Multivariate Regular Variation with Application to Financial Time Series Models

Richard A. Davis
Colorado State University

Bojan Basrak
Eurandom

Thomas Mikosch
University of Groningen

Outline

+ Characteristics of some financial time series
  - IBM returns
  - NZ-USA exchange rate
+ Models for log-returns
  - GARCH
  - stochastic volatility
+ Regular variation
  - univariate case
  - multivariate case
+ Applications of multivariate regular variation
  - Stochastic recurrence equations (GARCH)
  - limit behavior of sample correlations

Characteristics of Some Financial Time Series

Define $X_t = \ln(P_t) - \ln(P_{t-1})$ (log returns)

- heavy tailed

$$P(|X_1| > x) \sim C\, x^{-\alpha}, \quad 0 < \alpha < 4.$$

- uncorrelated

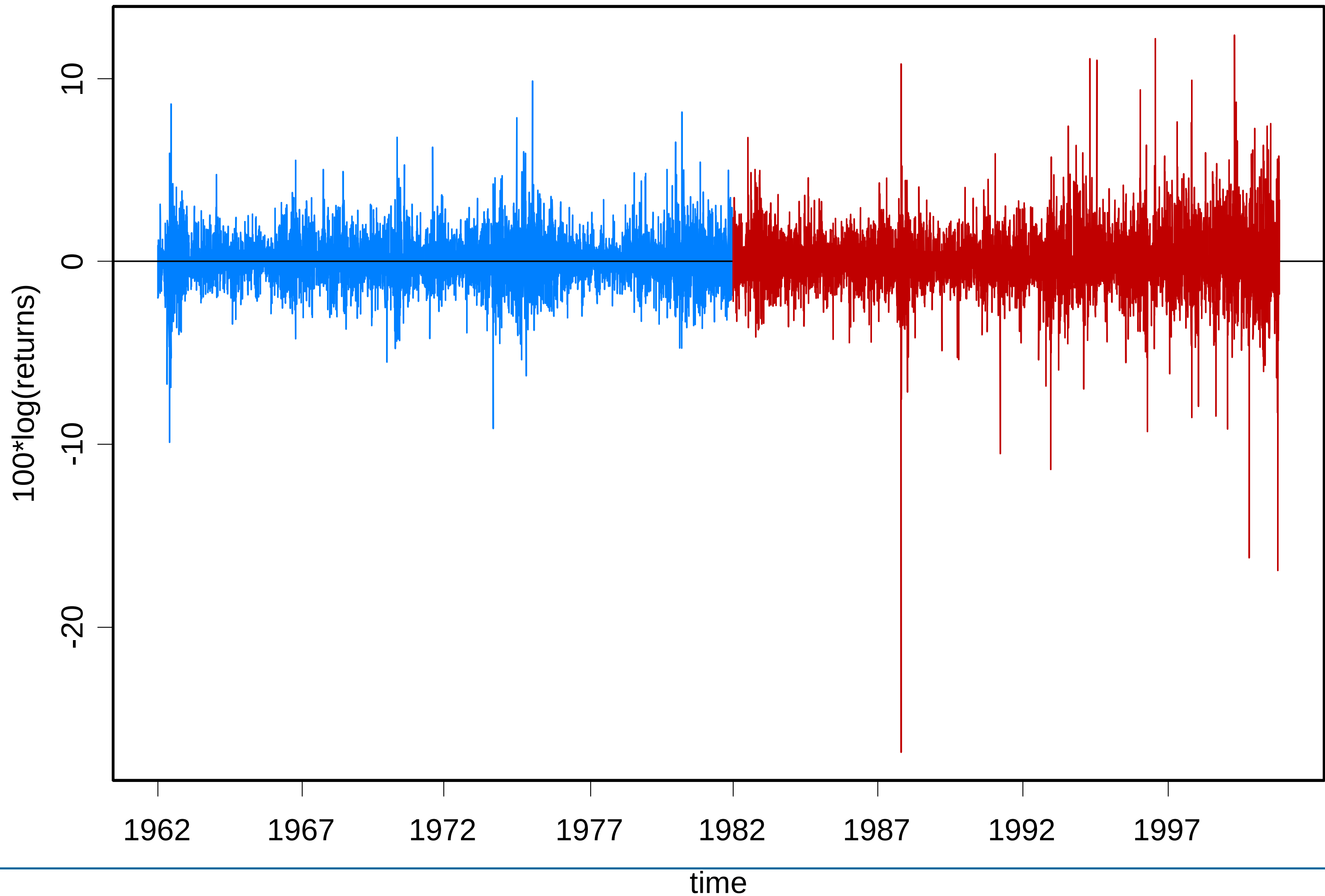
$\hat{\rho}_X(h)$ near 0 for all lags $h > 0$ (MGD sequence)

- $|X_t|$ and X_t^2 have slowly decaying autocorrelations

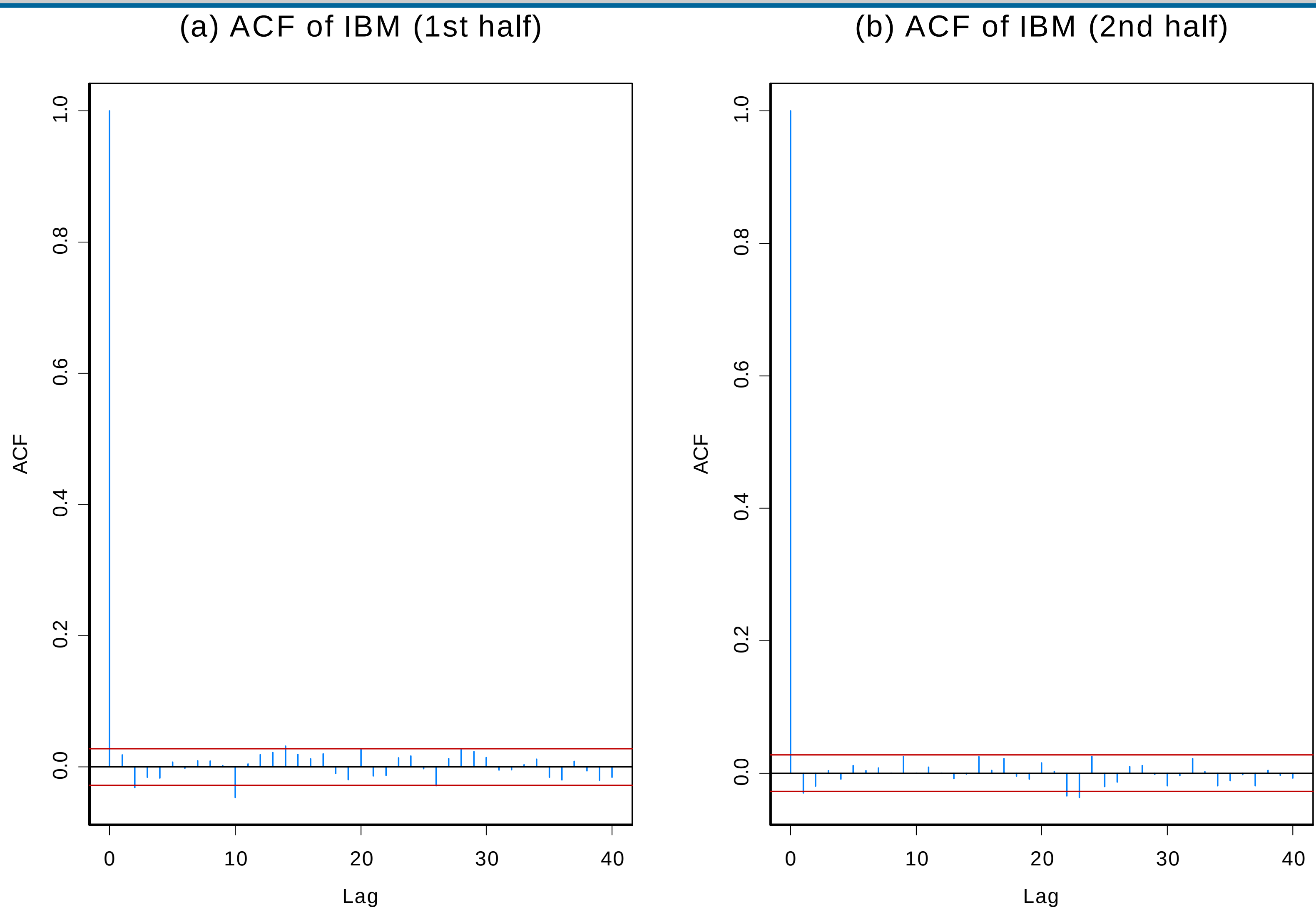
$\hat{\rho}_{|X|}(h)$ and $\hat{\rho}_{X^2}(h)$ converge to 0 *slowly* as h increases.

- process exhibits 'stochastic volatility'.

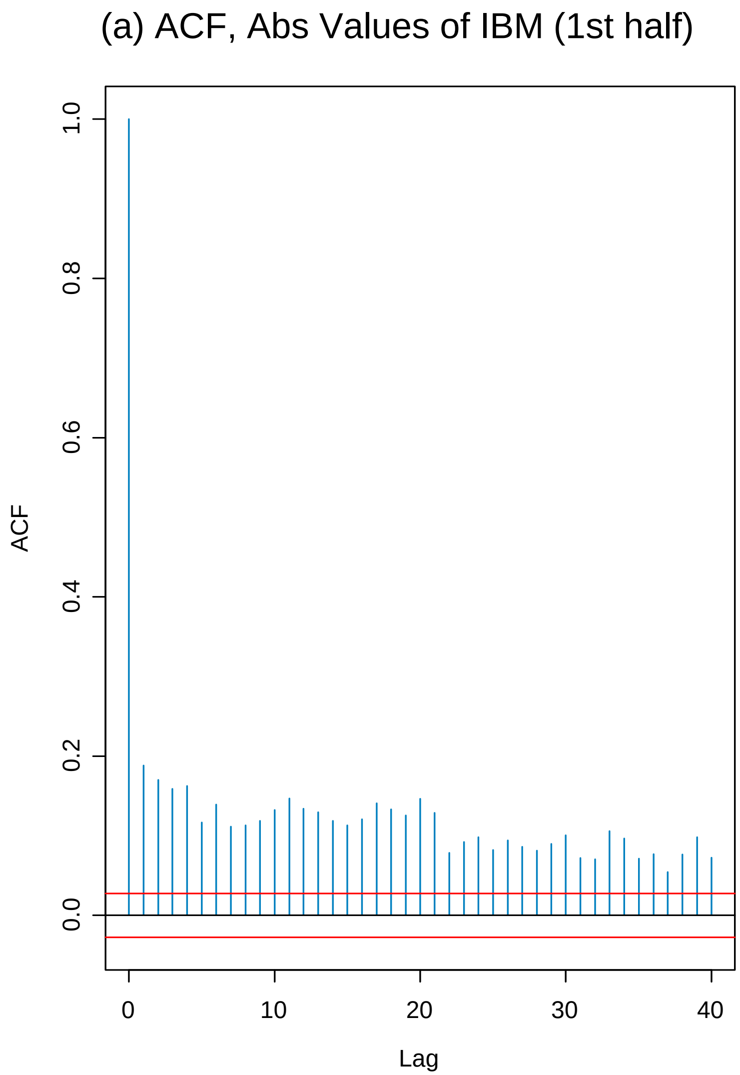
Log returns for IBM 1/3/62-11/3/00 (blue=1961-1981)

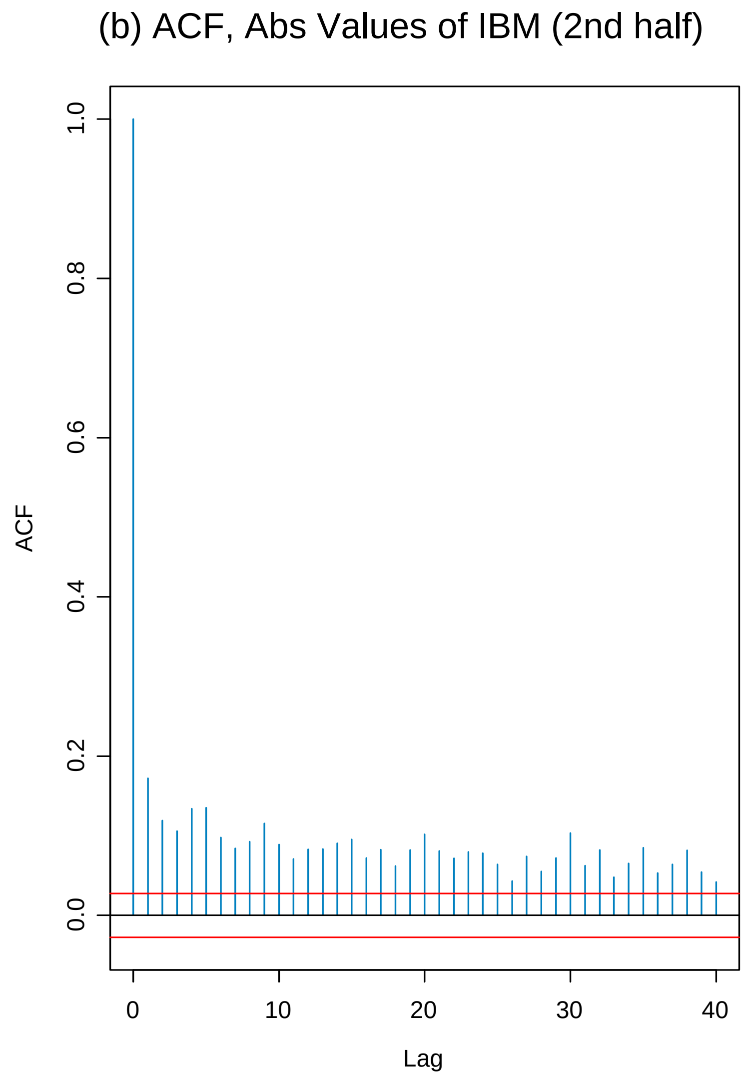


Sample ACF IBM (a) 1962-1981, (b) 1982-2000

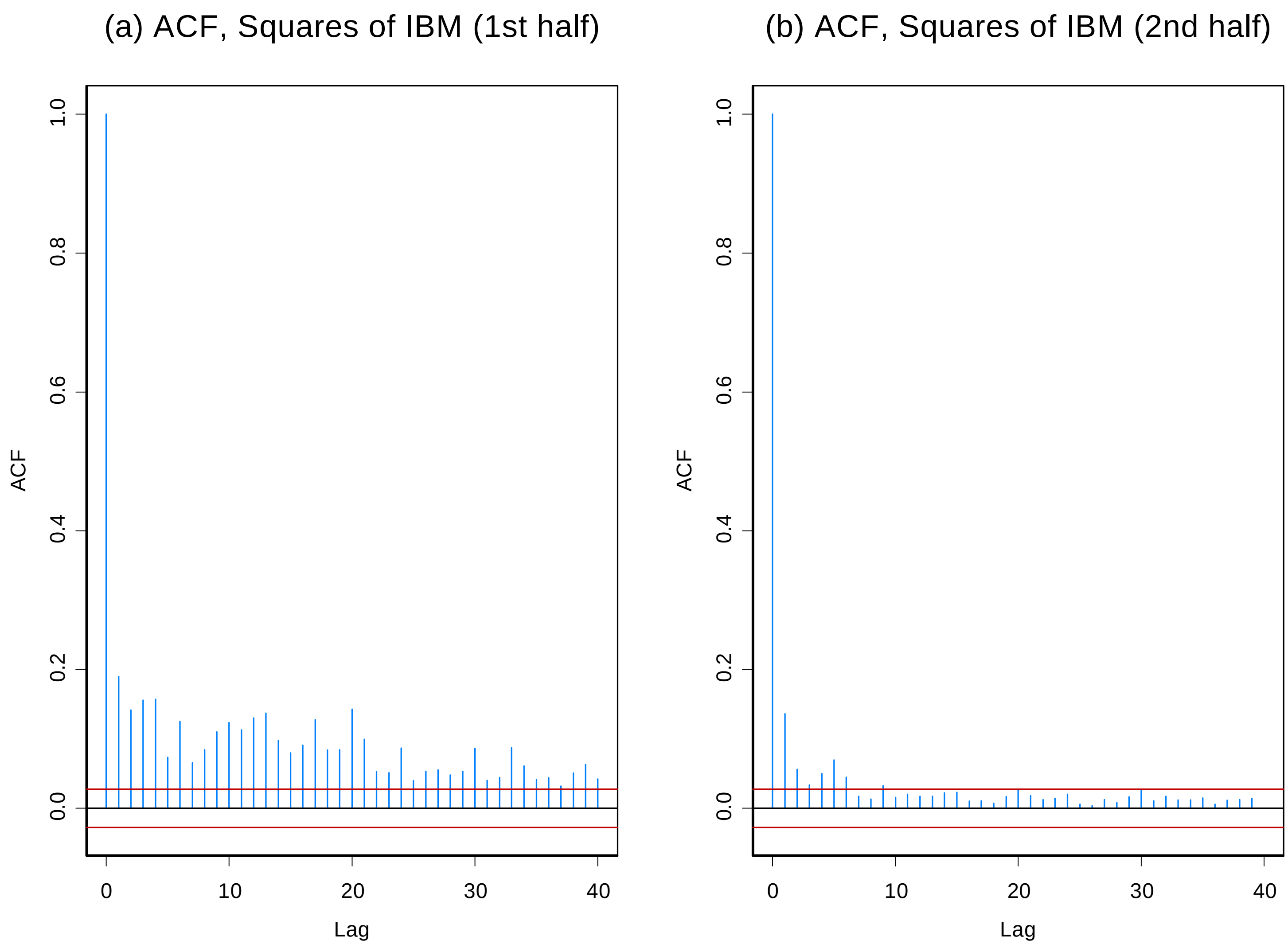


Sample ACF of abs values for IBM (a) 1961-1981, (b) 1982-2000

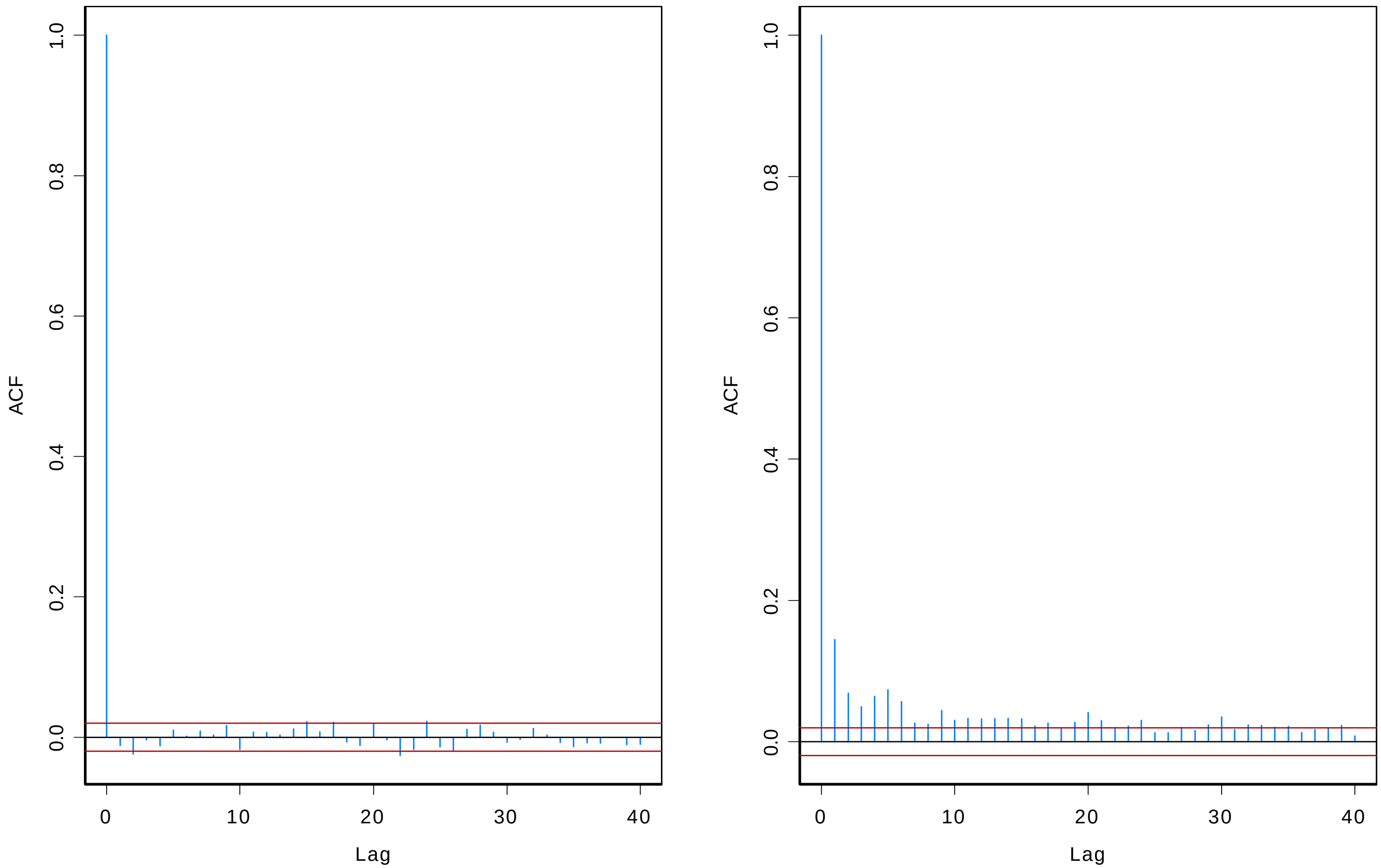




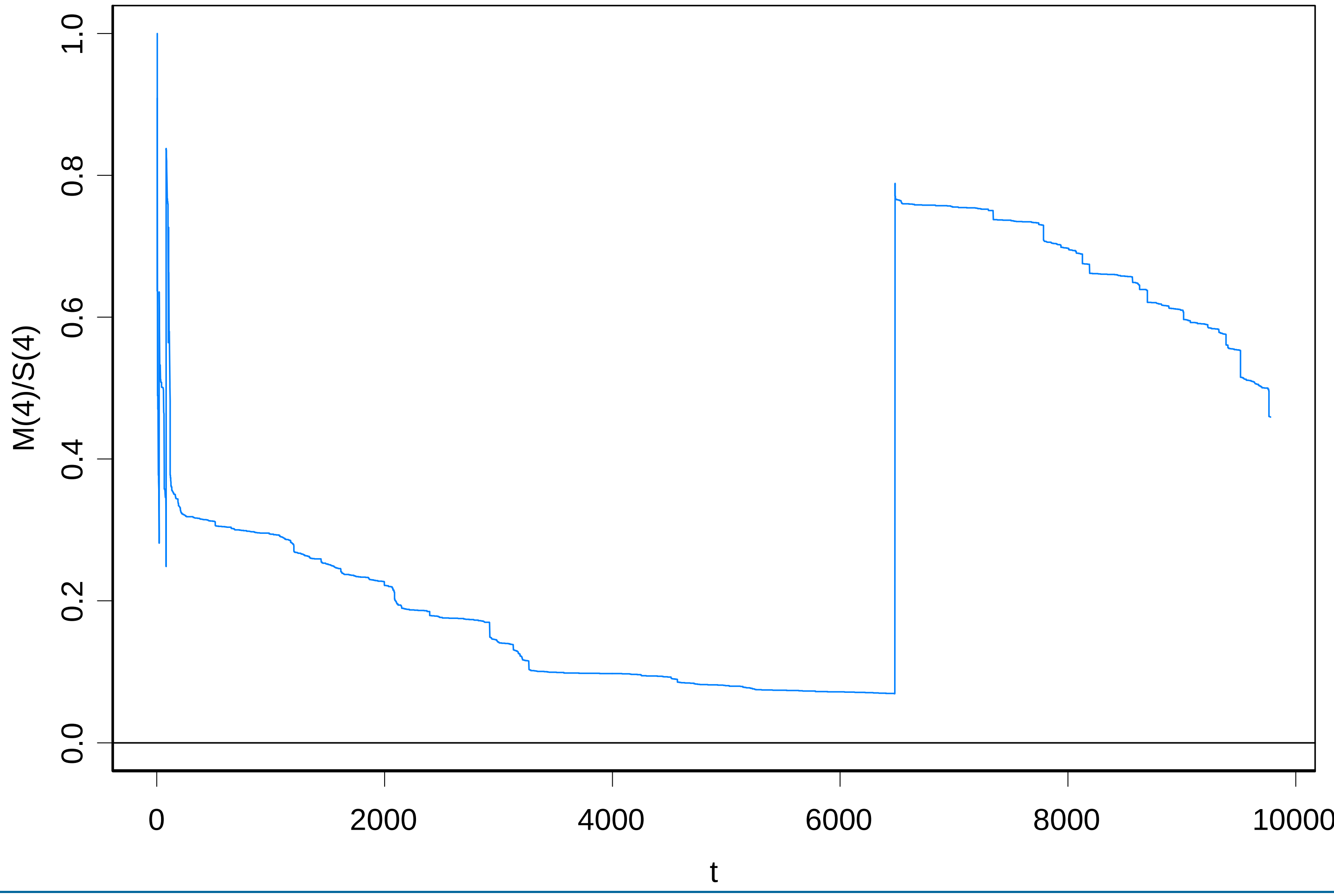
Sample ACF of squares for IBM (a) 1961-1981, (b) 1982-2000



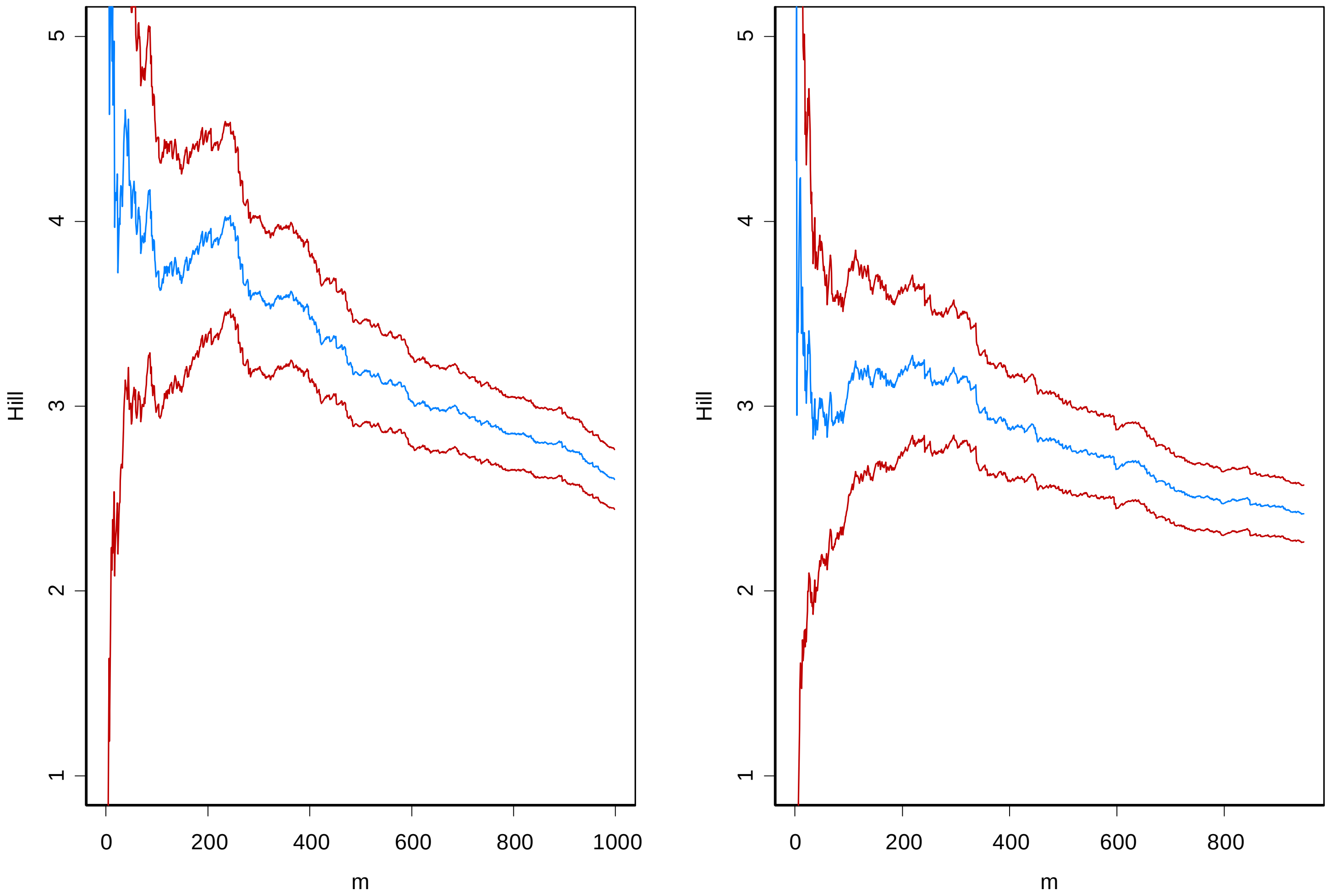
Sample ACF of original data and squares for IBM 1962-2000



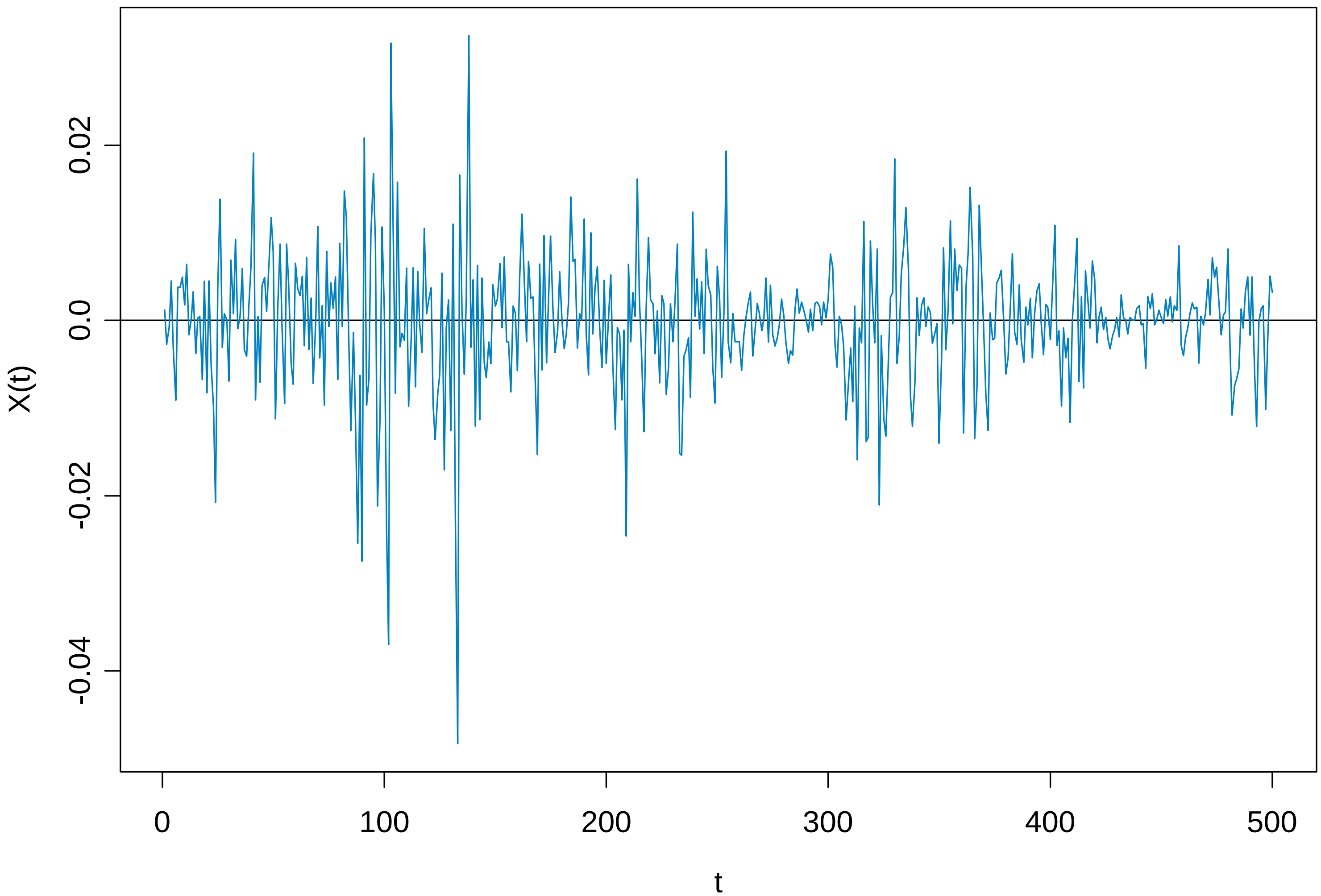
Plot of $M_t(4)/S_t(4)$ for IBM



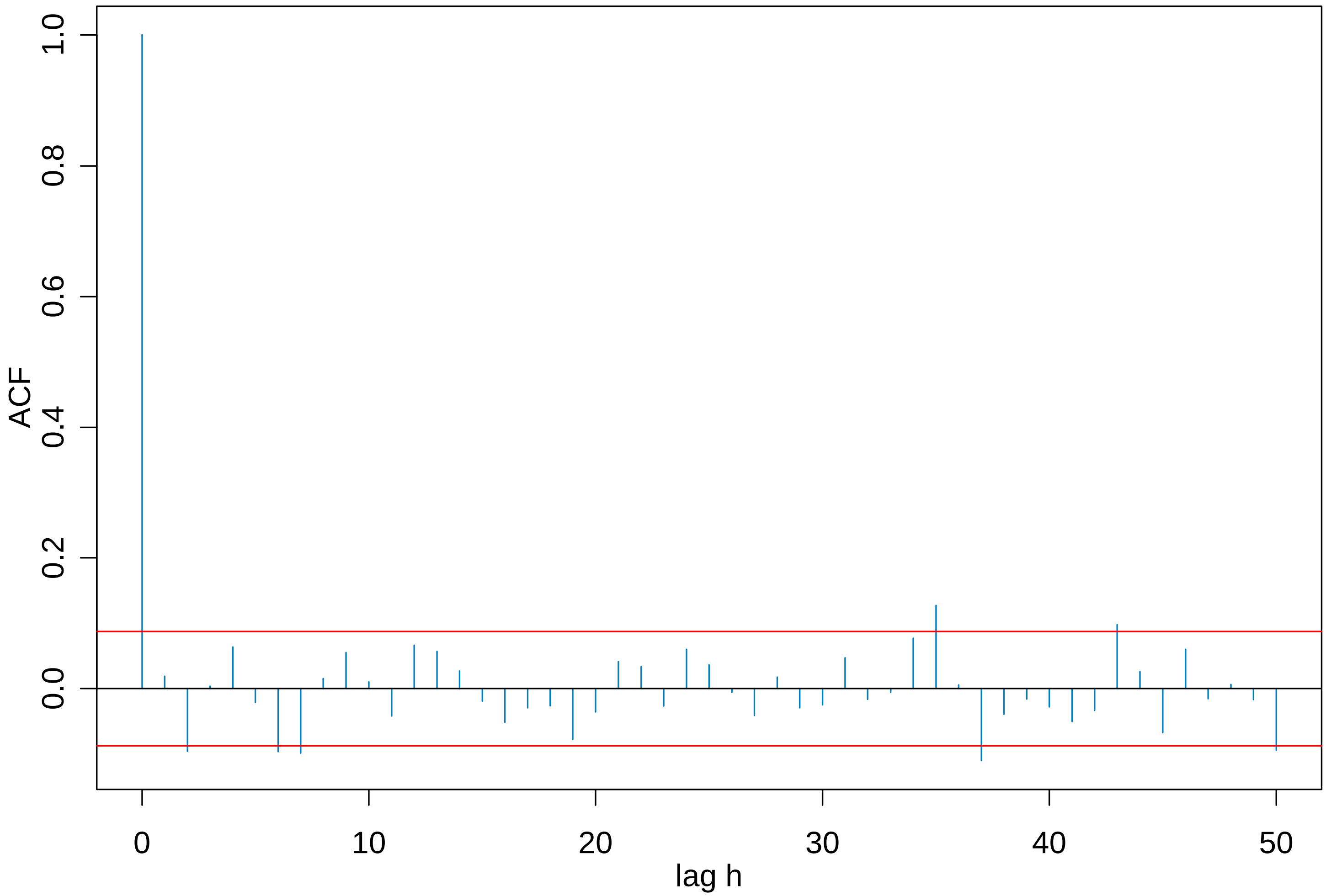
Hill's plot of tail index for IBM (1962-1981, 1982-2000)



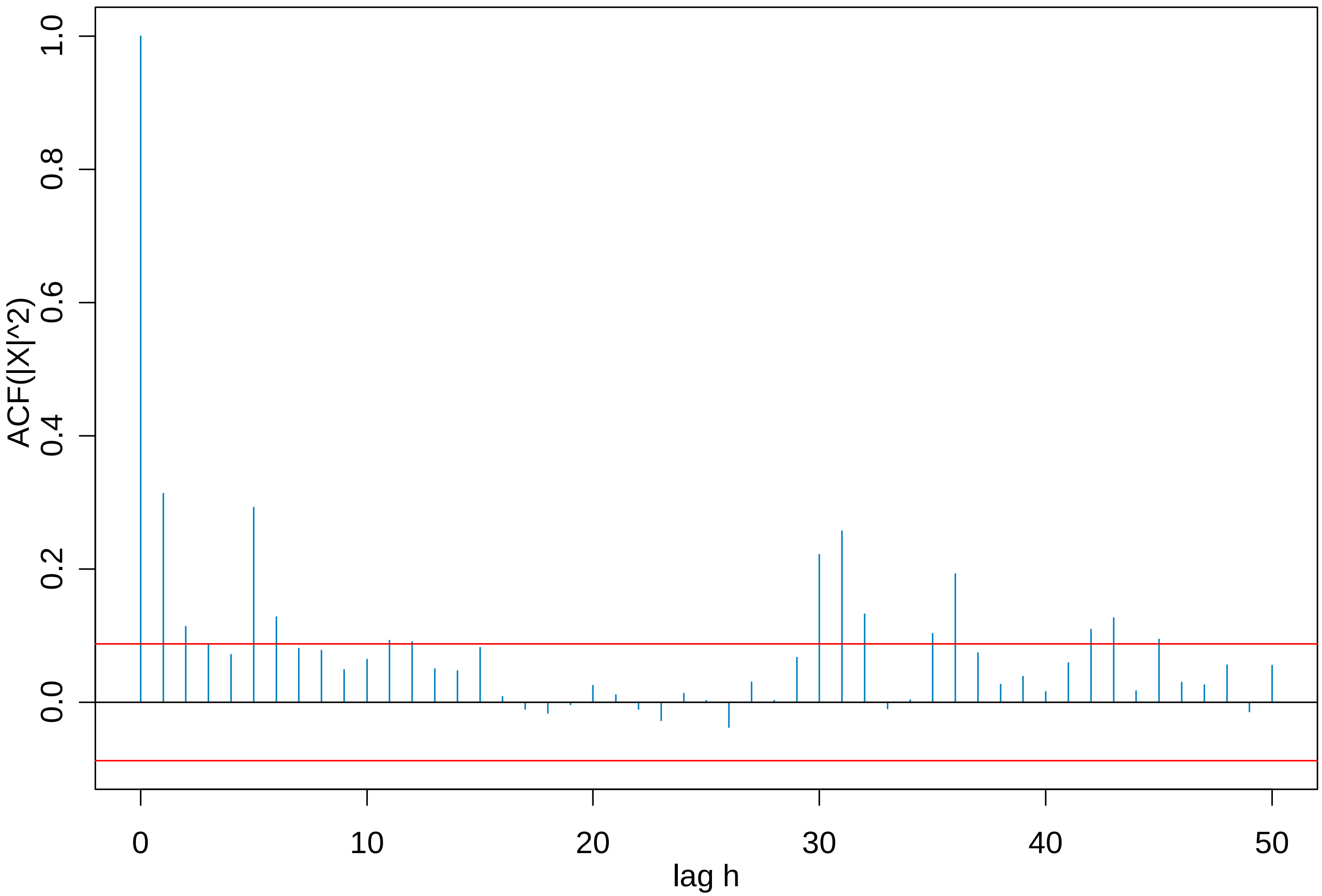
500-daily log-returns of NZ/US exchange rate



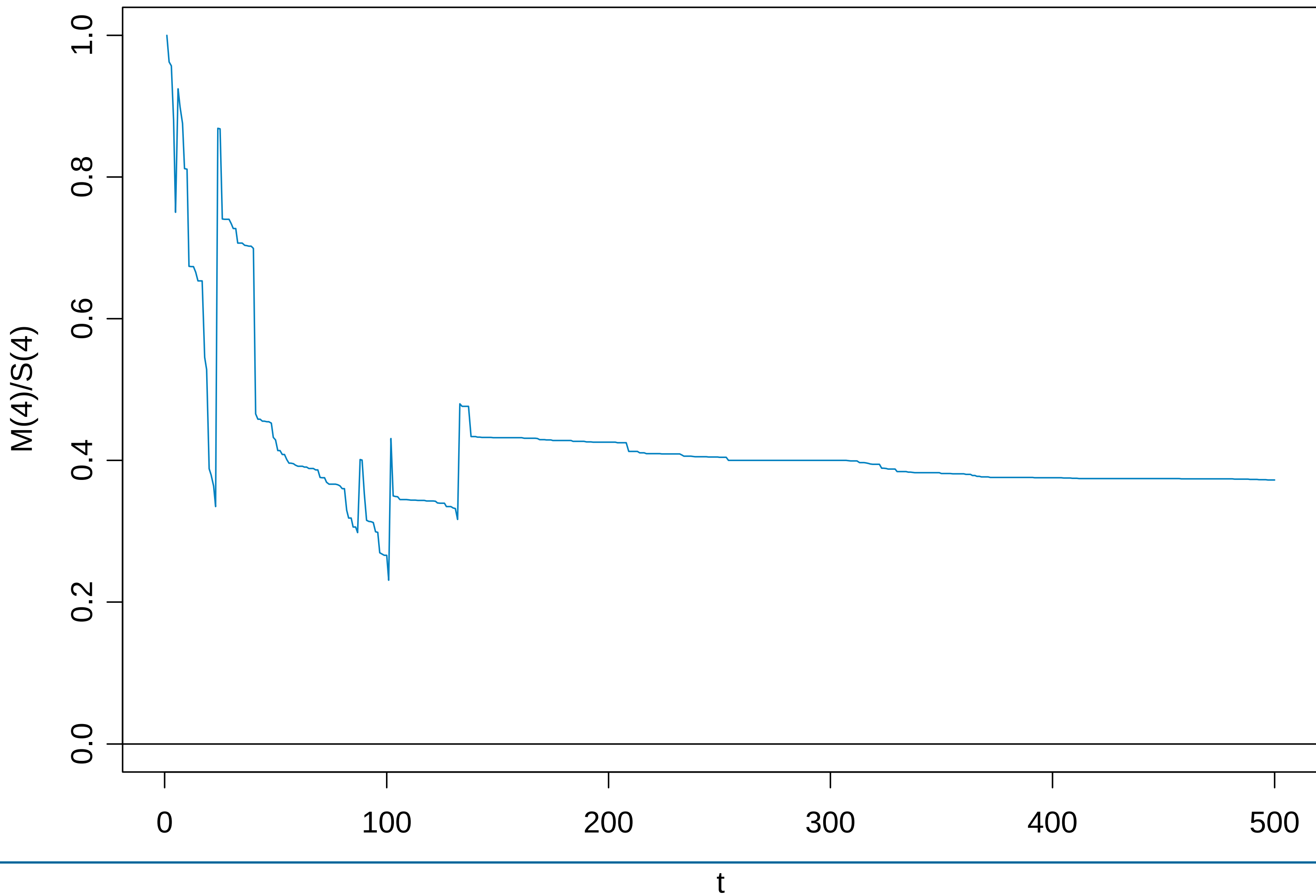
ACF of X(t)=log-returns of NZ/US exchange rate



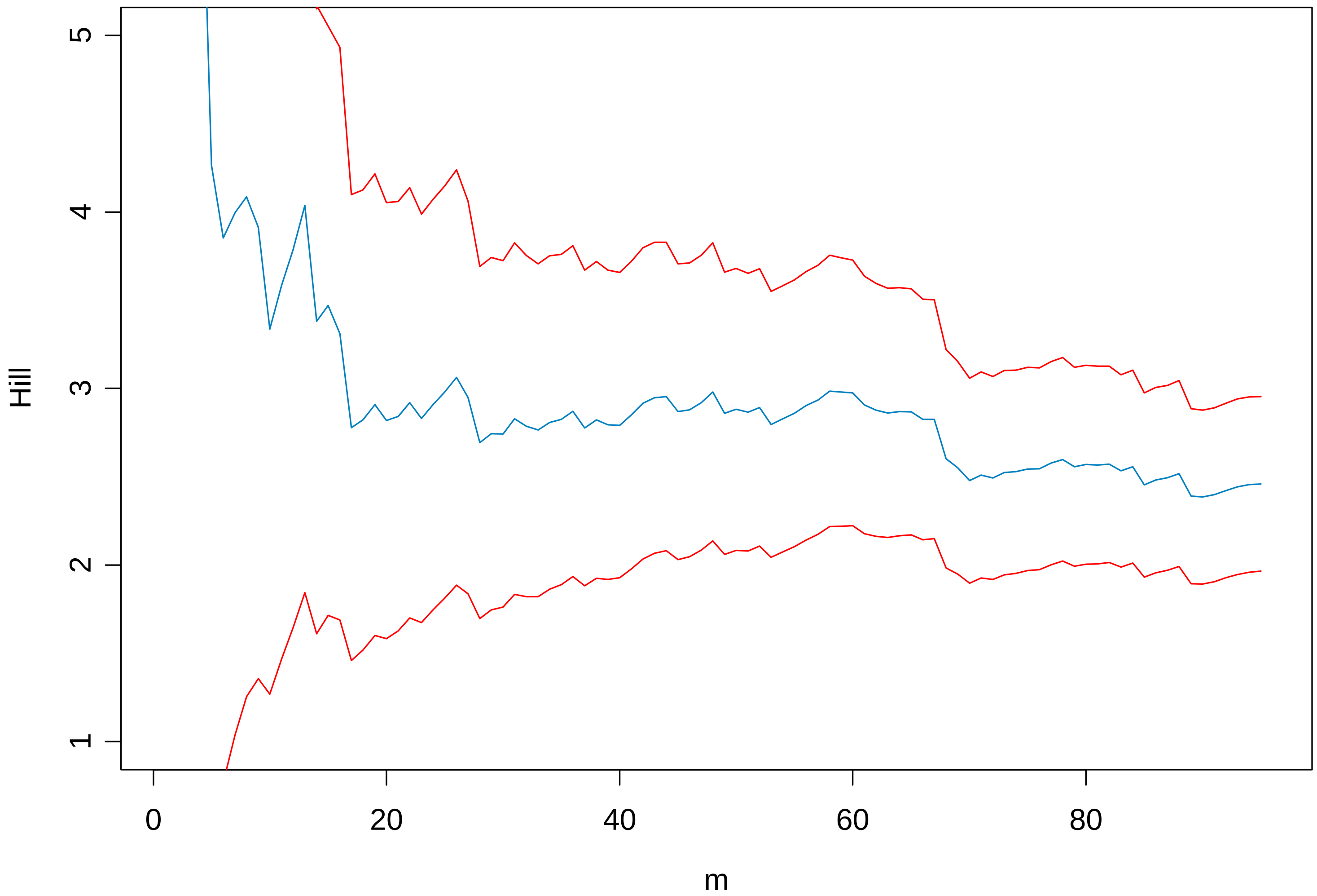
ACF of $X^2(t)$



Plot of $M_t(4)/S_t(4)$



Hill's plot of tail index



Models for Log(returns)

Basic model

$$X_t = \ln(P_t) - \ln(P_{t-1}) \quad \text{(log returns)}$$
$$= \sigma_t Z_t ,$$

where

- $\{Z_t\}$ is IID with mean 0, variance 1 (if exists). (e.g. N(0,1) or a *t*-distribution with ν df.)
- $\{\sigma_t\}$ is the volatility process
- σ_t and Z_t are independent.

Properties:

- $EX_t = 0$, $\text{Cov}(X_t, X_{t+h}) = 0$, $h>0$ (uncorrelated if $\text{Var}(X_t) < \infty$)
- conditional heteroscedastic (condition on σ_t).

Models for Log(returns)-cont

$X_t = \sigma_t Z_t$ (observation eqn in state-space formulation)

Two classes of models for volatility:

(i) GARCH(p,q) process (General AutoRegressive Conditional Heteroscedastic-observation-driven specification)

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_p X_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_q \sigma_{t-q}^2 \quad .$$

Special case: ARCH(1):

$$\begin{aligned} X_t^2 &= (\alpha_0 + \alpha_1 X_{t-1}^2) Z_t^2 \\ &= \alpha_1 Z_t^2 X_{t-1}^2 + \alpha_0 Z_t^2 \\ &= A_t X_{t-1}^2 + B_t \qquad \text{(stochastic recursion eqn)} \end{aligned}$$

$$\rho_{X^2}(h) = \alpha_1^h, \text{ if } \alpha_1^2 < 1/3.$$

Models for Log(returns)-cont

GARCH(2,1): $X_t = \sigma_t Z_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \alpha_2 X_{t-2}^2 + \beta_1 \sigma_{t-1}^2$.

Then $\mathbf{Y}_t = (X_t^2, X_{t-1}^2, \sigma_t^2)'$ follows the SRE given by

$$\begin{bmatrix} X_t^2 \\ X_{t-1}^2 \\ \sigma_t^2 \end{bmatrix} = \begin{bmatrix} \alpha_1 Z_t^2 & \alpha_2 Z_t^2 & \beta_1 Z_t^2 \\ 1 & 0 & 0 \\ \alpha_1 & \alpha_2 & \beta_1 \end{bmatrix} \begin{bmatrix} X_{t-1}^2 \\ X_{t-2}^2 \\ \sigma_{t-1}^2 \end{bmatrix} + \begin{bmatrix} \alpha_0 Z_t^2 \\ 0 \\ 0 \end{bmatrix}$$

Questions:

- Existence of a unique stationary soln to the SRE?
- Distributional properties of the stationary distribution?
- Moment properties of the process? Finite variance?

Models for Log(returns)-cont

$X_t = \sigma_t Z_t$ (observation eqn in state-space formulation)

(ii) stochastic volatility process (parameter-driven specification)

$$\log \sigma_t^2 = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}, \quad \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty, \{\varepsilon_t\} \sim \text{IID N}(0, \sigma^2)$$

$$\rho_{X^2}(h) = Cor(\sigma_t^2, \sigma_{t+h}^2) / EZ_1^4$$

Regular Variation — univariate case

Definition: The random variable X is regularly varying with index α if

$$P(|X|> t\,x)/P(|X|>t) \to x^{-\alpha} \text{ and } P(X> t)/P(|X|>t) \to p,$$

or, equivalently, if

$$P(X> t\,x)/P(|X|>t) \to px^{-\alpha} \text{ and } P(X< -t\,x)/P(|X|>t) \to qx^{-\alpha},$$

where $0 \le p \le 1$ and $p+q=1$.

Equivalence:

X is RV(α) *if and only if* $P(X \in t \bullet)/P(|X|>t) \to_v \mu(\bullet)$

($\to_v$ vague convergence of measures on $\mathbb{R}\backslash\{0\}$). In this case,

$$\mu(dx) = \big(p\alpha\, x^{-\alpha-1}\, I(x>0) + q\alpha\, (-x)^{-\alpha-1}\, I(x<0)\big)\, dx$$

Note: $\mu(tA) = t^{-\alpha}\mu(A)$.

Regular Variation — univariate case

Another formulation:

Define the ± 1 valued rv θ, $P(\theta = 1) = p$, $P(\theta = -1) = 1 - p = q$.

Then

X is RV(α) *if and only if*

$$\frac{P(|X| > t\,x,\ X/|X| \in S)}{P(|X| > t)} \to x^{-\alpha} P(\theta \in S)$$

or

$$\frac{P(|X| > t\,x,\ X/|X| \in \bullet)}{P(|X| > t)} \to_v x^{-\alpha} P(\theta \in \bullet)$$

($\to_v$ vague convergence of measures on $S^0 = \{-1,1\}$).

Regular Variation—multivariate case

Multivariate regular variation of $\mathbf{X}=(X_1, \ldots, X_m)$: There exists a random vector $\theta \in S^{m-1}$ such that

$$P(|\mathbf{X}| > t\,x,\ \mathbf{X}/|\mathbf{X}| \in \bullet\ )/P(|\mathbf{X}|>t) \to_v x^{-\alpha} P(\ \theta \in \bullet\ )$$

($\to_v$ vague convergence on S^{m-1}, unit sphere in R^m) .

- $P(\ \theta \in \bullet)$ is called the spectral measure
- α is the index of $\mathbf{X}$.

Equivalence:

$$\frac{P(\ \mathbf{X} \in t\bullet)}{P(|\ \mathbf{X}\ | > t\ )} \to_v \mu(\bullet)$$

μ is a measure on R^m which satisfies of $x > 0$ and A bounded away from 0,

$$\mu(xB) = x^{-\alpha}\mu(xA).$$

Regular Variation—multivariate case

Examples: Let X_1, X_2 be positive regularly varying with index α

1. If X_1 and X_2 are iid, then $\mathbf{X}= (X_1, X_2)$ is multivariate regularly varying with index α and spectral distribution

$$P(\theta =(0,1)) = P(\theta =(1,0)) =.5$$ (mass on axes).

Interpretation: Unlikely that X_1 and X_2 are very large at the same time.

2. If $X_1 = X_2$, then $\mathbf{X}= (X_1, X_2)$ is multivariate regularly varying with index α and spectral distribution

$$P(\theta = (1/\text{sqrt}(2), 1/\text{sqrt}(2))) = 1.$$

Regular Variation—multivariate case

Another equivalence? Suppose $\mathbf{X} > \mathbf{0}.$

MRV $\Leftrightarrow$ all linear combinations of **X** are regularly varying

i.e., if and only if

$$P(\mathbf{c}^T\mathbf{X} > t)/P(\mathbf{1}^T\mathbf{X} > t) \to w(\mathbf{c}),$$ exists for all real-valued **c**,

in which case,

$$w(t\mathbf{c}) = t^{-\alpha} w(\mathbf{c}).$$

($\Rightarrow$) true (use vague convergence with $A_{\mathbf{c}}=\{\mathbf{y}: \mathbf{c}^T\mathbf{y} > 1\}$, i.e.,

$$\frac{P(\mathbf{X} \in tA_{\mathbf{c}})}{P(\mathbf{1}^T\mathbf{X} > t)} = \frac{P(\mathbf{c}^T\mathbf{X} > t)}{P(|\mathbf{X}| > t)} \frac{P(|\mathbf{X}| > t)}{P(\mathbf{1}^T\mathbf{X} > t)} \to \frac{\mu(A_{\mathbf{c}})}{\mu(A_{\mathbf{1}})} =: w(\mathbf{c})$$

Regular Variation—multivariate case

($\Leftarrow$) established by Basrak, Davis and Mikosch (2000) for α **not** an even integer—case of even integer is unknown.

Idea of argument: Define the measures

$$m_t(\bullet) = P(\mathbf{X} \in t\bullet)/P(\mathbf{1}^T\mathbf{X} > t)$$

- By assumption we know that for fixed $\mathbf{x}$, $m_t(A_\mathbf{x}) \to \mu(A_\mathbf{x})$
- $\{m_t\}$ is tight: For B bded away from 0, $\sup_t m_t(B) < \infty$.
- Do subsequential limits of $\{m_t\}$ coincide?

  If $m_{t'} \to_v \mu_1$ and $m_{t''} \to_v \mu_2$, then

  $\mu_1(A_\mathbf{x}) = \mu_2(A_\mathbf{x})$ for all $\mathbf{x} \neq \mathbf{0}$.

Problem: Need $\mu_1 = \mu_2$ but only have equality on $A_\mathbf{x}$ not a π-system.

Overcome this using transform theory.

Applications of Multivariate Regular Variation

- Domain of attraction for sums of iid random vectors (Rvaceva, 1962). That is, when does the partial sum

$$a_n^{-1}\sum_{t=1}^{n} \mathrm{X}_\mathrm{t}$$

converge for some constants a_n?

- Domain of attraction for componentwise maxima of iid random vectors (Resnick, 1987). Limit behavior of

$$a_n^{-1}\bigvee_{t=1}^{n} \mathbf{X}_\mathrm{t}$$

- Weak convergence of point processes with iid points.
- Solution to stochastic recurrence equations, $\mathbf{Y}_t = \mathbf{A}_t \mathbf{Y}_{t-1} + \mathbf{B}_t$
- Weak convergence of sample autocovaraınces.

Point Processes With IID Vectors

Theorem Let $\{\mathbf{X}_t\}$ be an iid sequence of random vectors that are multivariate regularly varying. Then we have the following point process convergence

$$N_n := \sum_{t=1}^{n} \varepsilon_{\mathbf{X}_t/a_n} \xrightarrow{d} N := \sum_{j=1}^{\infty} \varepsilon_{P_i \theta_i},$$

where $\{a_n\}$ satisfies $nP(|\mathbf{X}_t| > a_n) \to 1$, and N is a Poisson process with intensity measure μ.

- $\{P_i\}$ are Poisson pts corresponding to the radial part (intensity measure $\alpha\, x^{-\alpha-1}\,(dx)$.
- $\{\theta_i\}$ are iid with the spectral distribution given by the MRV.

Applications—stochastic recurrence equations

$$\mathbf{Y}_t = \mathbf{A}_t\, \mathbf{Y}_{t-1} + \mathbf{B}_t, \qquad (\mathbf{A}_t, \mathbf{B}_t) \sim \text{IID},$$

$\mathbf{A}_t$ $d\times d$ random matrices, $\mathbf{B}_t$ random d-vectors

Examples

ARCH(1): $X_t = (\alpha_0 + \alpha_1 X^2_{t-1})^{1/2} Z_t, \quad \{Z_t\} \sim \text{IID}$. Then the squares follow an SRE with $Y_t = X_t^2,\ \ A_t = \alpha_1 Z_t^2,\ \ B_t = \alpha_0 Z_t^2$.

GARCH(2,1): $X_t = \sigma_t Z_t,\ \ \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \alpha_2 X_{t-2}^2 + \beta_1 \sigma_{t-1}^2$.
Then $\mathbf{Y}_t = (X_t^2, X_{t-1}^2, \sigma_t^2)'$ follows the SRE given by

$$\begin{bmatrix} X_t^2 \\ X_{t-1}^2 \\ \sigma_t^2 \end{bmatrix} = \begin{bmatrix} \alpha_1 Z_t^2 & \alpha_2 Z_t^2 & \beta_1 Z_t^2 \\ 1 & 0 & 0 \\ \alpha_1 & \alpha_2 & \beta_1 \end{bmatrix} \begin{bmatrix} X_{t-1}^2 \\ X_{t-2}^2 \\ \sigma_{t-1}^2 \end{bmatrix} + \begin{bmatrix} \alpha_0 Z_t^2 \\ 0 \\ 0 \end{bmatrix}$$

Stochastic Recurrence Equations (cont)

Regular variation of the marginal distribution (Kesten)

Assume **A** and **B** have non-negative entries and

- $E\,\|\mathbf{A}_1\|^{\varepsilon} < 1$ for some $\varepsilon > 0$
- $\mathbf{A}_1$ has no zero rows a.s.
- W.P. 1, $\{\ln \rho(\mathbf{A}_1 \ldots \mathbf{A}_n)$: is dense in $\mathbf{R}$ for some $n, \mathbf{A}_1 \ldots \mathbf{A}_n > 0\}$
- There exists a $\kappa_0 > 0$ such that $E\|A\|^{\kappa_0} \ln^+\|A\| < \infty$ and

$$E\left(\min_{i=1,\ldots,d} \sum_{j=1}^{d} A_{ij}\right)^{\kappa_0} \geq d^{\kappa_0/2}$$

Then there exists a $\kappa_1 \in (0, \kappa_0]$ such that **all** linear combinations of **Y** are regularly varying with index κ_1. (Also need $E\,|B|^{\kappa_1} < \infty$.)

Application to GARCH

Proposition: Let $(\mathbf{Y}_t)$ be the soln to the SRE based on the *squares* of a GARCH model. Assume

- Top Lyapunov exponent $\gamma < 0$. (See Bougerol and Picard`92)
- Z has a positive density on $(-\infty, \infty)$ with all moments finite or $E|Z|^h = \infty$, for all $h \geq h_0$ and $E|Z|^h < \infty$ for all $h < h_0$.
- Not all the GARCH parameters vanish.

Then $(\mathbf{Y}_t)$ is *strongly mixing* with geometric rate and all finite dimensional distributions are *multivariate regularly varying* with index κ_1.

Corollary: The corresponding GARCH process is strongly mixing and has all finite dimensional distributions that are MRV with index $\kappa = 2\kappa_1$.

Application to GARCH (cont)

Remarks:

1. Kesten's result applied to an iterate of $\mathbf{Y}_t$, i.e., $\mathbf{Y}_{tm} = \tilde{\mathbf{A}}_t \mathbf{Y}_{(t-1)m} + \tilde{\mathbf{B}}_t$

2. Determination of κ is difficult. Explicit expressions only known in two(?) cases.

- ARCH(1): $E|\alpha_1 Z^2|^{\kappa/2} = 1$.

α_1	.312	.577	1.00	1.57
κ	8.00	4.00	2.00	1.00

- GARCH(1,1): $E|\alpha_1 Z^2 + \beta_1|^{\kappa/2} = 1$ (Mikosch and St→ric→)
- For IGARCH ($\alpha_1 + \beta_1 = 1$), then $\kappa = 2 \Rightarrow$ infinite variance.
- Can estimate κ empirically by replacing expectations with sample moments.

Summary for GARCH(p,q)

$\kappa \in (0,2)$:

$$(\hat{\rho}_X(h))_{h=1,\ldots,m} \xrightarrow{d} (V_h / V_0)_{h=1,\ldots,m},$$

$\kappa \in (2,4)$:

$$\left(n^{1-2/\kappa}\hat{\rho}_X(h)\right)_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1}(0)\left(V_h\right)_{h=1,\ldots,m}.$$

$\kappa \in (4,\infty)$:

$$\left(n^{1/2}\hat{\rho}_X(h)\right)_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1}(0)\left(G_h\right)_{h=1,\ldots,m}.$$

Remark: Similar results hold for the sample ACF based on $|X_t|$ and X_t^2.

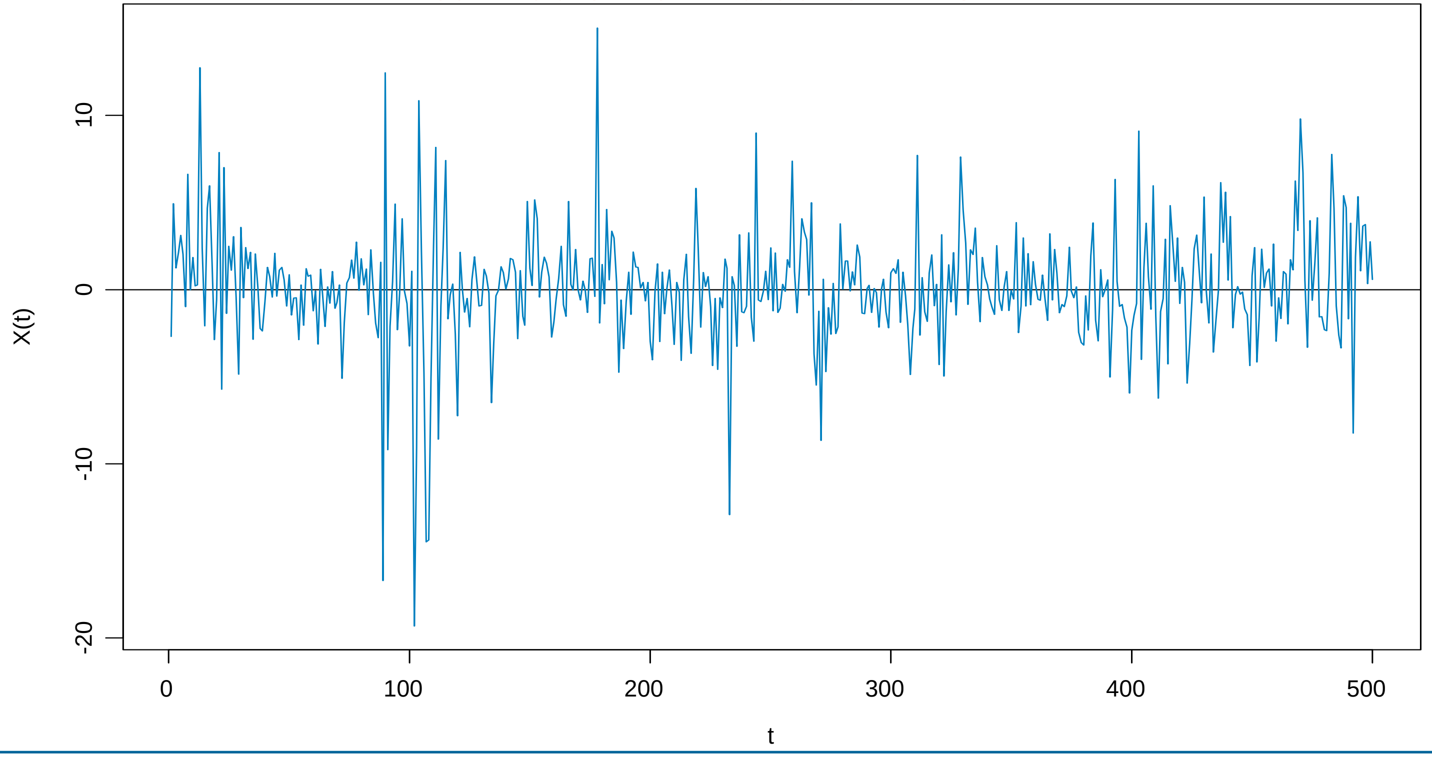
Realization of GARCH Process

Fitted GARCH(1,1) model for NZ-USA exchange:

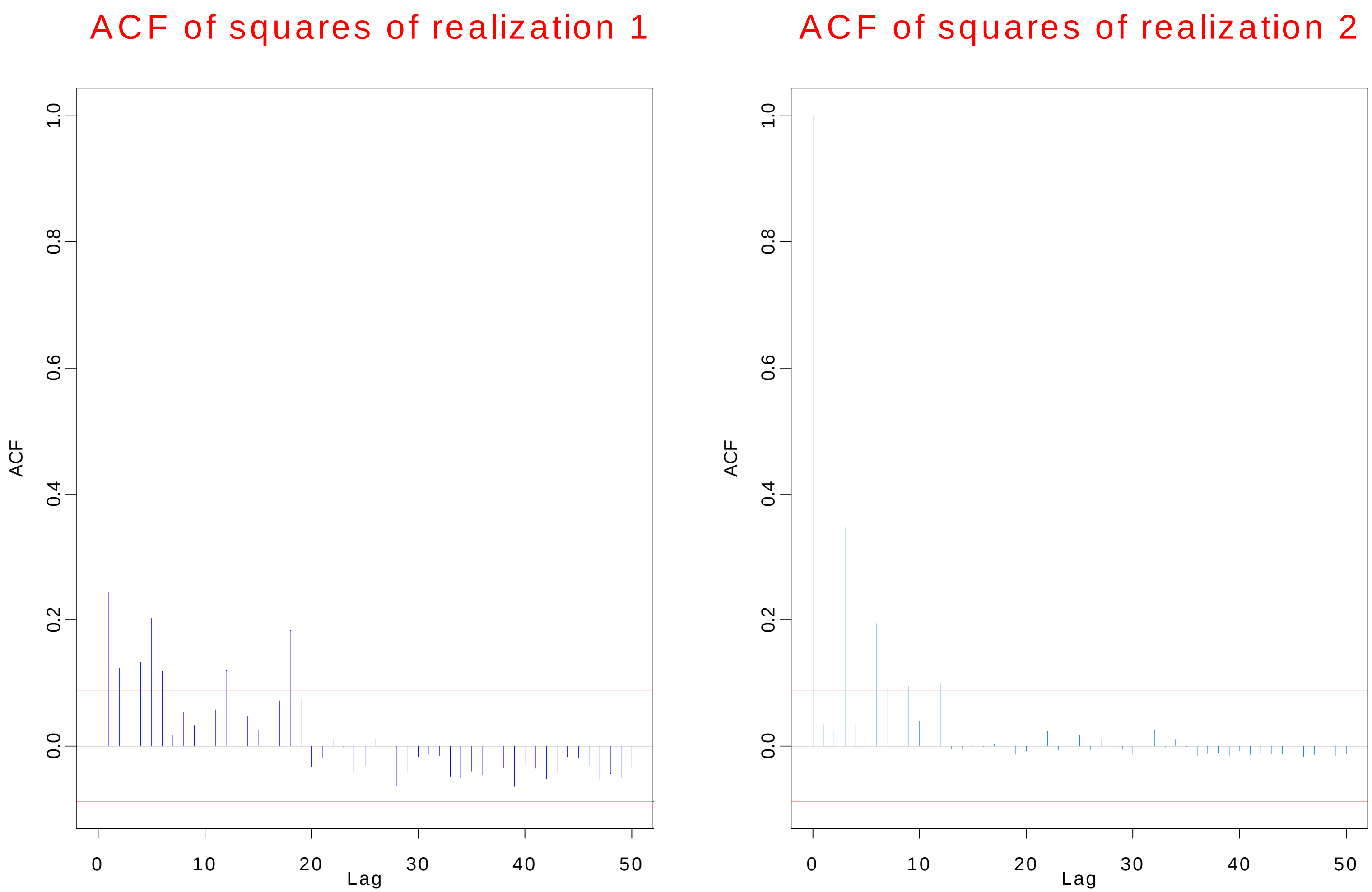
$$X_t = \sigma_t Z_t, \quad \sigma_t^2 = (6.70)10^{-7} + .1519 X_{t\text{-}1}^2 + .772 \sigma_{t\text{-}1}^2$$

$(Z_t) \sim$ IID t-distr with 5 df. κ is approximately 3.8

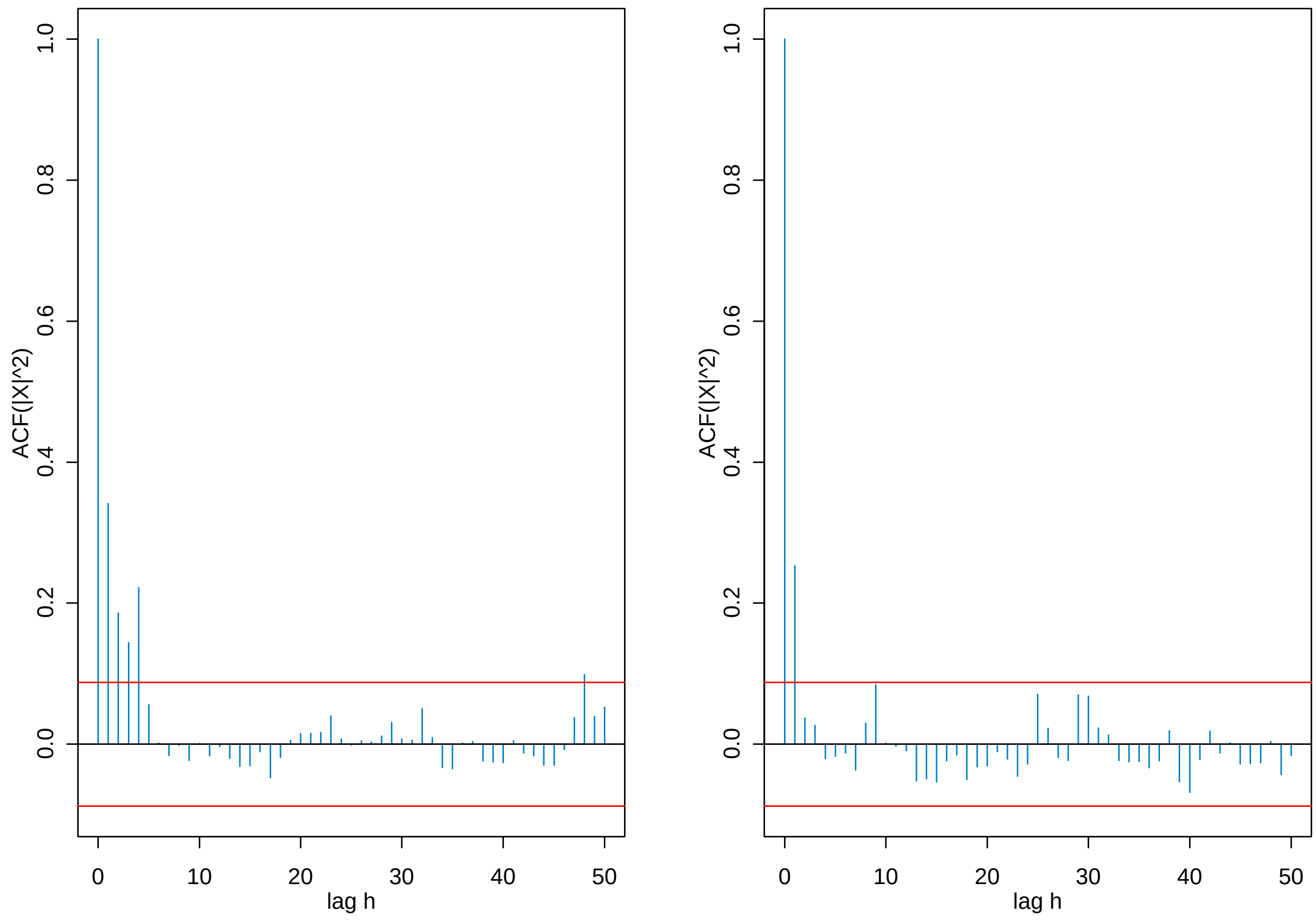
Realization of fitted GARCH



ACF of Fitted GARCH(1,1) Process

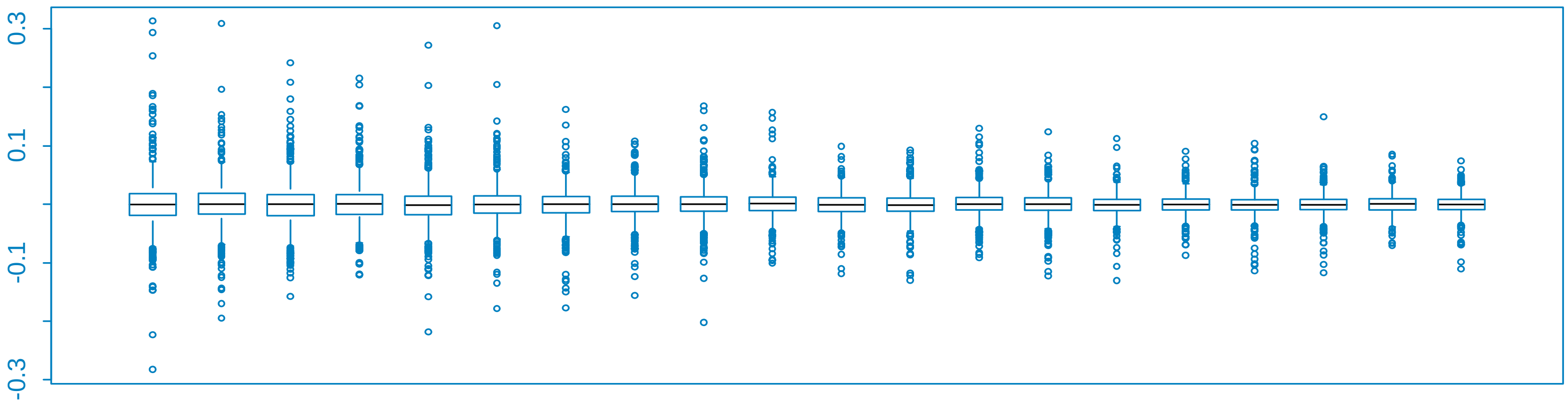


ACF of 2 realizations of an (ARCH)^2: $X_t=(.001+.7\,X_{t-1})^{1/2}\,Z_t$

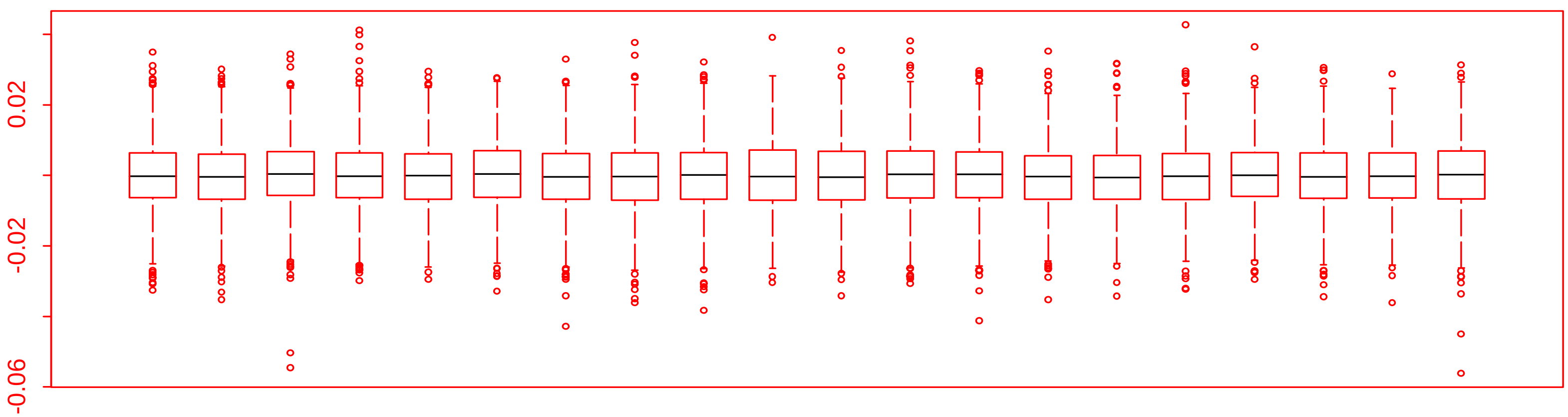


Sample ACF for GARCH and SV Models (1000 reps)

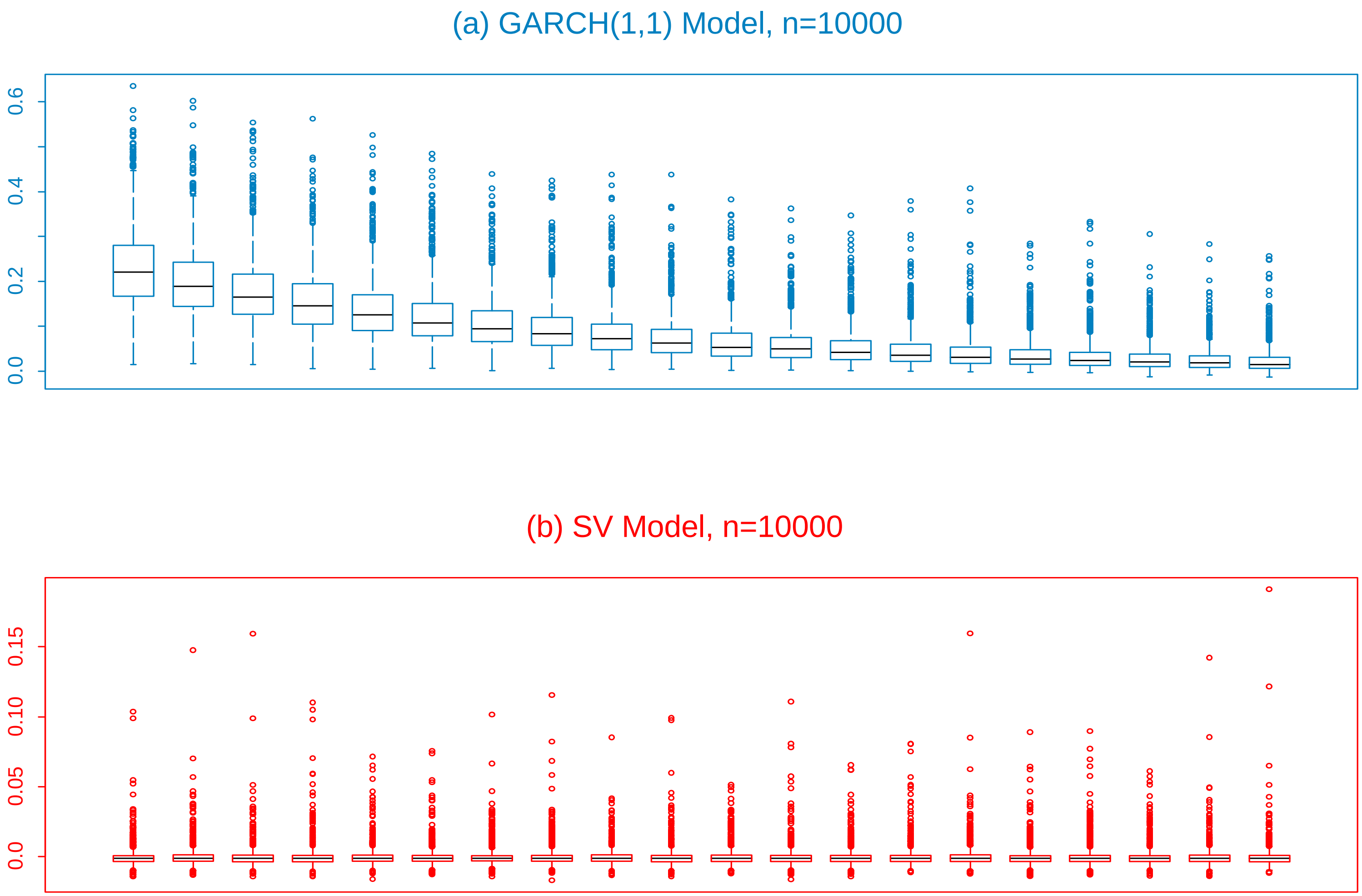
(a) GARCH(1,1) Model, n=10000



(b) SV Model, n=10000



Sample ACF for Squares of GARCH and SV (1000 reps)



Sample ACF for Squares of GARCH and SV (1000 reps)