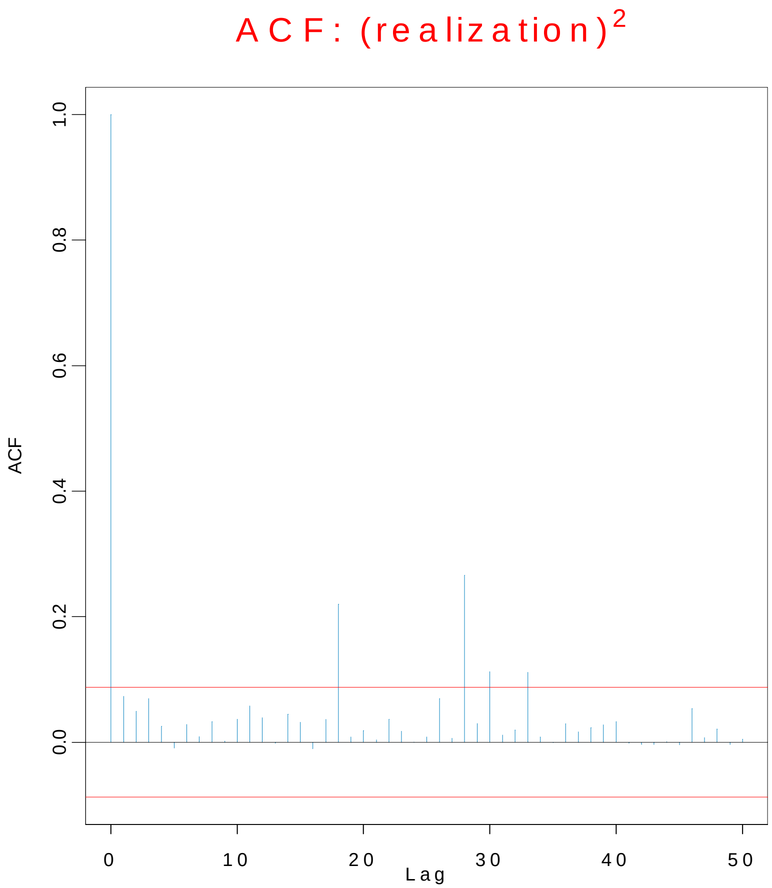
$$(\varepsilon_t) \sim WN(0, 4.2432).$$

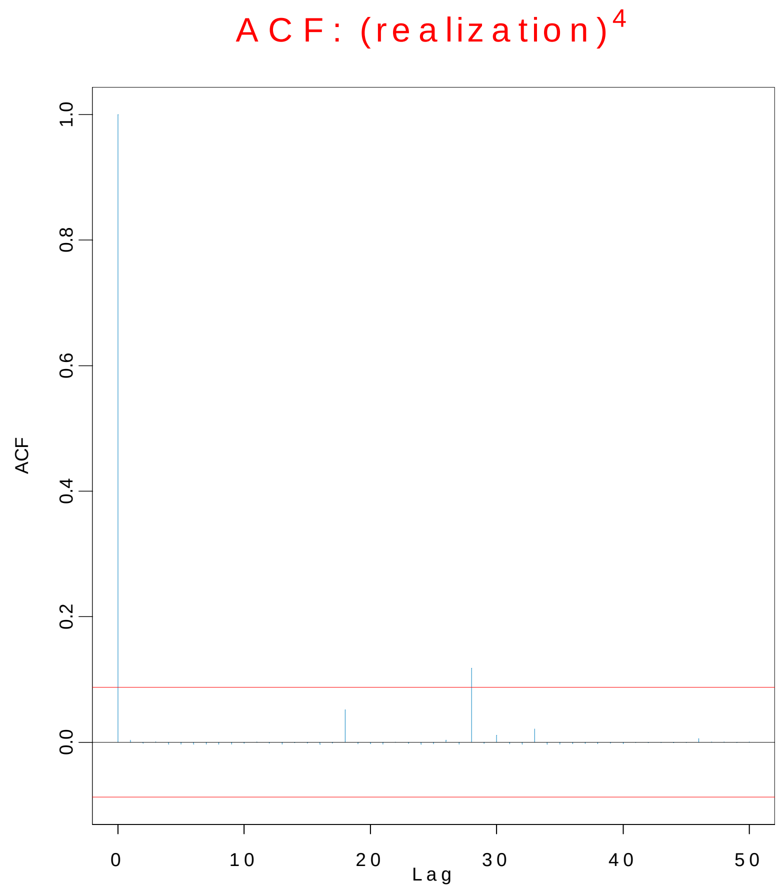
Stochastic Volatility Models (cont)

Simulation of SVM model: Took ε_t to be distributed according to log of a t random variable with 3 df (suitable normalized).

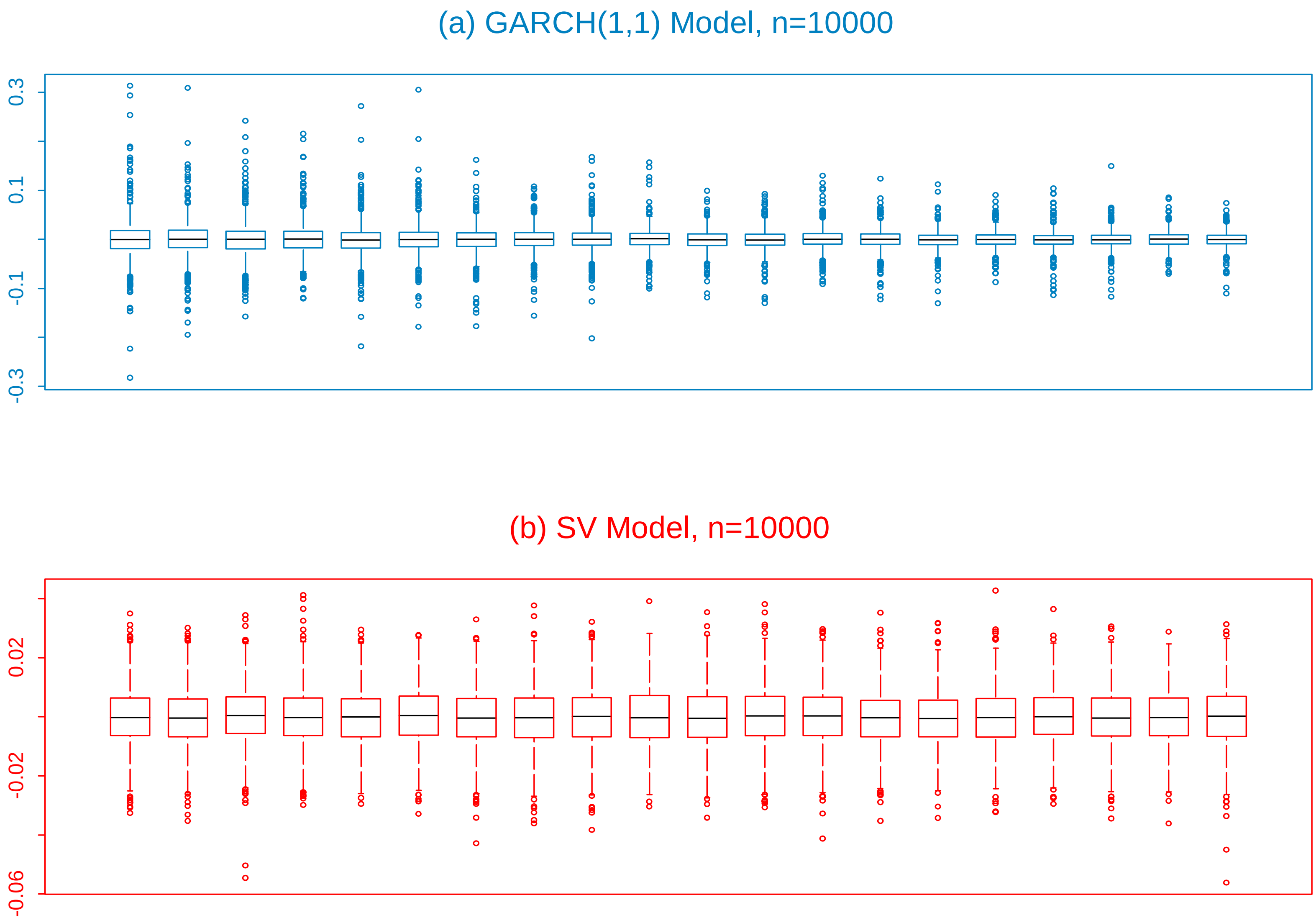


Stochastic Volatility Models (cont)

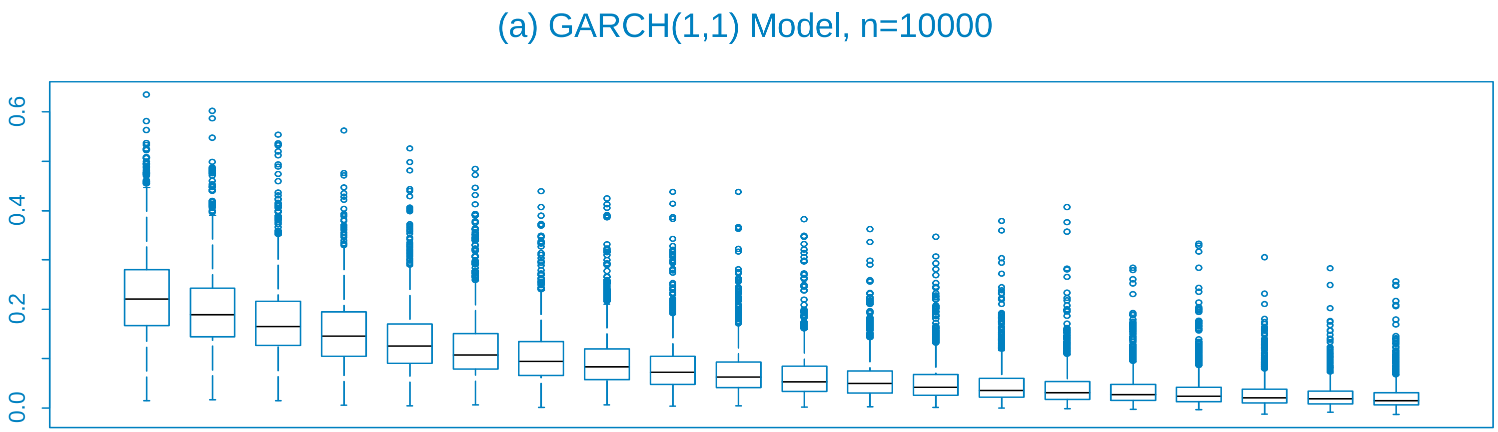


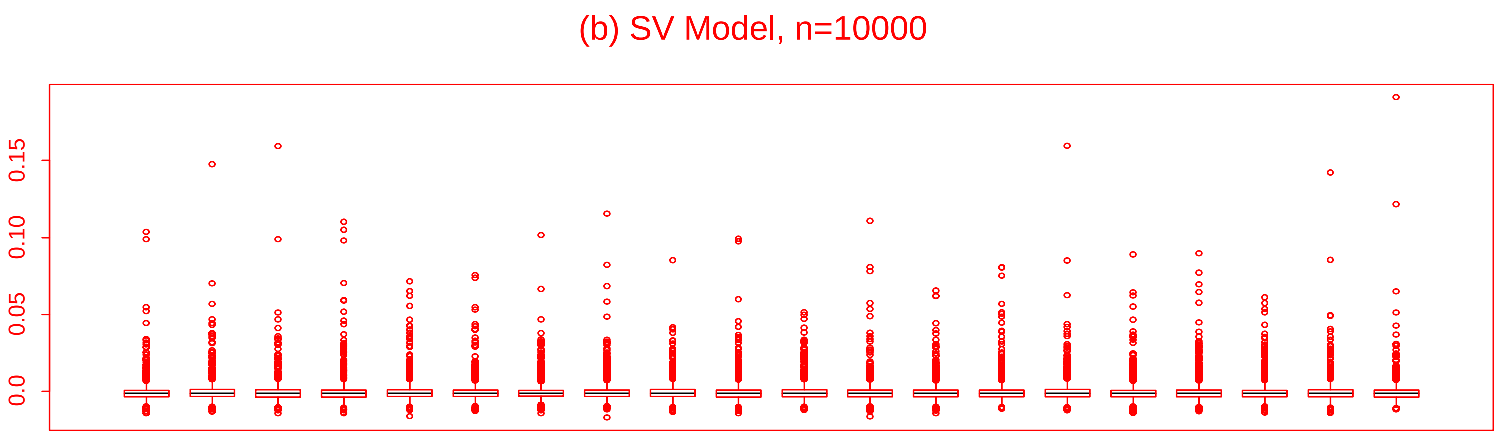


Sample ACF for GARCH and SV Models (1000 reps)



Sample ACF for Squares of GARCH and SV (1000 reps)





Sample ACF for Squares of GARCH and SV (1000 reps)