Maximum Likelihood Estimation for Allpass Models

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http://www.stat.colostate.edu/~rdavis/talks/IMA01.pdf

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☞ Introduction

- properties of financial time series
- motivating example
- all-pass models and their properties

☞ Estimation

- likelihood approximation
- MLE and LAD
- asymptotic results
- order selection

☞ Empirical results

- simulation
- NZ/USA exchange rates

☞ Noninvertible MA processes

- preliminaries
- a two-step estimation procedure
- Microsoft trading volume

☞ Summary

Financial Time Series

☞ Log returns, $X_t = 100*(\ln(P_t) - \ln(P_{t-1}))$, of financial assets often exhibit:

- heavy-tailed marginal distributions

  $P(|X_1| > x) \sim C\, x^{-\alpha}, \quad 0 < \alpha < 4.$

- lack of serial correlation

  $\hat{\rho}_X(h)$ near 0 for all lags $h > 0$ (MGD sequence)

- $|X_t|$ and X_t^2 have slowly decaying autocorrelations

  $\hat{\rho}_{|X|}(h)$ and $\hat{\rho}_{X^2}(h)$ converge to 0 *slowly* as $h \to \infty$

- process exhibits 'stochastic volatility'

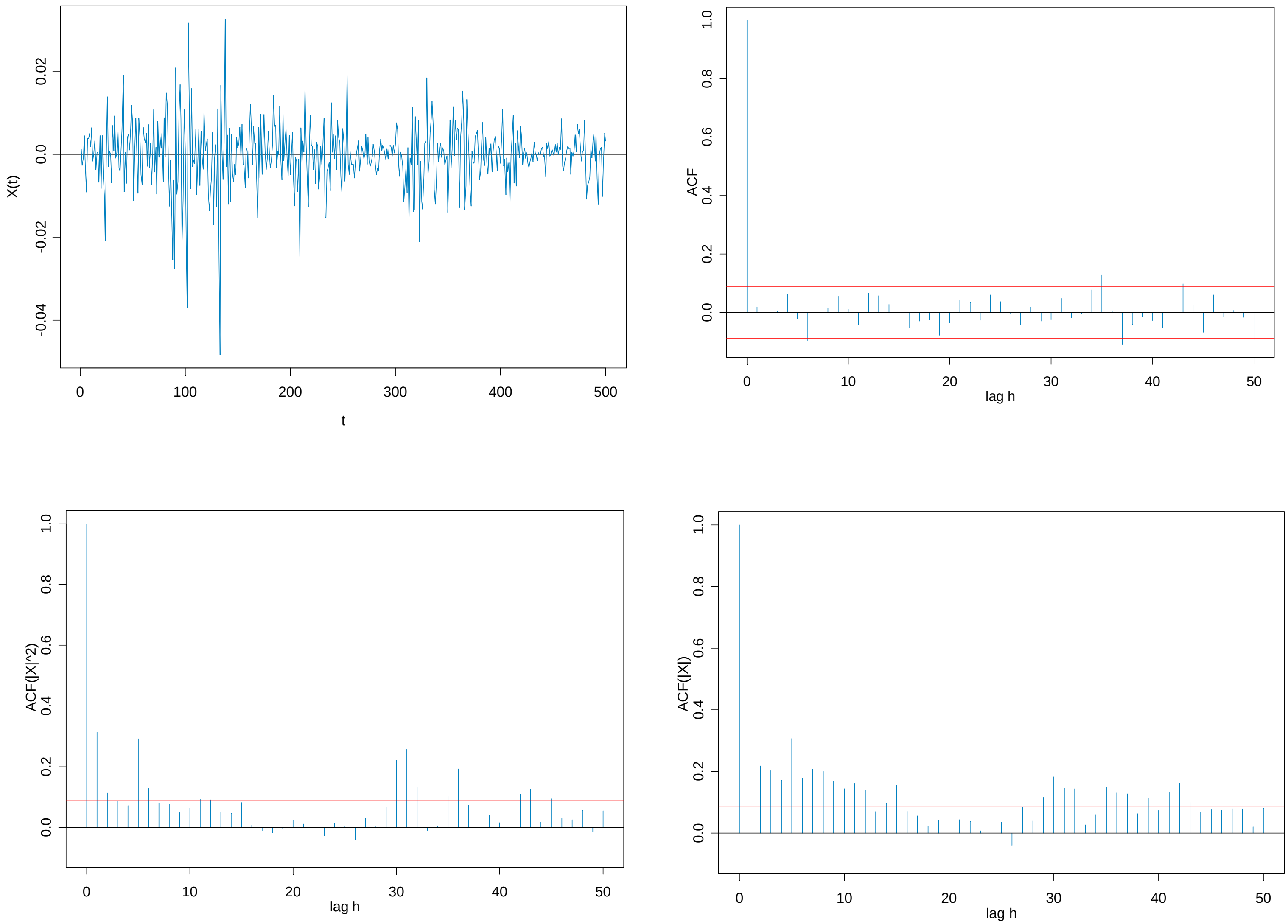
☞ Nonlinear models $X_t = \sigma_t Z_t$, $\{Z_t\} \sim \text{IID}(0,1)$

- ARCH and its variants (Engle `82; Bollerslev, Chou, and Kroner 1992)

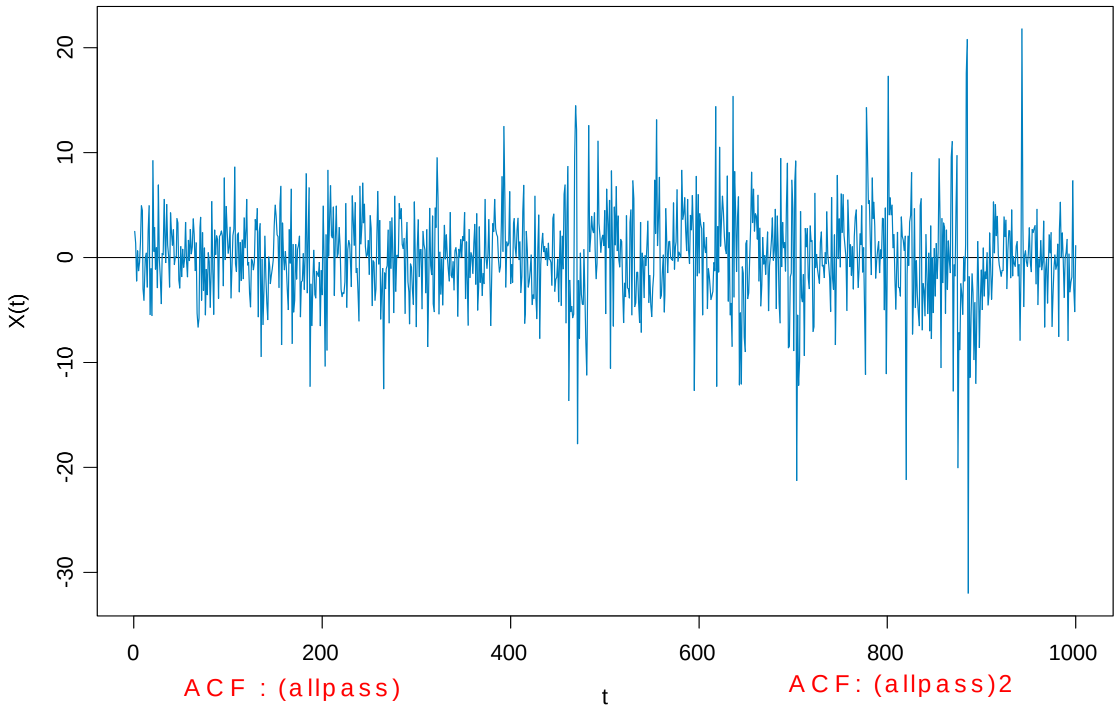
- Stochastic volatility (Clark 1973; Taylor 1986)

Motivating example

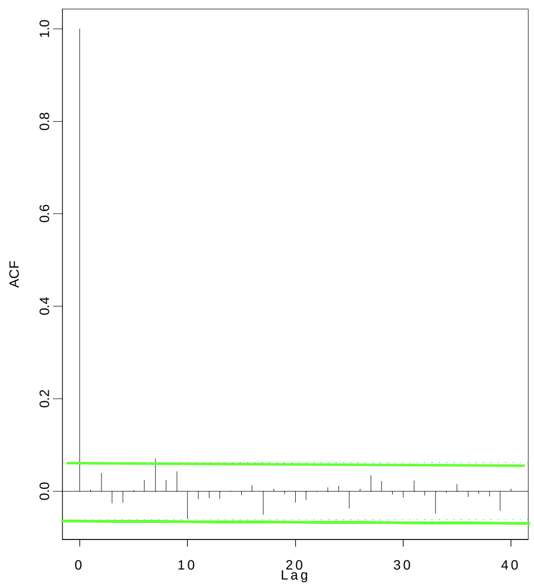
500-daily log-returns of NZ/US exchange rate



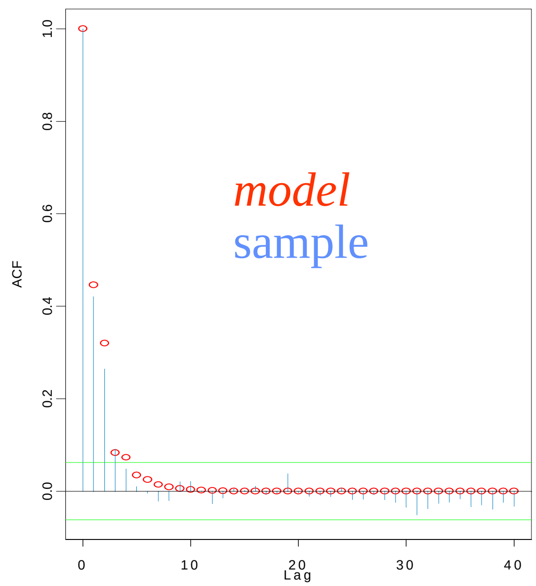
All-pass model of order 2 (t3 noise)



A C F : (allpass)



A C F: (allpass)2



All-pass Models

Causal AR polynomial: $\phi(z)=1-\phi_1 z - \cdots - \phi_p z^p$, $\phi(z) \neq 0$ for $|z|\leq 1$.

Define MA polynomial:

$$\theta(z) = -z^p \phi(z^{-1})/\phi_p = -(z^p - \phi_1 z^{p-1} - \cdots - \phi_p)/\phi_p$$

$$\neq 0 \text{ for } |z|\geq 1$$ (MA polynomial is non-invertible).

Model for data $\{X_t\}$: $\phi(B)X_t = \theta(B) Z_t$, $\{Z_t\} \sim$ IID (non-Gaussian)

$$B^k X_t = X_{t-k}$$

Examples:

All-pass(1): $X_t - \phi X_{t-1} = Z_t - \phi^{-1} Z_{t-1}$, $|\phi| < 1$.

All-pass(2): $X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t + \phi_1/\phi_2 Z_{t-1} - 1/\phi_2 Z_{t-2}$

Properties:

- causal, non-invertible ARMA with MA representation

$$X_t = \frac{B^p \phi(B^{-1})}{-\phi_p \phi(B)} Z_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

- uncorrelated (flat spectrum)

$$f_X(\omega) = \frac{\left|e^{-ip\omega}\right|^2 \left|\phi(e^{i\omega})\right|^2}{\phi_p^2 \left|\phi(e^{-i\omega})\right|^2} \frac{\sigma^2}{2\pi} = \frac{\sigma^2}{\phi_p^2 2\pi}$$

- zero mean
- data are dependent if noise is non-Gaussian (e.g. Breidt & Davis 1991).
- squares and absolute values are correlated.
- X_t is heavy-tailed if noise is heavy-tailed.

Estimation for All-Pass Models

☞ Second-order moment techniques do not work

- least squares
- Gaussian likelihood

☞ Higher-order cumulant methods

- Giannakis and Swami (1990)
- Chi and Kung (1995)

☞ Non-Gaussian likelihood methods

- likelihood approximation
- quasi-likelihood
- least absolute deviations
- minimum dispersion

Approximating the likelihood

Data: $(X_1, \ldots, X_n)$

Model: $$X_t = \phi_{01}X_{t-1} + \cdots + \phi_{0p}X_{t-p} - (Z_{t-p} - \phi_{01}Z_{t-p+1} - \cdots - \phi_{0p}Z_t)/\phi_{0r}$$

where ϕ_{0r} is the last non-zero coefficient among the ϕ_{0j}'s.

Noise: $z_{t-p} = \phi_{01}z_{t-p+1} + \cdots + \phi_{0p}z_t - (X_t - \phi_{01}X_{t-1} - \cdots - \phi_{0p}X_{t-p})$,

where $z_t = Z_t / \phi_{0r}$.

More generally define,

$$z_{t-p}(\phi) = \begin{cases} 0, & \text{if } t = n+p,\ldots,n+1, \\ \phi_1 z_{t-p+1}(\phi) + \cdots + \phi_p z_t(\phi) - \phi(B)X_t, & \text{if } t = n,\ldots,p+1. \end{cases}$$

Note: $z_t(\phi_0)$ is a close approximation to z_t (initialization error)

Assume that Z_t has density function f_σ and consider the vector

$$\mathbf{z} = (\underbrace{X_{1-p},\dots,X_0, z_{1-p}(\phi),\dots,z_0(\phi)}, \underbrace{z_1(\phi),\dots,z_{n-p+1}(\phi),\dots,z_n(\phi)})'$$

independent pieces

Joint density of $\mathbf{z}$:

$$h(\mathbf{z}) = h_1(X_{1-p},\dots,X_0, z_{1-p}(\phi),\dots,z_0(\phi))$$

$$\bullet\left(\prod_{t=1}^{n-p} f_\sigma(\phi_q z_t(\phi))\,|\,\phi_q\,|\right) h_2(z_{n-p+1}(\phi),\dots,z_n(\phi)),$$

and hence the joint density of the data can be approximated by

$$h(\mathbf{x}) = \left(\prod_{t=1}^{n-p} f_\sigma(\phi_q z_t(\phi))\,|\,\phi_q\,|\right)$$

where $q=\max\{0 \le j \le p: \phi_j \neq 0\}$.

Log-likelihood:

$$L(\phi,\sigma) = -(n-p)\ln(\sigma / |\phi_q|) + \sum_{t=1}^{n-p} \ln f(\sigma^{-1}\phi_q z_t(\phi))$$

where $f_\sigma(z) = \sigma^{-1} f(z/\sigma)$.

Least absolute deviations: choose Laplace density

$$f(z) = \frac{1}{\sqrt{2}} \exp(-\sqrt{2}\,|z|)$$

and log-likelihood becomes

$$\text{constant} - (n-p)\ln\kappa - \sum_{t=1}^{n-p} \sqrt{2}\,|z_t(\phi)|/\kappa, \quad \kappa = \sigma/|\phi_q|$$

Concentrated Laplacian likelihood

$$l(\phi) = \text{constant} - (n-p)\ln \sum_{t=1}^{n-p} |z_t(\phi)|$$

Maximizing $l(\phi)$ is equivalent to minimizing the absolute deviations

$$m_{\mathrm{n}}(\phi) = \sum_{t=1}^{n-p} |z_t(\phi)|.$$

Assumptions

- ☞ Assume $\{Z_t\}$ iid $f_\sigma(z)=\sigma^{-1}f(\sigma^{-1}z)$ with
  - σ a scale parameter
  - mean 0, variance σ^2
- ☞ For f known, use maximum likelihood
  - further smoothness assumptions (integrability, symmetry, etc.) on f
  - Fisher information: $\tilde{I} = \sigma^{-2}\int (f'(z))^2 / f(z)dz$
- ☞ For f unknown, use quasi-likelihood
- ☞ Least absolute deviations
  - assume f has median 0
  - assume f continuous in neighborhood of 0
  - act as if f = Laplace to get criterion function

Results

☞ Let $\gamma(h)$ = ACVF of AR model with AR poly $\phi_0(.)$ and

$$\Gamma_p = [\gamma(j-k)]_{j,k=1}^{p}$$

☞ Maximum likelihood:

$$\sqrt{n}(\hat{\phi}_{\mathrm{MLE}} - \phi_0) \xrightarrow{D} N\left(0, \frac{1}{2(\sigma^2 \tilde{I} - 1)} \sigma^2 \Gamma_p^{-1}\right)$$

☞ Least absolute deviations:

$$\sqrt{n}(\hat{\phi}_{\mathrm{LAD}} - \phi_0) \xrightarrow{D} N\left(0, \frac{\mathrm{Var}(|Z_1|)}{2\sigma^4 f_\sigma^2(0)} \sigma^2 \Gamma_p^{-1}\right)$$

Further comments on MLE

Let $\alpha=(\phi_1, \ldots, \phi_p, \sigma/|\phi_1|, \beta_1, \ldots, \beta_q)$, where $\beta_1, \ldots, \beta_q$ are the parameters of pdf f.

Set

☞ $$\hat{\mathrm{I}} = \sigma_0^{-2}\int (f'(z;\beta_0))^2 / f(z;\beta_0)dz$$

☞ $$\hat{\mathrm{K}} = \alpha_{0,p+1}^{-2}\left\{\int z^2 (f'(z;\beta_0))^2 / f(z;\beta_0)dz - 1\right\}$$

☞ $$\mathrm{L} = -\alpha_{0,p+1}^{-1}\int z\frac{f'(z;\beta_0)}{f(z;\beta_0)}\frac{\partial f(z;\beta_0)}{\partial \beta_0}dz$$

☞ $$\mathrm{I}_f(\beta_0) = \int \frac{1}{f(z;\beta_0)}\frac{\partial f(z;\beta_0)}{\partial \beta_0}\frac{\partial f^T(z;\beta_0)}{\partial \beta_0}dz$$ (Fisher Information)

Under smoothness conditions on f wrt $\beta_1, \ldots, \beta_q$ we have

$$\sqrt{n}(\hat{\alpha}_{\mathrm{MLE}} - \alpha_0) \xrightarrow{D} N(0, \Sigma^{-1}),$$

where

$$\Sigma^{-1} = \begin{bmatrix} \frac{1}{2(\sigma_0^2 \hat{I} - 1)} \Gamma_p^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\hat{K} - L' I_f^{-1} L)^{-1} & -\hat{K}^{-1} L' (I_f - L\hat{K}^{-1} L')^{-1} \\ \mathbf{0} & -(I_f - L\hat{K}^{-1} L')^{-1} L\hat{K}^{-1} & (I_f - L\hat{K}^{-1} L')^{-1} \end{bmatrix}$$

Note: $\hat{\phi}_{\mathrm{MLE}}$ is asymptotically independent of $\hat{\alpha}_{\mathrm{p+1,MLE}}$ and $\hat{\beta}_{\mathrm{MLE}}$

Identifiability in LAD case?

- Minimizer may not be unique.

- Gaussian case: $\{Z_t\}$ iid $\mathrm{N}(0,\sigma_0^2\phi_{0p}^{-2}) = \mathrm{N}(0,\sigma_1^2\phi_{1p}^{-2})$, so

$$E\,|\,z_1(\phi_1)\,| = E\left|\frac{Z_1\sigma_1}{\sigma_0\phi_{1p}}\right| = E\left|\frac{Z_1\sigma_0}{\sigma_0\phi_{0p}}\right| = E\,|\,z_1(\phi_0)\,|$$

- Consider $\{c_j\}$ with at least two non-zero elements and

$$\sum_{j=-\infty}^{\infty}|c_j| < \infty \text{ and } \sum_{j=-\infty}^{\infty} c_j^2 = 1$$

Jian and Pawitan (1998) show

$$E\,\left|\sum_{j=-\infty}^{\infty} c_j Z_j\right| > E\,|Z_1|$$

holds for Laplace, Student's t, contaminated normal, etc.

- Non-Gaussian case: $E\,|\,z_1(\phi_1)\,| = E\left|\dfrac{\phi_0(B^{-1})\phi_1(B)}{\phi_{0p}\phi_1(B^{-1})\phi_0(B)} Z_t\right| > E\,|\,z_1(\phi_0)\,|$

Central Limit Theorem (LAD case)

- Think of $\mathbf{u} = n^{1/2}(\phi - \phi_0)$ as an element of $\mathbb{R}^p$

- Define

$$S_n(\mathbf{u}) = \sum_{t=1}^{n-p} \left(| z_t(\phi_0 + n^{-1/2}\mathbf{u}) | - | z_t(\phi_0) | \right)$$

$$= m_n(\phi_0 + n^{-1/2}\mathbf{u}) - \sum_{t=1}^{n-p} | z_t(\phi_0) |$$

- Then $S_n(\mathbf{u}) \rightarrow S(\mathbf{u})$ in distribution on $C(\mathbb{R}^p)$, where

$$S(\mathbf{u}) = \frac{f_\sigma(0)}{|\phi_{0r}|} \mathbf{u}'\Gamma_p \mathbf{u} + \mathbf{u}'\mathbf{N}, \ \ \mathbf{N} \sim N(\mathbf{0}, \frac{2\mathrm{Var}(|Z_1|)}{\phi_{0r}^2 \sigma^2} \Gamma_p),$$

- Hence,

$$\arg\min S_n(\mathbf{u}) = n^{1/2}(\hat{\phi}_{LAD} - \phi_0)$$

$$\xrightarrow[D]{} \arg\min S(\mathbf{u})$$

$$= -\frac{|\phi_{0r}| \Gamma_p^{-1}}{2 f_\sigma(0)} \mathbf{N} \sim N(\mathbf{0}, \frac{\mathrm{Var}(|Z_1|)}{2\sigma^4 f_\sigma^4(0)} \sigma^2 \Gamma_p^{-1})$$

Asymptotic Results (LAD case):

Theorem 1. Let $\{Y_t\}$ be the linear process

$$Y_t = \sum_{j=-\infty}^{\infty} c_j z_{t-j},$$

where $c_0=0$, $\sum_{j=-\infty}^{\infty} |c_j| < \infty$, $\{z_t\} \sim \text{IID}(0,\sigma^2)$, $\text{median}(z_1)=0$, $g(0)>0$ (g density of z_1). Then

$$S_n = \sum_{t=1}^{n-p} \left(| z_t - n^{-1/2} Y_t | - | z_t | \right)$$

$$\to \text{Var}(Y_1) g(0) + N$$

where $N \sim \text{N}(0, \gamma^*(0) + 2\sum_{h\geq 1} \gamma^*(h))$ and $\gamma^*(h)$ is the covariance function for $Y_t \,\text{sgn}(z_t)$

Key idea:

$$S_n = \sum_{t=1}^{n-p} \left(| z_t \text{ -} n^{\text{-}1/2} Y_t | \text{ - } | z_t | \right)$$

$$= -n^{-1/2} \sum_{t=1}^{n-p} Y_t \operatorname{sgn}(z_t)$$

$$+ 2 \sum_{t=1}^{n-p} (n^{-1/2} Y_t - z_t) \left\{ 1_{\{0 < z_t < n^{-1/2} Y_t\}} - 1_{\{n^{-1/2} Y_t < z_t < 0\}} \right\}$$

$$\rightarrow N + Var(Y_1) g(0)$$

Theorem 2. On $C(\mathbb{R}^p)$,

$$S_n(\mathbf{u}) = \sum_{t=1}^{n-p}\left(|z_t(\phi_0 + n^{-1/2}\mathbf{u})| - |z_t(\phi_0)|\right)$$

$$\to S(\mathbf{u}),$$

where

$$S(\mathbf{u}) = \frac{f_\sigma(0)}{|\phi_{0r}|}\mathbf{u}'\Gamma_p\mathbf{u} + \mathbf{u}'\mathbf{N},$$

$$\mathbf{N} \sim N\left(\mathbf{0}, \frac{2Var(|Z_1|)}{\phi_{0r}^2\sigma^2}\Gamma_p\right),$$

and Γ_p is the covariance matrix of a causal AR(p).

Limit theory for LAD estimate. Note that

$$\hat{\phi}_{\text{LAD}} = \phi_0 + \hat{\mathbf{u}}_n / \sqrt{n}$$

so that $\hat{\mathbf{u}}_n = \sqrt{n}(\hat{\phi}_{\text{LAD}} - \phi_0) = \arg\min S_n(\mathbf{u})$

$$\to \hat{\mathbf{u}} = \arg\min S(\mathbf{u}).$$

Minimizing S, we find that the minimizer or limit random variable is

$$\hat{\mathbf{u}}_n = \sqrt{n}(\hat{\phi}_{\text{LAD}} - \phi_0) \to -\frac{|\phi_{0r}|\Gamma_p^{-1}}{2 f_\sigma(0)}\mathbf{N}$$

$$-\frac{|\phi_{0r}|\Gamma_p^{-1}}{2 f_\sigma(0)}\mathbf{N} \sim N(\mathbf{0}, \frac{Var(|Z_1|)}{2\sigma^4 f_\sigma^2(0)}\sigma^2\Gamma_p^{-1})$$

Asymptotic Covariance Matrix

- For LS estimators of AR(p):

$$\sqrt{n}(\hat{\phi}_{\mathrm{LS}} - \phi_0) \xrightarrow{D} N(0, \sigma^2 \Gamma_p^{-1})$$

- For LAD estimators of AR(p):

$$\sqrt{n}(\hat{\phi}_{\mathrm{LAD}} - \phi_0) \xrightarrow{D} N(0, \frac{1}{4\sigma^2 f^2(0)} \sigma^2 \Gamma_p^{-1})$$

- For LAD estimators of AP(p):

$$\sqrt{n}(\hat{\phi}_{\mathrm{LAD}} - \phi_0) \xrightarrow{D} N(0, \frac{\mathrm{Var}(|Z_1|)}{2\sigma^4 f_\sigma^2(0)} \sigma^2 \Gamma_p^{-1})$$

- For MLE estimators of AP(p):

$$\sqrt{n}(\hat{\phi}_{\mathrm{MLE}} - \phi_0) \xrightarrow{D} N(0, \frac{1}{2(\sigma^2 \hat{I} - 1)} \sigma^2 \Gamma_p^{-1})$$

Laplace: (LAD=MLE)

$$\frac{\mathrm{Var}(|Z_1|)}{2\sigma^4 f_\sigma^2(0)} = \frac{1}{2} = \frac{1}{2(\sigma^2 \hat{I} - 1)}$$

Students t_ν, $\nu > 2$:

LAD: $$\frac{\mathrm{Var}(|Z_1|)}{2\sigma^4 f_\sigma^2(0)} = \frac{\Gamma^2(\nu/2)(\nu-2)\pi}{2\Gamma^2((\nu+1)/2)} - \frac{2(\nu-2)^2}{(\nu-1)^2}$$

MLE: $$\frac{1}{2(\sigma^2 \hat{I} - 1)} = \frac{(\nu-2)(\nu+3)}{12}$$

Student's t_3:

LAD: .7337

MLE: 0.5

ARE: .7337/.5=1.4674

Order Selection:

Partial ACF From the previous result, if true model is of order r and fitted model is of order p > r, then

$$n^{1/2}\hat{\phi}_{p,LAD} \rightarrow N(0, \frac{\mathrm{Var}(|Z|)}{2\sigma^4 f_\sigma^2(0)})$$

where $\hat{\phi}_{p,LAD}$ is the pth element of $\hat{\phi}_{LAD}$.

Procedure:

1. Fit high order (P-th order), obtain residuals and estimate scalar,

$$\theta^2 = \frac{\mathrm{Var}(|Z_1|)}{2\sigma^4 f_\sigma^2(0)},$$

by empirical moments of residuals and density estimates.

2. Fit AP models of order p=1,2, . . . , P via LAD and obtain p-th coefficient $\hat{\phi}_{p,p}$ for each.

3. Choose model order r as the smallest order beyond which the estimated coefficients are statistically insignificant.

Note: Can replace $\hat{\phi}_{p,p}$ with $\hat{\phi}_{p,\mathrm{MLE}}$ if using MLE. In this case for p > r

$$n^{1/2}\hat{\phi}_{p,MLE} \to N\left(0, \frac{1}{2(\sigma^2 \hat{I} - 1)}\right).$$

AIC: 2*p* or not 2*p*?

- An approximately unbiased estimate of the Kullback-Leiber index of fitted to true model:

$$AIC(p) := -2L_X(\hat{\phi}, \hat{\kappa}) + \frac{\mathrm{Var}(|Z_1|)}{E|Z_1|\sigma^2 f_\sigma(0)}\, p$$

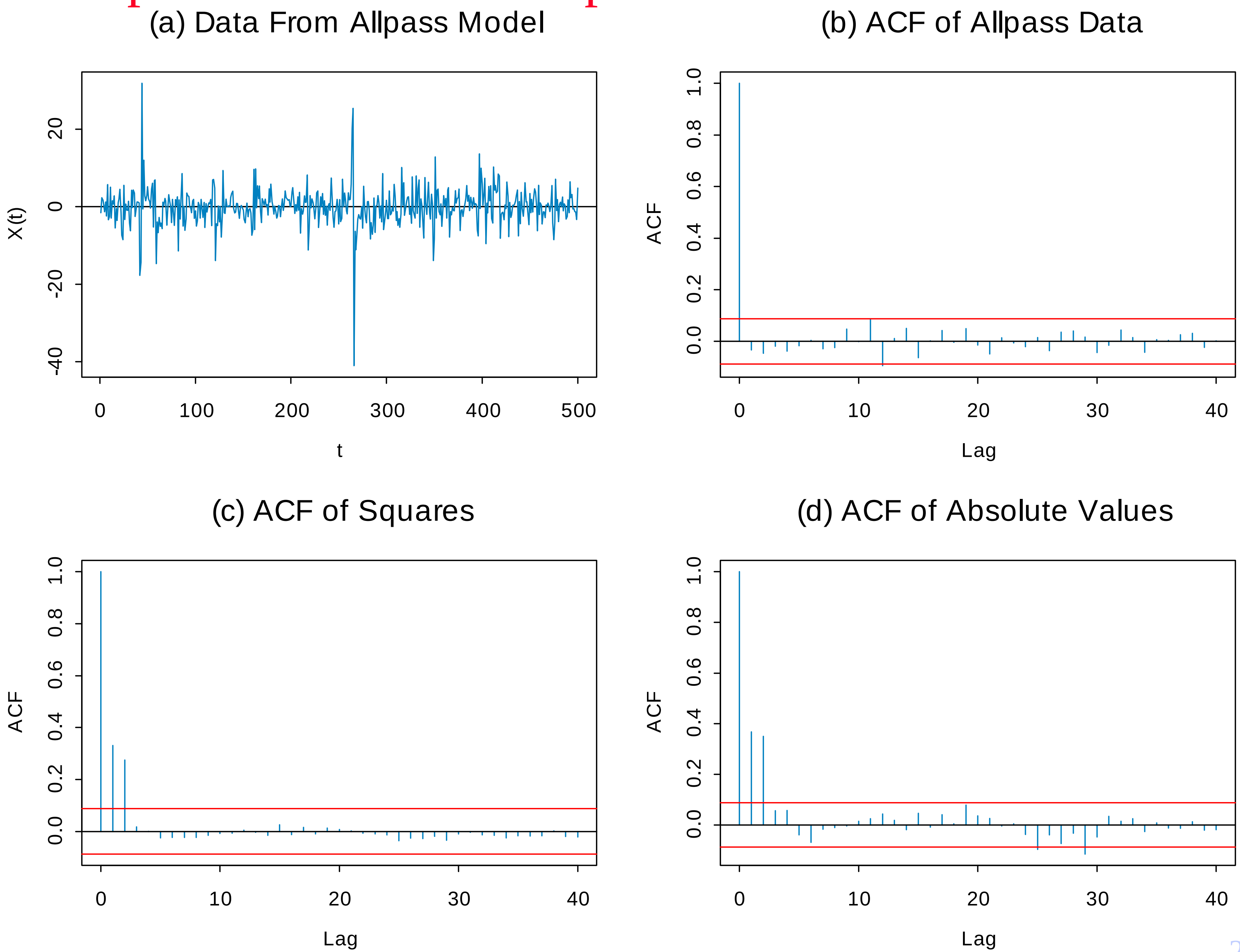
- Penalty term for Laplace case:

$$\frac{\mathrm{Var}(|Z_1|)}{E|Z_1|\sigma^2 f_\sigma(0)}\, p = \frac{\sigma^2/2}{(\sigma/\sqrt{2})\sigma^2(1/\sqrt{2}\sigma)}\, p = p$$

- Estimated penalty term:

$$\frac{\mathrm{var}(|z_t(\hat{\phi})|)}{ave\{|z_t(\hat{\phi})|\}\hat{f}_{z_t(\hat{\phi})}(0)}\, p \xrightarrow[P]{} \frac{\mathrm{Var}(|Z_1|)}{E|Z_1|\sigma^2 f_\sigma(0)}\, p$$

Sample realization of all-pass of order 2



Estimates:

$$\hat{\phi}_1 = .297(.0381),\ \hat{\phi}_2 = .374(.0381)$$

Standard errors computed as $\hat{\theta}\, sqrt\{(1-\hat{\phi}_2^2)/500\}$

where $\hat{\theta} = .919$

Order selection:

- cut-off value for PACF is 1.96*.908/sqrt(500)=.0796
- $AIC(p) := -2L_X(\hat{\phi}, \hat{\kappa}) + 1.896\, p$

	1	2	3	4	5
phi_p	0.289	0.374	0.009	0.011	0.01
AIC(p)	2451	2346	2347	2348	2350

	6	7	8	9	10
	0.047	0.034	-0.05	0.083	0.021
	2348	2349	2345	2343	2345

Simulation results:

- 1000 replicates of all-pass models
- model order parameter value

model order	parameter value
1	$\phi_1 = .5$
2	$\phi_1 = .3,\ \phi_2 = .4$

- noise distribution is t with 3 d.f.
- sample sizes n=500, 5000
- estimation method is LAD

To guard against being trapped in local minima, we adopted the following strategy.

- 250 random starting values were chosen at *random*. For model of order p, k-th starting value was computed recursively as follows:

  1. Draw $\phi_{11}^{(k)}, \phi_{22}^{(k)}, \ldots, \phi_{pp}^{(k)}$ iid uniform (-1,1).
  2. For j=2, …, p, compute

$$\begin{bmatrix} \phi_{j1}^{(k)} \\ \vdots \\ \phi_{j,j-1}^{(k)} \end{bmatrix} = \begin{bmatrix} \phi_{j-1,1}^{(k)} \\ \vdots \\ \phi_{j-1,j-1}^{(k)} \end{bmatrix} - \phi_{jj}^{(k)} \begin{bmatrix} \phi_{j-1,j-1}^{(k)} \\ \vdots \\ \phi_{j-1,1}^{(k)} \end{bmatrix}$$

- Select top 10 based on minimum function evaluation.

- Run Hooke and Jeeves with each of the 10 starting values and choose best optimized value.

	Asymptotic		Empirical			
N	mean	**std dev**	mean	**std dev**	%coverage	rel eff*
500	ϕ_1=.5	**.0332**	.4979	**.0397**	94.2	11.8
5000	ϕ_1=.5	**.0105**	.4998	**.0109**	95.4	9.3

	Asymptotic		Empirical		
N	mean	**std dev**	mean	**std dev**	%coverage
500	ϕ_1=.3	**.0351**	.2990	**.0456**	92.5
	ϕ_2=.4	**.0351**	.3965	**.0447**	92.1
5000	ϕ_1=.3	**.0111**	.3003	**.0118**	95.5
	ϕ_2=.4	**.0111**	.3990	**.0117**	94.7

*Efficiency relative to maximum absolute residual kurtosis:

$$\left| \frac{1}{n-p} \sum_{t=1}^{n-p} \left(\frac{z_t(\phi)}{v_2^{1/2}} \right)^4 - 3 \right|, \qquad v_2 = \frac{1}{n-p} \sum_{t=1}^{n-p} (z_t(\phi) - \bar{z}(\phi))^2$$

MLE Simulations Results using t-distr(3.5)

	Asymptotic		Empirical		
N	mean	**std dev**	mean	**std dev**	%coverage
500	ϕ_1=.5	**.0349**	.4983	**.0421**	91.7
	ν=3.5	**.5853**	3.449	**.4527**	92.5
5000	ϕ_1=.5	**.0110**	.4997	**.0088**	95.0
	ν=3.5	**.1851**	3.449	**.1341**	96.7

	Asymptotic		Empirical		
N	mean	**std dev**	mean	**std dev**	%coverage
500	ϕ_1=.3	**.0369**	.2969	**.0451**	89.6
	ϕ_2=.4	**.0369**	.3973	**.0446**	90.6
	ν=3.5	**.5853**	3.556	**.5685**	92.4
5000	ϕ_1=.3	**.0117**	.3002	**.0099**	94.7
	ϕ_2=.4	**.0117**	.4001	**.0106**	93.6
	ν=3.5	**.1764**	3.510	**.1764**	94.7

Minimum Dispersion Estimator: Minimize the objective fcn

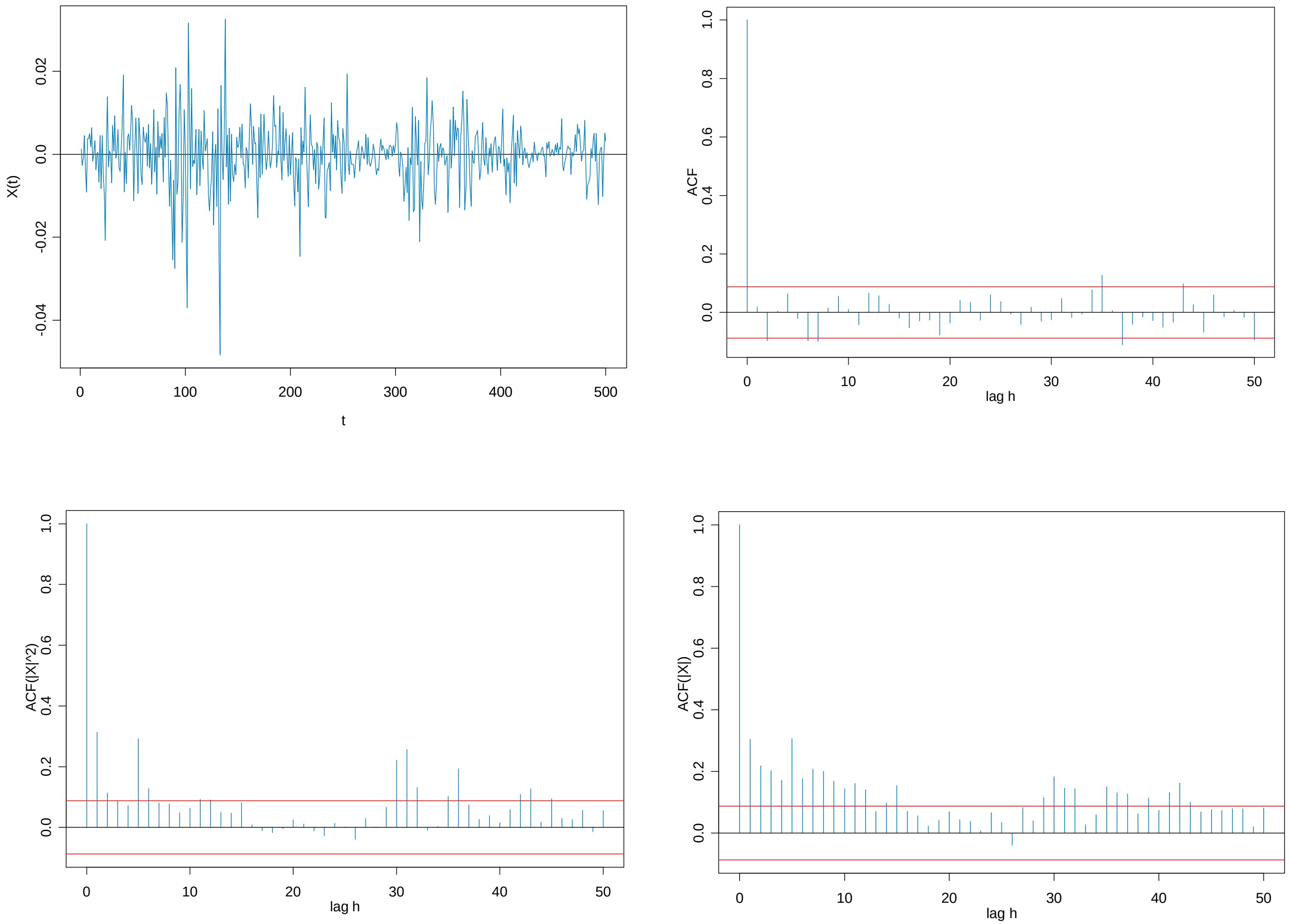
$$S(\phi) = \sum_{t=1}^{n-p} \left(\frac{t}{n-p+1} - \frac{1}{2} \right) z_{(t)}(\phi)$$

where $\{z_{(t)}(\phi)\}$ are the ordered $\{z_t(\phi)\}$.

		Empirical		Empirical LAD	
N		mean	**std dev**	mean	**std dev**
500	ϕ_1=.5	.4978	**.0315**	.4979	**.0397**
5000	ϕ_1=.5	.4997	**.0094**	.4998	**.0109**
500	ϕ_1=.3	.2988	**.0374**	.2990	**.0456**
	ϕ_2=.4	.3957	**.0360**	.3965	**.0447**
5000	ϕ_1=.3	.3007	**.0101**	.3003	**.0118**
	ϕ_2=.4	.3993	**.0104**	.3990	**.0117**

Application to financial data

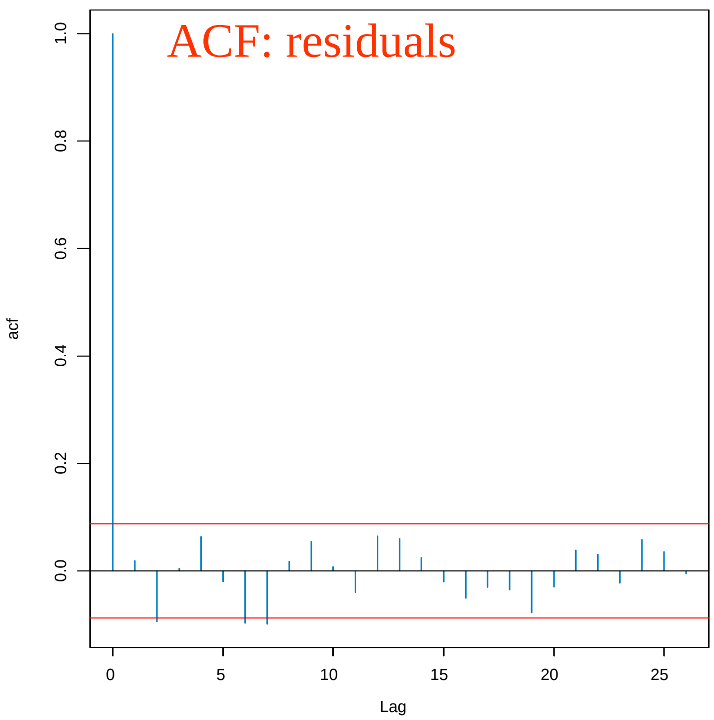
500-daily log-returns of NZ/US exchange rate

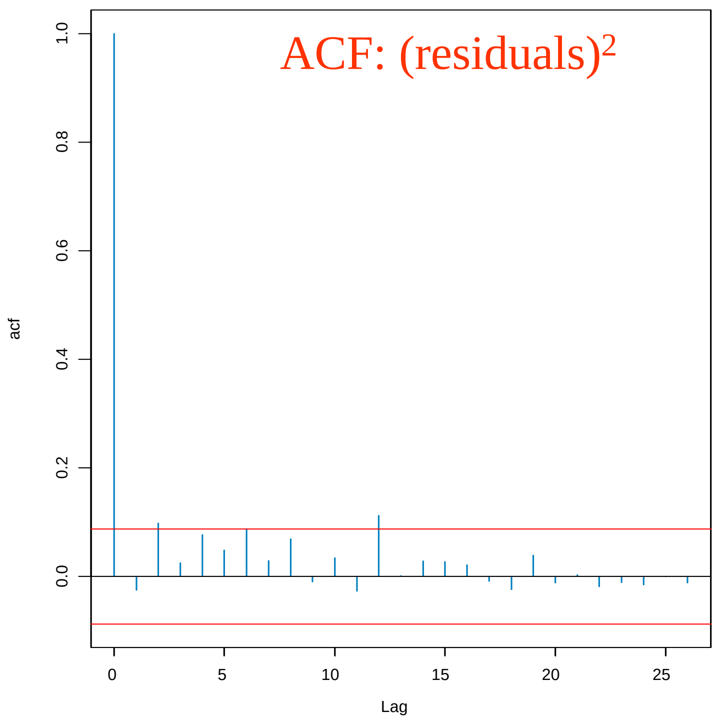


All-pass model fitted to NZ-USA exchange rates (using LAD):

Order = 6, ϕ_1=-.367, ϕ_2=-.750, ϕ_3=-.391, ϕ_4=.088, ϕ_5= -.193, ϕ_6=-.096

(AIC had local minima at p=6 and 10)





Noninvertible MA models with heavy tailed noise

$$X_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q},$$

a. $\{Z_t\} \sim \text{IID}(\alpha)$ with Pareto tails

b. $\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$

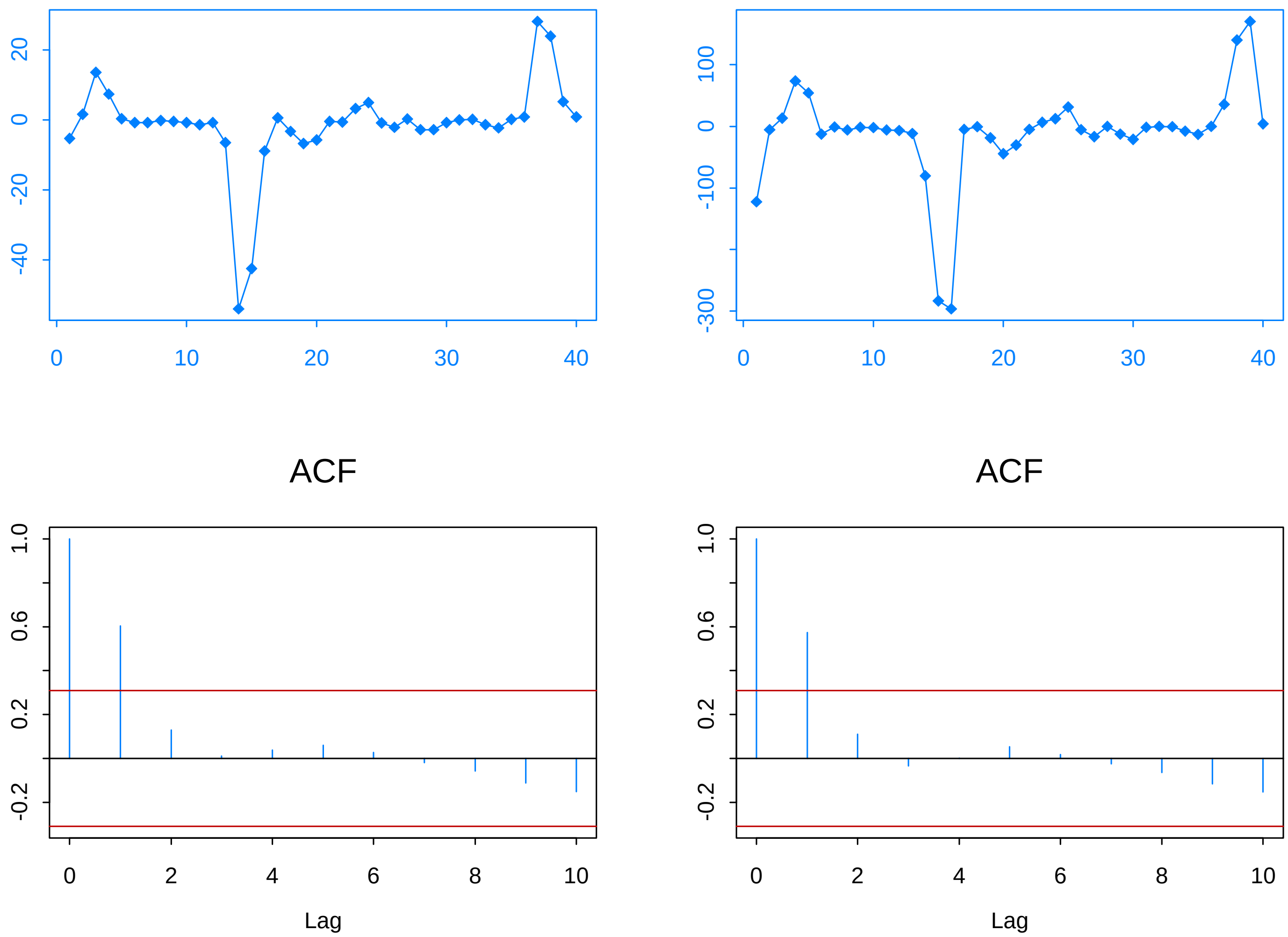
No zeros inside the unit circle $\Rightarrow$ invertible

Some zero(s) inside the unit circle $\Rightarrow$ noninvertible

Realizations of an invertible and noninvertible MA(2) processes

Model: $X_t = \theta_*(B)\, Z_t$, $\{Z_t\} \sim \text{IID}(\alpha = 1)$, where

$\theta_i(B) = (1 + 1/2B)(1 + 1/3B)$ and $\theta_{ni}(B) = (1 + 2B)(1 + 3B)$



Application of all-pass to noninvertible MA model fitting

Suppose $\{X_t\}$ follows the noninvertible MA model

$$X_t = \theta_i(B)\,\theta_{ni}(B)\,Z_t\,, \quad \{Z_t\} \sim \text{IID}.$$

Step 1: Let $\{U_t\}$ be the residuals obtained by fitting a purely invertible MA model, i.e.,

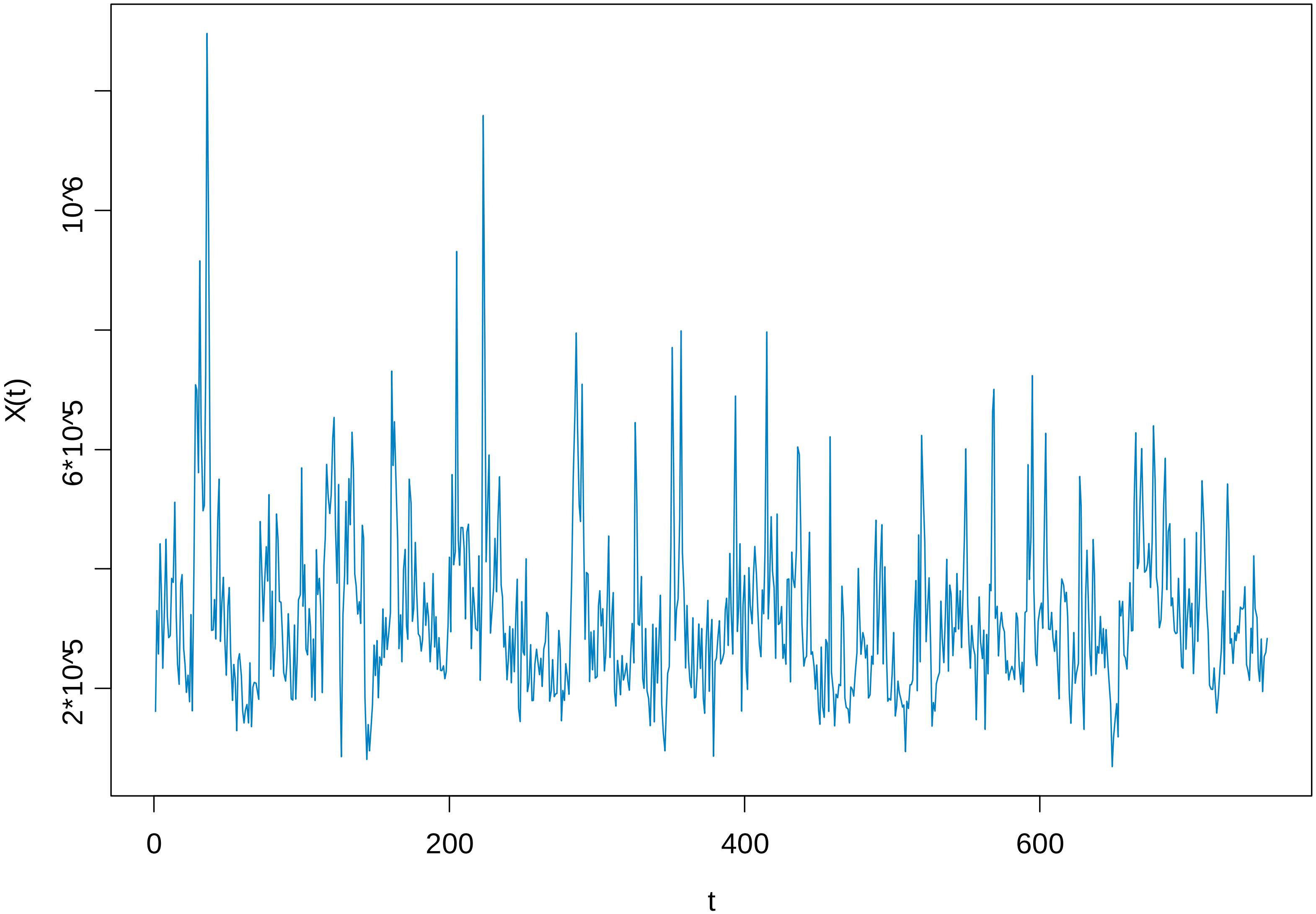
$$\begin{aligned} X_t &= \hat{\theta}(B)U_t \\ &\approx \theta_i(B)\tilde{\theta}_{ni}(B)U_t, \quad (\tilde{\theta}_{ni} \text{ is the invertible version of } \theta_{ni}). \end{aligned}$$

So $$U_t \approx \frac{\theta_{ni}(B)}{\tilde{\theta}_{ni}(B)} Z_t$$

Step 2: Fit a purely causal AP model to $\{U_t\}$

$$\tilde{\theta}_{ni}(B)U_t = \theta_{ni}(B)Z_t.$$

Volumes of Microsoft (MSFT) stock traded over 755 transaction days (6/3/96 to 5/28/99)



Analysis of MSFT:

Step 1: Log(volume) follows MA(4).

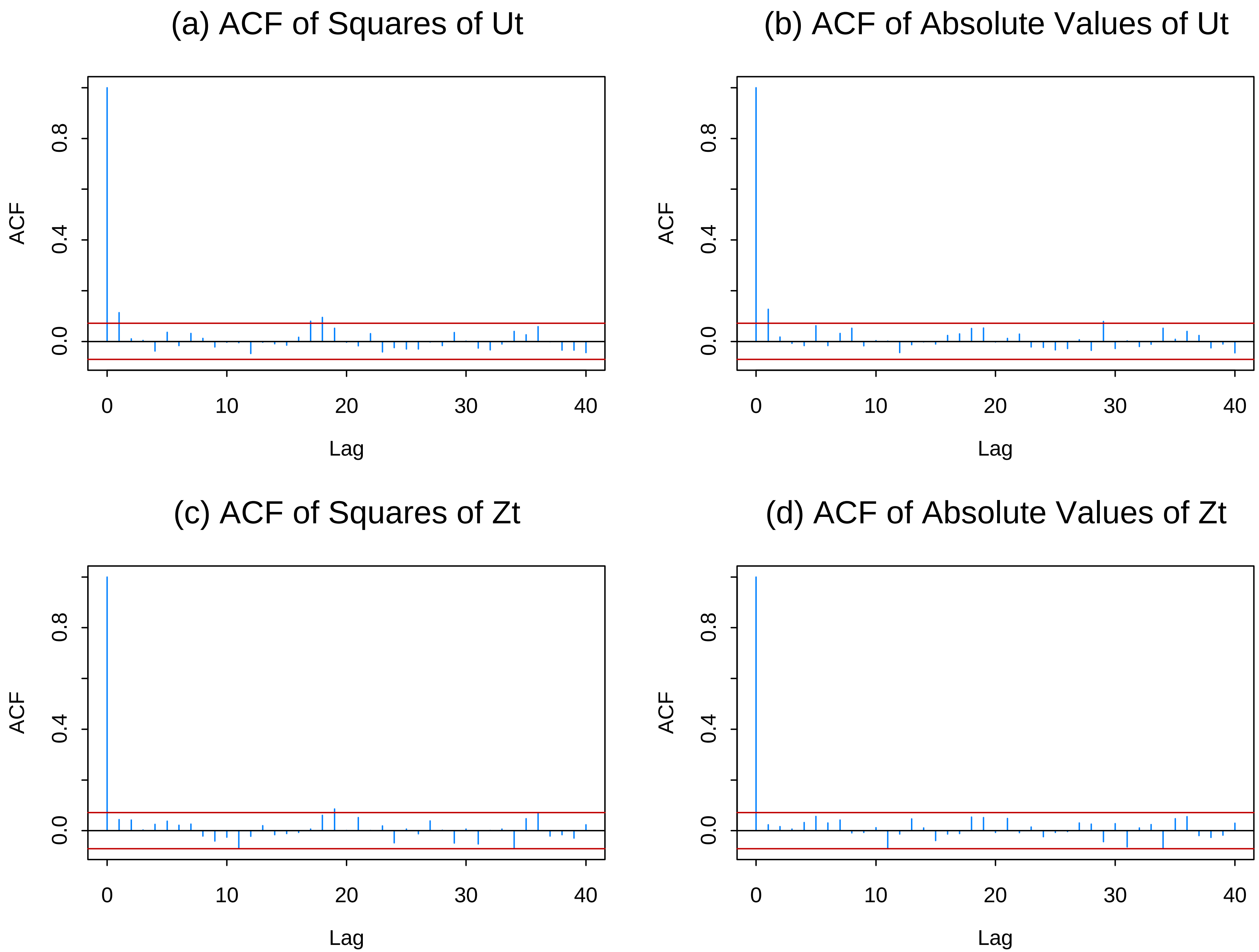
$$X_t = (1 + .513B + .277B^2 + .270B^3 + .202B^4)\, U_t \qquad \text{(invertible MA(4))}$$

Step 2: All-pass model of order 4 fitted to $\{U_t\}$ using MLE (t-dist):

$$(1 + .184B + .132B^2 - .833B^3 - .314B^4)U_t$$
$$= (1 + 2.65B - .418B^2 - .586B^3 - 3.18B^4)Z_t.$$

Conclude that $\{X_t\}$ follows a noninvertible MA(4) which after refitting has the form:

$$X_t = (1 + 1.34B + 1.374B^2 + 2.54B^3 + 4.96B^4)\, Z_t\ ,\ \{Z_t\} \sim \text{IID } t(6.3)$$


(a) ACF of Squares of Ut
(b) ACF of Absolute Values of Ut
(c) ACF of Squares of Zt
(d) ACF of Absolute Values of Zt
ACF
0.0
0.4
0.8
0
10
20
30
40
Lag

Summary: Microsoft Trading Volume

☞ Two-step fit of noninvertible MA(4):

- invertible MA(4): residuals not iid
- causal AP(4); residuals iid

☞ Direct fit of purely noninvertible MA(4):

$$(1+1.34B+1.374B^2+2.54B^3+4.96B^4)$$

☞ For MCHP, invertible MA(4) fits.

Summary

☞ All-pass models and their properties

- linear time series with “nonlinear” behavior

☞ Estimation

- likelihood approximation
- MLE and LAD
- order selection

☞ Emprirical results

- simulation study
- AP(6) for NZ/USA exchange rates

☞ Noninvertible moving average processes

- two-step estimation procedure using all-pass
- noninvertible MA(4) for Microsoft trading volume

Further Work

☞ Least absolute deviations

- further simulations
- order selection
- heavy-tailed case
- other smooth objective functions (e.g., min dispersion)

☞ Maximum likelihood

- Gaussian mixtures
- simulation studies
- applications

☞ Noninvertible moving average modeling

- initial estimates from two-step all-pass procedure
- adaptive procedures