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Define $X_t = \ln (P_t) - \ln (P_{t-1})$ (log returns)

• heavy tailed

$$P(|X_1| > x) \sim C x^{-\alpha}, \quad 0 < \alpha < 4.$$  

• uncorrelated

$$\hat{\rho}_x(h) \text{ near } 0 \text{ for all lags } h > 0 \text{ (MGD sequence)}$$

• $|X_t|$ and $X_t^2$ have slowly decaying autocorrelations

$$\hat{\rho}_{|X|}(h) \text{ and } \hat{\rho}_{X^2}(h) \text{ converge to } 0 \text{ slowly as } h \text{ increases.}$$

• process exhibits ‘stochastic volatility’. 
Sample ACF IBM (a) 1962-1981, (b) 1982-2000

(a) ACF of IBM (1st half)

(b) ACF of IBM (2nd half)
Sample ACF of abs values for IBM (a) 1961-1981, (b) 1982-2000

(a) ACF, Abs Values of IBM (1st half)

(b) ACF, Abs Values of IBM (2nd half)
Sample ACF of squares for IBM (a) 1961-1981, (b) 1982-2000
Sample ACF of original data and squares for IBM 1962-2000
Plot of $M_t(4)/S_t(4)$ for IBM
500-daily log-returns of NZ/US exchange rate
ACF of $X(t) = \text{log}-\text{returns of NZ/US exchange rate}$
ACF of $X^2(t)$
Plot of $M_t(4)/S_t(4)$
Hill’s plot of tail index
Models for Log(returns)

**Basic model**

\[ X_t = \ln (P_t) - \ln (P_{t-1}) \quad \text{(log returns)} \]

\[ = \sigma_t Z_t , \]

where

- \( \{Z_t\} \) is IID with mean 0, variance 1 (if exists). (e.g. N(0,1) or a \( t \)-distribution with \( \nu \) df.)
- \( \{\sigma_t\} \) is the volatility process
- \( \sigma_t \) and \( Z_t \) are independent.

**Properties:**

- \( \text{E}X_t = 0, \text{Cov}(X_t, X_{t+h}) = 0, h>0 \) (uncorrelated if \( \text{Var}(X_t) < \infty \))
- conditional heteroscedastic (condition on \( \sigma_t \)).
Models for Log(returns)-cont

\[ X_t = \sigma_t Z_t \] (observation eqn in state-space formulation)

Two classes of models for volatility:

(i) GARCH(p,q) process (General AutoRegressive Conditional Heteroscedastic-observation-driven specification)

\[ \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_p X_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_q \sigma_{t-q}^2. \]

Special case: ARCH(1):

\[ X_t^2 = (\alpha_0 + \alpha_1 X_{t-1}^2)Z_t^2 \]
\[ = \alpha_1 Z_t^2 X_{t-1}^2 + \alpha_0 Z_t^2 \]
\[ = A_t X_{t-1}^2 + B_t \] (stochastic recursion eqn)

\[ \rho_{X^2}(h) = \alpha_1^h, \text{ if } \alpha_1^2 < 1/3. \]
Models for Log(returns)-cont

GARCH(2,1): \[ X_t = \sigma_t Z_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \alpha_2 X_{t-2}^2 + \beta_1 \sigma_{t-1}^2. \]

Then \( Y_t = (X_t^2, X_{t-1}^2, \sigma_t^2)' \) follows the SRE given by

\[
\begin{bmatrix}
X_t^2 \\
X_{t-1}^2 \\
\sigma_t^2
\end{bmatrix}
= \begin{bmatrix}
\alpha_1 Z_t^2 & \alpha_2 Z_t^2 & \beta_1 Z_t^2 \\
1 & 0 & 0 \\
\alpha_1 & \alpha_2 & \beta_1
\end{bmatrix}
\begin{bmatrix}
X_{t-1}^2 \\
X_{t-2}^2 \\
\sigma_{t-1}^2
\end{bmatrix}
+ \begin{bmatrix}
\alpha_0 Z_t^2 \\
0 \\
0
\end{bmatrix}
\]

Questions:

- Existence of a unique stationary soln to the SRE?
- Distributional properties of the stationary distribution?
- Moment properties of the process? Finite variance?
Models for Log(returns)-cont

\( X_t = \sigma_t Z_t \) (observation eqn in state-space formulation)

(ii) stochastic volatility process (parameter-driven specification)

\[
\log \sigma_t^2 = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}, \quad \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty, \{ \varepsilon_t \} \sim \text{IID } N(0, \sigma^2)
\]

\[
\rho_{X^2}(h) = \text{Cor}(\sigma_t^2, \sigma_{t+h}^2) / EZ_1^4
\]
**Regular Variation — univariate case**

**Definition:** The random variable $X$ is regularly varying with index $\alpha$ if

$$P(|X|> tx)/P(|X|>t) \to x^{-\alpha} \quad \text{and} \quad P(X> t)/P(|X|>t) \to p,$$

or, equivalently, if

$$P(X> tx)/P(|X|>t) \to px^{-\alpha} \quad \text{and} \quad P(X< -tx)/P(|X|>t) \to qx^{-\alpha},$$

where $0 \leq p \leq 1$ and $p+q=1.$

**Equivalence:**

$X$ is RV($\alpha$) *if and only if* $P(X \in t \bullet)/P(|X|>t) \to_{\nu} \mu(\bullet)$ ($\to_{\nu}$ vague convergence of measures on $\mathbb{R}\{0\}$). In this case,

$$\mu(dx) = (p \alpha x^{-\alpha-1} I(x>0) + q \alpha (-x)^{-\alpha-1} I(x<0)) \ dx$$

**Note:** $\mu(tA) = t^\alpha \mu(A).$
Regular Variation — univariate case

Another formulation:

Define the ± 1 valued rv $\theta$, $P(\theta = 1) = p$, $P(\theta = -1) = 1 - p = q$.

Then 

$X$ is $RV(\alpha)$ if and only if

$$
\frac{P(|X| > tx, X/|X| \in S)}{P(|X| > t)} \to x^{-\alpha}P(\theta \in S)
$$

or

$$
\frac{P(|X| > tx, X/|X| \in \bullet)}{P(|X| > t)} \to_v x^{-\alpha}P(\theta \in \bullet)
$$

($\to_v$ vague convergence of measures on $S^0 = \{-1,1\}$).
Regular Variation—multivariate case

Multivariate regular variation of $X = (X_1, \ldots, X_m)$: There exists a random vector $\theta \in S^{m-1}$ such that

$$P(|X| > tx, X/|X| \in \cdot)/P(|X| > t) \xrightarrow{\nu} x^{-\alpha} P(\theta \in \cdot)$$

($\xrightarrow{\nu}$ vague convergence on $S^{m-1}$, unit sphere in $\mathbb{R}^m$).

- $P(\theta \in \cdot)$ is called the spectral measure
- $\alpha$ is the index of $X$.

Equivalence:

$$\frac{P( X \in t\cdot)}{P(|X| > t)} \xrightarrow{\nu} \mu(\cdot)$$

$\mu$ is a measure on $\mathbb{R}^m$ which satisfies of $x > 0$ and $A$ bounded away from 0,

$$\mu(xB) = x^{-\alpha} \mu(xA).$$
Examples: Let $X_1, X_2$ be positive regularly varying with index $\alpha$

1. If $X_1$ and $X_2$ are iid, then $X = (X_1, X_2)$ is multivariate regularly
varying with index $\alpha$ and spectral distribution

$$P(\theta = (0,1)) = P(\theta = (1,0)) = .5 \text{ (mass on axes).}$$

Interpretation: Unlikely that $X_1$ and $X_2$ are very large at the
same time.

2. If $X_1 = X_2$, then $X = (X_1, X_2)$ is multivariate regularly varying
with index $\alpha$ and spectral distribution

$$P(\theta = (1/\sqrt{2}, 1/\sqrt{2})) = 1.$$
Another equivalence? Suppose $X > 0$.

MRV $\iff$ all linear combinations of $X$ are regularly varying

i.e., if and only if

$$P(c^T X > t)/P(1^T X > t) \to w(c),$$

exists for all real-valued $c$,

in which case,

$$w(tc) = t^{-\alpha} w(c).$$

($\Rightarrow$) true (use vague convergence with $A_c = \{y : c^T y > 1\}$, i.e.,

$$\frac{P(X \in tA_c)}{P(1^T X > t)} = \frac{P(c^T X > t)}{P(|X| > t)} \frac{P(1^T X > t)}{\mu(A_1)} \to \frac{\mu(A_c)}{\mu(A_1)} =: w(c).$$
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<th>Regular Variation—multivariate case</th>
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(⇐) established by Basrak, Davis and Mikosch (2000) for $\alpha$ not an even integer—case of even integer is unknown.

Idea of argument: Define the measures

$$m_t(\cdot) = \frac{P(X \in t \cdot)}{P(1^T X > t)}$$

- By assumption we know that for fixed $x$, $m_t(A_x) \rightarrow \mu(A_x)$
- $\{m_t\}$ is tight: For $B$ bded away from 0, $\sup_t m_t(B) < \infty$.
- Do subsequential limits of $\{m_t\}$ coincide?

If $m_t' \rightarrow_v \mu_1$ and $m_t'' \rightarrow_v \mu_2$, then

$$\mu_1(A_x) = \mu_2(A_x) \quad \text{for all } x \neq 0.$$  

Problem: Need $\mu_1 = \mu_2$ but only have equality on $A_x$ not a $\pi$-system.

Overcome this using transform theory.
Applications of Multivariate Regular Variation

- Domain of attraction for sums of iid random vectors (Rvaceva, 1962). That is, when does the partial sum

\[ a_n^{-1} \sum_{t=1}^{n} X_t \]

converge for some constants \( a_n \)?

- Domain of attraction for componentwise maxima of iid random vectors (Resnick, 1987). Limit behavior of

\[ a_n^{-1} \bigvee_{t=1}^{n} X_t \]

- Weak convergence of point processes with iid points.

- Solution to stochastic recurrence equations, \( Y_t = A_t Y_{t-1} + B_t \)

- Weak convergence of sample autocovariances.
**Theorem** Let \( \{X_t\} \) be an iid sequence of random vectors that are multivariate regularly varying. Then we have the following point process convergence

\[
N_n := \sum_{t=1}^{n} \mathcal{E}_{X_t/a_n} \xrightarrow{d} N := \sum_{j=1}^{\infty} \mathcal{E}_{P_j \theta_i},
\]

where \( \{a_n\} \) satisfies \( nP(|X_t| > a_n) \to 1 \), and \( N \) is a Poisson process with intensity measure \( \mu \).

- \( \{P_i\} \) are Poisson pts corresponding to the radial part (intensity measure \( \alpha x^{-\alpha-1} \, dx \)).
- \( \{\theta_i\} \) are iid with the spectral distribution given by the MRV.
Applications—stochastic recurrence equations

\[ Y_t = A_t Y_{t-1} + B_t, \quad (A_t, B_t) \sim \text{IID}, \]
\[ A_t \text{ } d \times d \text{ random matrices, } B_t \text{ random } d \text{-vectors} \]

Examples

ARCH(1): \[ X_t = (\alpha_0 + \alpha_1 X_{t-1}^2)^{1/2} Z_t, \quad \{Z_t\} \sim \text{IID}. \]
Then the squares follow an SRE with \[ Y_t = X_t^2, \quad A_t = \alpha_1 Z_t^2, \quad B_t = \alpha_0 Z_t^2. \]

GARCH(2,1): \[ X_t = \sigma_t Z_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \alpha_2 X_{t-2}^2 + \beta_1 \sigma_{t-1}^2. \]
Then \[ Y_t = (X_t^2, X_{t-1}^2, \sigma_t^2)' \] follows the SRE given by

\[
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1 & 0 & 0 \\
\alpha_1 & \alpha_2 & \beta_1
\end{bmatrix}
\begin{bmatrix}
X_{t-1}^2 \\
X_{t-2}^2 \\
\sigma_{t-1}^2
\end{bmatrix} +
\begin{bmatrix}
\alpha_0 Z_t^2 \\
0 \\
0
\end{bmatrix}
\]
Stochastic Recurrence Equations (cont)

Regular variation of the marginal distribution (Kesten)

Assume $A$ and $B$ have non-negative entries and

• $E \|A_1\|^{\varepsilon} < 1$ for some $\varepsilon > 0$

• $A_1$ has no zero rows a.s.

• W.P. 1, $\{\ln \rho(A_1 \ldots A_n):$ is dense in $\mathbb{R}$ for some $n, A_1 \ldots A_n > 0\}$

• There exists a $\kappa_0 > 0$ such that $E\|A\|^\kappa_0 \ln^+ \|A\| < \infty$ and

$$
E\left( \min_{i=1, \ldots, d} \sum_{j=1}^{d} A_{ij} \right)^{\kappa_0} \geq d^{\kappa_0/2}
$$

Then there exists a $\kappa_1 \in (0, \kappa_0]$ such that all linear combinations of $Y$ are regularly varying with index $\kappa_1$. (Also need $E|B|^{\kappa_i} < \infty$.)

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## Application to GARCH

**Proposition:** Let \((Y_t)\) be the solution to the SRE based on the squares of a GARCH model. Assume

- Top Lyapunov exponent \(\gamma < 0\). (See Bougerol and Picard`92)
- \(Z\) has a positive density on \((-\infty, \infty)\) with all moments finite or \(E|Z|^h = \infty\), for all \(h \geq h_0\) and \(E|Z|^h < \infty\) for all \(h < h_0\).
- Not all the GARCH parameters vanish.

Then \((Y_t)\) is *strongly mixing* with geometric rate and all finite dimensional distributions are *multivariate regularly varying* with index \(\kappa_1\).

**Corollary:** The corresponding GARCH process is strongly mixing and has all finite dimensional distributions that are MRV with index \(\kappa = 2\kappa_1\).
Remarks:
1. Kesten’s result applied to an iterate of $Y_t$, i.e., $Y_{tm} = \tilde{A}_t Y_{(t-1)m} + \tilde{B}_t$

2. Determination of $\kappa$ is difficult. Explicit expressions only known in two(?) cases.

- **ARCH(1):** $E|\alpha_1 Z|^2|^{\kappa/2} = 1$.

<table>
<thead>
<tr>
<th>$\alpha_1$</th>
<th>.312</th>
<th>.577</th>
<th>1.00</th>
<th>1.57</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa$</td>
<td>8.00</td>
<td>4.00</td>
<td>2.00</td>
<td>1.00</td>
</tr>
</tbody>
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- **GARCH(1,1):** $E|\alpha_1 Z^2 + \beta_1|^{\kappa/2} = 1$ (Mikosch and St$ar{r}$ab)  
  - For IGARCH ($\alpha_1 + \beta_1 = 1$), then $\kappa = 2 \Rightarrow$ infinite variance.
  - Can estimate $\kappa$ empirically by replacing expectations with sample moments.
Summary for GARCH(p,q)

\( \kappa \in (0,2) : \)
\[
(\hat{\rho}_X(h))_{h=1,\ldots,m} \xrightarrow{d} (V_h / V_0)_{h=1,\ldots,m},
\]

\( \kappa \in (2,4) : \)
\[
\left( n^{1-2/\kappa} \hat{\rho}_X(h) \right)_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1}(0)(V_h)_{h=1,\ldots,m}.
\]

\( \kappa \in (4,\infty) : \)
\[
\left( n^{1/2} \hat{\rho}_X(h) \right)_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1}(0)(G_h)_{h=1,\ldots,m}.
\]

Remark: Similar results hold for the sample ACF based on \(|X_t|\) and \(X_t^2\).
Realization of fitted GARCH

Fitted GARCH(1,1) model for NZ-USA exchange:

\[ X_t = \sigma_t Z_t, \quad \sigma_t^2 = (6.70)10^{-7} + 0.1519X_{t-1}^2 + 0.772\sigma_{t-1}^2 \]

\( (Z_t) \sim \text{IID } t\text{-distr with 5 df.} \quad \kappa \text{ is approximately } 3.8 \)
ACF of Fitted GARCH(1,1) Process

ACF of squares of realization 1

ACF of squares of realization 2
ACF of 2 realizations of an (ARCH)^2: $X_t = (0.001 + 0.7 X_{t-1})^{1/2} Z_t$
Sample ACF for GARCH and SV Models (1000 reps)

(a) GARCH(1,1) Model, n=10000

(b) SV Model, n=10000
Sample ACF for Squares of GARCH and SV (1000 reps)

(a) GARCH(1,1) Model, n=10000

(b) SV Model, n=10000
Sample ACF for Squares of GARCH and SV (1000 reps)

(c) GARCH(1,1) Model, n=100000

(d) SV Model, n=100000