Sample Autocorrelations for Financial Time Series Models

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Outline

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- Applications
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Characteristics of Some Financial Time Series

Define \( X_t = 100 \times (\ln (P_t) - \ln (P_{t-1})) \) (log returns)

- heavy tailed

\[ P(\left| X_1 \right| > x) \sim C \ x^{-\alpha}, \quad 0 < \alpha < 4. \]

- uncorrelated

\[ \hat{\rho}_x (h) \] near 0 for all lags \( h > 0 \) (MGD sequence)

- \( |X_t| \) and \( X_t^2 \) have slowly decaying autocorrelations

\[ \hat{\rho}_{|X|} (h) \) and \( \hat{\rho}_{X^2} (h) \) converge to 0 slowly as \( h \) increases.

- process exhibits ‘stochastic volatility’.
Sample ACF IBM (a) 1962-1981, (b) 1982-2000

(a) ACF of IBM (1st half)

(b) ACF of IBM (2nd half)
Sample ACF of abs values for IBM (a) 1961-1981, (b) 1982-2000
Sample ACF of squares for IBM (a) 1961-1981, (b) 1982-2000

(a) ACF, Squares of IBM (1st half)

(b) ACF, Squares of IBM (2nd half)
Sample ACF of original data and squares for IBM 1962-2000
Plot of $M_t(4)/S_t(4)$ for IBM
500-daily log-returns of NZ/US exchange rate
ACF of $X(t)=\text{log-returns of NZ/US exchange rate}$
ACF of $X^2(t)$
Plot of $M_t(4)/S_t(4)$
Hill’s plot of tail index
Models for Log(returns)

Basic model

\[ X_t = 100 \times (\ln (P_t) - \ln (P_{t-1})) \]  \hspace{1cm} \text{(log returns)}

\[ = \sigma_t Z_t , \]

where

- \{Z_t\} is IID with mean 0, variance 1 (if exists). (e.g. N(0,1) or a \(t\)-distribution with \(\nu\) df.)
- \{\sigma_t\} is the volatility process
- \(\sigma_t\) and \(Z_t\) are independent.
Models for Log(returns)-cont

\[ X_t = \sigma_t Z_t \]  (observation eqn in state-space formulation)

Examples of models for volatility:

(i) GARCH(p,q) process (observation-driven specification)

\[ \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_p X_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_q \sigma_{t-q}^2. \]

Special case: ARCH(1), \[ X_t^2 = (\alpha_0 + \alpha_1 X_{t-1}^2)Z_t^2. \]

\[ \rho_{X^2}(h) = \alpha_1^h, \text{ if } \alpha_1^2 < 1/3. \]

(ii) stochastic volatility process (parameter-driven specification)

\[ \log \sigma_t^2 = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}, \quad \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty, \{ \varepsilon_t \} \sim \text{IID } N(0, \sigma^2) \]

\[ \rho_{X^2}(h) = \text{Cor}(\sigma_t^2, \sigma_{t+h}^2) / EZ_1^4 \]
**Linear Processes**

**Model:** \( X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j} \) \( \{Z_t\} \sim \text{IID}, \ P(\mid Z_t \mid > x) \sim C x^{-\alpha}, \ 0<\alpha<2. \)

**Properties:**

- \( P(\mid X_t \mid > x) \sim C_2 x^{-\alpha} \)
- Define \( \rho(h) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} \big/ \sum_{j=-\infty}^{\infty} \psi_j^2 \).

**Case \( \alpha > 2: \)**
\[
 n^{1/2} (\hat{\rho}(h) - \rho(h)) \xrightarrow{d} \sum_{j=1}^{\infty} (\rho(h+j)+\rho(h-j)-2\rho(j)\rho(h)) N_j, \ \{N_t\} \sim \text{IIDN}
\]

**Case \( 0 < \alpha < 2: \)**
\[
 (n / \ln n)^{1/\alpha} (\hat{\rho}(h) - \rho(h)) \xrightarrow{d} \sum_{j=1}^{\infty} (\rho(h+j)+\rho(h-j)-2\rho(j)\rho(h)) S_j / S_0, \quad \{S_t\} \sim \text{IID stable (\alpha)}, \ S_0 \text{ stable (\alpha/2)}
\]
Background Results—multivariate regular variation

**Multivariate regular variation of** $\mathbf{X}=(X_1, \ldots, X_m)$: There exists a random vector $\mathbf{\theta} \in S^{m-1}$ such that

$$P(|\mathbf{X}| > tx, \mathbf{X}/|\mathbf{X}| \in \bullet) / P(|\mathbf{X}| > t) \rightarrow_v x^{-\alpha} P(\mathbf{\theta} \in \bullet)$$

($\rightarrow_v$ vague convergence on $S^{m-1}$).

- $P(\mathbf{\theta} \in \bullet)$ is called the spectral measure
- $\alpha$ is the index of $\mathbf{X}$.

**Equivalence:** There exist positive constants $a_n$ and a measure $\mu$,

$$nP(\mathbf{X}/ a_n \in \bullet) \rightarrow_v \mu(\bullet)$$

In this case, one can choose $a_n$ and $\mu$ such that

$$\mu((x, \infty) \times B) = x^{-\alpha} P(\mathbf{\theta} \in B)$$
Another equivalence?

MRV $\iff$ all linear combinations of $\mathbf{X}$ are regularly varying

i.e., if and only if

$$P(c^T \mathbf{X} > t)/P(1^T \mathbf{X} > t) \to w(c), \text{ exists for all real-valued } c,$$

in which case,

$$w(tc) = t^{-\alpha}w(c).$$

$(\Rightarrow)$ true

$(\Leftarrow)$ established by Basrak, Davis and Mikosch (2000) for $\alpha$ not an even integer—case of even integer is unknown.
Theorem (Davis & Hsing `95, Davis & Mikosch `97). Let \( \{X_t\} \) be a stationary sequence of random vectors. Suppose

(i) finite dimensional distributions are jointly regularly varying (let \((\theta_{-k}, \ldots, \theta_k)\) be the vector in \(S^{(2k+1)m-1}\) in the definition).

(ii) mixing condition \(|a_n|\) or strong mixing.

(iii) \( \lim_{k \to \infty} \limsup_{n \to \infty} P(\bigvee_{k \leq |\ell| \leq r_n} |X_k| > a_n y | |X_0| > a_n y) = 0. \)

Then

\[
\gamma = \lim_{k \to \infty} E\left(\left|\theta_0^{(k)}\right|^\alpha - \bigvee_{j=1}^k \left|\theta_j^{(k)}\right|\right)_+ / E\left|\theta_0^{(k)}\right|^\alpha
\]

exists. If \(\gamma > 0\), then

\[
N_n := \sum_{t=1}^n \varepsilon_{X_t / a_n} \xrightarrow{d} N := \sum_{i=1}^\infty \sum_{j=1}^\infty \varepsilon_{P_i Q_{ij}},
\]
where

- $(P_i)$ are points of a Poisson process on $(0, \infty)$ with intensity function \( \nu(dy) = \gamma \alpha y^{-\alpha-1} dy \).

- $\sum_{j=1}^{\infty} \mathcal{E}_{Q_{ij}}$, $i \geq 1$, are iid point process with distribution $Q$, and $Q$ is the weak limit of

\[
\lim_{k \to \infty} E\left( |\theta_0^{(k)}|^{\alpha} - \bigvee_{j=1}^{k} |\theta_j^{(k)}| \right) + I_* \left( \sum_{|l| \leq k} \mathcal{E}_{\theta_l^{(k)}} \right) / E\left( |\theta_0^{(k)}|^{\alpha} - \bigvee_{j=1}^{k} |\theta_j^{(k)}| \right).
\]
Background Results—application to ACVF & ACF

Set-up: Let \( \{X_t\} \) be a stationary sequence and set

\[
X_t = X_t(m) = (X_t, \ldots, X_{t+m}).
\]

Suppose \( X_t \) satisfies the conditions of previous theorem. Then

\[
N_n := \sum_{t=1}^{n} \varepsilon_{X_t/a_n} \xrightarrow{d} N := \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varepsilon_{P_iQ_{ij}},
\]

Sample ACVF and ACF:

\[
\hat{\gamma}_X(h) = n^{-1} \sum_{t=1}^{n-h} X_t X_{t+h}, \ h \geq 0, \quad \text{ACVF}
\]

\[
\hat{\rho}_X(h) = \hat{\gamma}_X(h) / \hat{\gamma}_X(0), \ h \geq 1, \quad \text{ACF}
\]

If \( \text{EX}_0^2 < \infty \), then define \( \gamma_X(h) = \text{EX}_0 X_h \) and \( \rho_X(h) = \gamma_X(h) / \gamma_X(0) \).
Background Results—application to ACVF & ACF

(i) If $\alpha \in (0,2)$, then

$$
(na_n^{-2} \hat{\gamma}_X (h))_{h=0,\ldots,m} \xrightarrow{d} (V_h)_{h=0,\ldots,m}
$$

$$
(\hat{\rho}_X (h))_{h=1,\ldots,m} \xrightarrow{d} (V_h / V_0)_{h=1,\ldots,m},
$$

where

$$
V_h = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P_i^2 Q_{ij}^{(0)} Q_{ij}^{(h)}, \quad h = 0,\ldots,m.
$$

(ii) If $\alpha \in (2,4) +$ additional condition, then

$$
(na_n^{-2} (\hat{\gamma}_X (h) - \gamma_X (h)))_{h=0,\ldots,m} \xrightarrow{d} (V_h)_{h=0,\ldots,m}
$$

$$
(na_n^{-2} (\hat{\rho}_X (h) - \rho_X (h)))_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1} (0) (V_h - \rho_X (h) V_0)_{h=1,\ldots,m}.
$$
Applications—stochastic recurrence equations

\[ Y_t = A_t Y_{t-1} + B_t, \quad (A_t, B_t) \sim \text{IID}, \]

\[ A_t \text{ } d \times d \text{ random matrices, } B_t \text{ random } d\text{-vectors} \]

Examples

ARCH(1): \[ X_t = (\alpha_0 + \alpha_1 X_{t-1}^2)^{1/2} Z_t, \quad \{Z_t\} \sim \text{IID}. \] Then the squares follow an SRE with \[ Y_t = X_t^2, \quad A_t = \alpha_1 Z_t^2, \quad B_t = \alpha_0 Z_t^2. \]

GARCH(2,1): \[ X_t = \sigma_t Z_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \alpha_2 X_{t-2}^2 + \beta_1 \sigma_{t-1}^2. \] Then \[ Y_t = (X_t^2, X_{t-1}^2, \sigma_t^2)' \] follows the SRE given by

\[
\begin{bmatrix}
X_t^2 \\
X_{t-1}^2 \\
\sigma_t^2
\end{bmatrix} =
\begin{bmatrix}
\alpha_1 Z_t^2 & \alpha_2 Z_t^2 & \beta_1 Z_t^2 \\
1 & 0 & 0 \\
\alpha_1 & \alpha_2 & \beta_1
\end{bmatrix}
\begin{bmatrix}
X_{t-1}^2 \\
X_{t-2}^2 \\
\sigma_{t-1}^2
\end{bmatrix} +
\begin{bmatrix}
\alpha_0 Z_t^2 \\
0 \\
0
\end{bmatrix}
\]
Examples (tricks)

GARCH(1,1): \( X_t = \sigma_t Z_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2. \)
Although this process does not have a 1-dimensional SRE representation, the process \( \sigma_t^2 \) does. Iterating, we have

\[
\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 Z_{t-1}^2 + \beta_1 \sigma_{t-1}^2
\]

\[
= (\alpha_1 Z_{t-1}^2 + \beta_1) \sigma_{t-1}^2 + \alpha_0.
\]

Bilinear(1): \( X_t = aX_{t-1} + bX_{t-1} Z_{t-1} + Z_t, \quad \{Z_t\} \sim \text{IID} \)

\[
= Y_{t-1} + Z_t,
\]

\[
Y_t = A_t Y_{t-1} + B_t \quad A_t = a + bZ_t, \quad B_t = A_t Z_t
\]
Stochastic Recurrence Equations (cont)

\[ Y_t = A_t Y_{t-1} + B_t, \quad (A_t, B_t) \sim \text{IID} \]

Existence of stationary solution

- \( E \ln^+ \| A_1 \| < \infty \)
- \( E \ln^+ \| B_1 \| < \infty \)
- \( \inf n^{-1} E \ln \| A_1 \ldots A_n \| =: \gamma < 0 \) (\( \gamma \) – top Lyapunov exponent)

Ex. (d=1) \( E \ln |A_1| < 0 \).

Strong mixing

If \( E \| A_1 \|^\varepsilon < \infty, E |B_1|^\varepsilon < \infty \) for some \( \varepsilon > 0 \), then the SRE \( (Y_t) \) is geometrically ergodic \( \Rightarrow \) strong mixing with geometric rate (Meyn and Tweedie `93).
Regular variation of the marginal distribution (Kesten)

Assume $A$ and $B$ have non-negative entries and

- $E \|A_1\|^\varepsilon < 1$ for some $\varepsilon > 0$
- $A_1$ has no zero rows a.s.
- W.P. 1, $\{\ln \rho(A_1 \ldots A_n) : \text{is dense in } \mathbb{R} \text{ for some } n, A_1 \ldots A_n > 0\}$
- There exists a $\kappa_0 > 0$ such that $E\|A\|^{\kappa_0} \ln^+ \|A\| < \infty$ and
  $$E \left( \min_{i=1,\ldots,d} \sum_{j=1}^d A_{ij} \right)^{\kappa_0} \geq d^{\kappa_0/2}$$

Then there exists a $\kappa_1 \in (0, \kappa_0]$ such that all linear combinations of $Y$ are regularly varying with index $\kappa_1$. (Also need $E |B|^{\kappa_i} < \infty$.)
Proposition: Let \((Y_t)\) be the solution to the SRE based on the squares of a GARCH model. Assume

- Top Lyapunov exponent \(\gamma < 0\). (See Bougerol and Picard `92)
- \(Z\) has a positive density on \((-\infty, \infty)\) with all moments finite or \(E|Z|^h = \infty\), for all \(h \geq h_0\) and \(E|Z|^h < \infty\) for all \(h < h_0\).
- Not all the GARCH parameters vanish.

Then \((Y_t)\) is *strongly mixing* with geometric rate and all finite dimensional distributions are *multivariate regularly varying* with index \(\kappa_1\).

Corollary: The corresponding GARCH process is strongly mixing and has all finite dimensional distributions that are MRV with index \(\kappa = 2\kappa_1\).
Remarks:
1. Kesten’s result applied to an iterate of $Y_t$, i.e., $Y_{tm} = \tilde{A}_t Y_{(t-1)m} + \tilde{B}_t$

2. Determination of $\kappa$ is difficult. Explicit expressions only known in two(?) cases.
   - **ARCH(1):** $E|\alpha_1 Z^2|^{\kappa/2} = 1$.  
     
     $$\begin{array}{cccc} 
     \alpha_1 & .312 & .577 & 1.00 & 1.57 \\
     \kappa & 8.00 & 4.00 & 2.00 & 1.00 \\
     \end{array}$$
   
   - **GARCH(1,1):** $E|\alpha_1 Z^2 + \beta_1|^{\kappa/2} = 1$ (Mikosch and Starica)
   - **For IGARCH** ($\alpha_1 + \beta_1 = 1$), then $\kappa = 2 \Rightarrow$ infinite variance.
   - **Can estimate $\kappa$ empirically by replacing expectations with sample moments.**
Summary for GARCH(p,q)

\( \kappa \in (0,2) \):

\[ \left( \hat{\rho}_X(h) \right)_{h=1,\ldots,m} \xrightarrow{d} (V_h / V_0)_{h=1,\ldots,m}, \]

\( \kappa \in (2,4) \):

\[ \left( n^{1-2/\kappa} \hat{\rho}_X(h) \right)_{h=1,\ldots,m} \xrightarrow{d} \gamma^{-1}_X(0)(V_h)_{h=1,\ldots,m}. \]

\( \kappa \in (4,\infty) \):

\[ \left( n^{1/2} \hat{\rho}_X(h) \right)_{h=1,\ldots,m} \xrightarrow{d} \gamma^{-1}_X(0)(G_h)_{h=1,\ldots,m}. \]

Remark: Similar results hold for the sample ACF based on \(|X_t|\) and \(X_t^2\).
Fitted GARCH(1,1) model for NZ-USA exchange:

\[ X_t = \sigma_t Z_t, \quad \sigma_t^2 = (6.70)10^{-7} + .1519X_{t-1}^2 + .772\sigma_{t-1}^2 \]

\( (Z_t) \sim \text{IID t-distr with 5 df.} \quad \kappa \text{ is approximately 3.8} \)
ACF of Fitted GARCH(1,1) Process

ACF of squares of realization 1

ACF of squares of realization 2
ACF of 2 realizations of an (ARCH)^2: \( X_t = (0.001 + 0.7 X_{t-1})^{1/2} Z_t \)
ACF of 2 realizations of an ARCH: $X_t = (0.001 + X_{t-1})^{1/2} Z_t$
Stochastic Volatility Models

**SVM:** \( X_t = \sigma_t Z_t \)

- \((Z_t) \sim \text{IID with mean 0 (if it exists)}\)
- \((\sigma_t)\) is a stationary process (2 log \(\sigma_t\) is a linear process) given by

\[
\log \sigma_t^2 = \sum_{j=-\infty}^{\infty} \psi_j \epsilon_{t-j}, \quad \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty, \quad (\epsilon_t) \sim \text{IID } N(0, \sigma^2).
\]

**Heavy tails:** Assume \(Z_t\) has Pareto tails with index \(\alpha\), i.e.,

\[
P(\{| Z_t | > z \}) \sim C z^{-\alpha} \Rightarrow P(\{| X_t | > z \}) \sim C E \sigma^\alpha z^{-\alpha}.
\]

Then if \(\alpha \in (0, 2)\),

\[
(n / \ln n)^{1/\alpha} \hat{\rho}_X(h) \xrightarrow{d} \frac{\| \sigma_1 \sigma_{h+1} \|_{\alpha}}{\| \sigma_1 \|_{\alpha}^2} \frac{S_h}{S_0}.
\]
Stochastic Volatility Models (cont)

Other powers:

1. Absolute values: \( \alpha \in (1,2) \),

\[
E|X_t| = E|\sigma_t|E|Z_t|, \ E|X_t X_{t+h}| = (E|\sigma_t \sigma_{t+h}|)(E|Z_t|E|Z_{t+h}|) \\
Cov(|X_t|, |X_{t+h}|) = Cov(\sigma_t, \sigma_{t+h})(E|Z|)^2 \\
Cor(|X_t|, |X_{t+h}|) = Cor(\sigma_t, \sigma_{t+h})(E|Z|)^2/ EZ^2 \\
= 0 \ (?).
\]

We obtain

\[
n(n \ln n)^{-1/\alpha}(\hat{\gamma}_{|X|}(h) - \gamma_{|X|}(h)) \xrightarrow{d} \|\sigma_1 \sigma_{h+1}\|_\alpha S_h
\]

and

\[
(n / \ln n)^{1/\alpha} \hat{\rho}_{|X|}(h) \xrightarrow{d} \frac{\|\sigma_1 \sigma_{h+1}\|_\alpha S_h}{\|\sigma_1\|^2 S_0}.
\]
2. Higher order: $\alpha \in (0,2)$

The squares are again a SV process and the results of the previous proposition apply. Namely,

$$
\frac{(n / \ln n)^{2/\alpha}}{\hat{\rho}_{X^2}(h)} \xrightarrow{d} \frac{\left\| \sigma_1^2 \sigma_{h+1}^2 \right\|_{\alpha/2}}{\left\| \sigma_1^2 \right\|_{\alpha/2}} \frac{S_h}{S_0}.
$$

In particular,

$$
\hat{\rho}_{X^2}(h) \xrightarrow{p} 0.
$$
Stochastic Volatility Models (cont)

(log $X^2$) - mean for NZ-USA exchange rates
Stochastic Volatility Models (cont)

ACF/PACF for \((\log X^2)\) suggests ARMA \((1,1)\) model:

\[
\mu = -11.5403, \quad Y_t = .9646Y_{t-1} + \varepsilon_t -.8709 \varepsilon_{t-1}, \ (\varepsilon_t) \sim WN(0,4.6653)
\]
The ARMA (1,1) model for log $X^2$ leads to the SV model

$$X_t = \sigma_t Z_t$$

with

$$2 \ln \sigma_t = -11.5403 + v_t + \varepsilon_t$$

$$v_t = .9646 v_{t-1} + \gamma_t, \quad (\gamma_t) \sim \text{WN}(0,.07253)$$

$$(\varepsilon_t) \sim \text{WN}(0,4.2432).$$
Simulation of SVM model: Took $\varepsilon_t$ to be distributed according to log of a $t$ random variable with 3 df (suitable normalized).

ACF: abs(realization)
Stochastic Volatility Models (cont)

**ACF:** $(\text{realization})^2$

**ACF:** $(\text{realization})^4$
Sample ACF for GARCH and SV Models (1000 reps)

(a) GARCH(1,1) Model, n=10000

(b) SV Model, n=10000
Sample ACF for Squares of GARCH and SV (1000 reps)

(a) GARCH(1,1) Model, n=10000

(b) SV Model, n=10000
Sample ACF for Squares of GARCH and SV (1000 reps)

(c) GARCH(1,1) Model, n=100000

(d) SV Model, n=100000