Applications of Multivariate Regular Variation and Point Process Theory to Financial Time Series Models

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Outline

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- Applications of multivariate regular variation
  - Stochastic recurrence equations (GARCH)
  - Point process convergence
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Characteristics of some financial time series

Define $X_t = \ln (P_t) - \ln (P_{t-1})$ (log returns)

- heavy tailed

$P(|X_1| > x) \sim C x^{-\alpha}, \quad 0 < \alpha < 4.$

- uncorrelated

$\hat{\rho}_X (h) \; \text{near 0 for all lags } h > 0$ (MGD sequence)

- $|X_t|$ and $X_t^2$ have slowly decaying autocorrelations

$\hat{\rho}_{|X|} (h)$ and $\hat{\rho}_{X^2} (h)$ converge to 0 slowly as $h$ increases.

- process exhibits ‘stochastic volatility’.
Sample ACF IBM (a) 1962-1981, (b) 1982-2000

(a) ACF of IBM (1st half)  
(b) ACF of IBM (2nd half)
Sample ACF of abs values for IBM (a) 1961-1981, (b) 1982-2000
Sample ACF of squares for IBM (a) 1961-1981, (b) 1982-2000

(a) ACF, Squares of IBM (1st half)

(b) ACF, Squares of IBM (2nd half)
Multiplicative models for log(returns)

**Basic model**

\[ X_t = \ln (P_t) - \ln (P_{t-1}) \quad \text{(log returns)} \]

\[ = \sigma_t Z_t, \]

where

- \{Z_t\} is IID with mean 0, variance 1 (if exists). (e.g. N(0,1) or a \(t\)-distribution with \(\nu\) df.)
- \{\sigma_t\} is the volatility process
- \(\sigma_t\) and \(Z_t\) are independent.

**Properties:**

- \(EX_t = 0, \text{Cov}(X_t, X_{t+h}) = 0, h>0 \) (uncorrelated if \(\text{Var}(X_t) < \infty\))
- conditional heteroscedastic (condition on \(\sigma_t\)).
Multiplicative models for log(returns)-cont

\[ X_t = \sigma_t Z_t \] (observation eqn in state-space formulation)

Two classes of models for volatility:

(i) GARCH(p,q) process (General AutoRegressive Conditional Heteroscedastic-observation-driven specification)

\[ \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_p X_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_q \sigma_{t-q}^2. \]

Special case: ARCH(1):

\[ X_t^2 = (\alpha_0 + \alpha_1 X_{t-1}^2)Z_t^2 \]
\[ = \alpha_1 Z_t^2 X_{t-1}^2 + \alpha_0 Z_t^2 \]
\[ = A_t X_{t-1}^2 + B_t \]

(stochastic recursion eqn)

\[ \rho_{X^2}(h) = \alpha_1^h, \text{ if } \alpha_1^2 < 1/3. \]
GARCH(2,1): \[ X_t = \sigma_t Z_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \alpha_2 X_{t-2}^2 + \beta_1 \sigma_{t-1}^2. \]

Then \[ Y_t = (\sigma_t^2, X_{t-1}^2)' \] follows the SRE given by

\[
\begin{bmatrix}
\sigma_t^2 \\
X_{t-1}^2
\end{bmatrix}
= \begin{bmatrix}
\alpha_1 Z_{t-1}^2 + \beta_1 & \alpha_2 \\
Z_{t-1}^2 & 0
\end{bmatrix}
\begin{bmatrix}
\sigma_{t-1}^2 \\
X_{t-2}^2
\end{bmatrix}
+ \begin{bmatrix}
\alpha_0 \\
0
\end{bmatrix}
\]

Questions:
- Existence of a unique stationary soln to the SRE?
- Regular variation of the joint distributions?
Multiplicative models for log(returns)-cont

\[ X_t = \sigma_t Z_t \text{ (observation eqn in state-space formulation)} \]

(ii) stochastic volatility process \text{(parameter-driven specification)}

\[
\log \sigma_t^2 = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}, \quad \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty, \{ \varepsilon_t \} \sim \text{IID } N(0, \sigma^2)
\]

\[
\rho_{X^2}(h) = \text{Cor}(\sigma_t^2, \sigma_{t+h}^2) / EZ_1^4
\]

Question:

- Joint distributions of process regularly varying if distr of \( Z_1 \) is regularly varying?
Regular variation — univariate case

**Definition:** The random variable $X$ is regularly varying with index $\alpha$ if

$$
P(|X|> t x)/P(|X|>t) \to x^{-\alpha} \quad \text{and} \quad P(X> t)/P(|X|>t) \to p,
$$
or, equivalently, if

$$
P(X> t x)/P(|X|>t) \to px^{-\alpha} \quad \text{and} \quad P(X< -t x)/P(|X|>t) \to qx^{-\alpha},
$$

where $0 \leq p \leq 1$ and $p+q=1$.

**Equivalence:**

$X$ is RV($\alpha$) *if and only if* $P(X \in t \cdot )/P(|X|>t) \to_v \mu(\cdot)$

($\to_v$ vague convergence of measures on $\mathbb{R}\{0\}$). In this case,

$$
\mu(dx) = \left(p \alpha x^{-\alpha-1} I(x>0) + q \alpha (-x)^{-\alpha-1} I(x<0)\right) dx
$$

**Note:** $\mu(tA) = t^{-\alpha} \mu(A)$ for every $t$ and $A$ bounded away from 0.
Another formulation (polar coordinates):

Define the $\pm 1$ valued rv $\theta$, $P(\theta = 1) = p$, $P(\theta = -1) = 1 - p = q$.

Then

$X$ is RV$(\alpha)$ if and only if

$$\frac{P(|X| > t \ x, X/|X| \in S)}{P(|X| > t)} \rightarrow x^{-\alpha} P(\theta \in S)$$

or

$$\frac{P(|X| > t \ x, X/|X| \in \bullet)}{P(|X| > t)} \rightarrow_v x^{-\alpha} P(\theta \in \bullet)$$

($\rightarrow_v$ vague convergence of measures on $S^0 = \{-1, 1\}$).
Regular variation—multivariate case

**Multivariate regular variation of** $X=(X_1, \ldots, X_m)$: There exists a random vector $\theta \in S^{m-1}$ such that

$$P(|X| > tx, X/|X| \in \cdot)/P(|X| > t) \xrightarrow{v} x^{-\alpha} P(\theta \in \cdot)$$

($\xrightarrow{v}$ vague convergence on $S^{m-1}$, unit sphere in $\mathbb{R}^m$).

- $P(\theta \in \cdot)$ is called the spectral measure
- $\alpha$ is the index of $X$.

**Equivalence:** 

$$\frac{P( X \in t\cdot)}{P(|X| > t)} \xrightarrow{v} \mu(\cdot)$$

$\mu$ is a measure on $\mathbb{R}^m$ which satisfies for $x > 0$ and $A$ bounded away from 0,

$$\mu(xB) = x^{-\alpha} \mu(xA).$$
1. If $X_1 > 0$ and $X_2 > 0$ are iid RV($\alpha$), then $X = (X_1, X_2)$ is multivariate regularly varying with index $\alpha$ and spectral distribution

$$P(\theta = (0,1)) = P(\theta = (1,0)) = .5 \text{ (mass on axes).}$$

**Interpretation:** Unlikely that $X_1$ and $X_2$ are very large at the same time.

**Figure:** plot of $(X_{t1}, X_{t2})$ for realization of 10,000.
2. If \( X_1 = X_2 > 0 \), then \( \mathbf{X} = (X_1, X_2) \) is multivariate regularly varying with index \( \alpha \) and spectral distribution

\[
P( \theta = (1/\sqrt{2}, 1/\sqrt{2}) ) = 1.
\]

AR(1): \( X_t = 0.9 \, X_{t-1} + Z_t \), \( \{Z_t\} \sim \text{IID symmetric stable (1.8)} \)

Distr of \( \theta \):

\[
\begin{align*}
\pm(1, 0.9) / \sqrt{1.81}, & \quad \text{W.P. } 0.9898 \\
\pm(0, 1), & \quad \text{W.P. } 0.0102
\end{align*}
\]

**Figure:** plot of \((X_t, X_{t+1})\) for realization of 10,000.
Applications of multivariate regular variation

• Domain of attraction for sums of iid random vectors (Rvaceva, 1962). That is, when does the partial sum

\[ a_n^{-1} \sum_{t=1}^{n} X_t \]

converge for some constants \( a_n \)?

• Spectral measure of random stable vectors.

• Domain of attraction for componentwise maxima of iid random vectors (Resnick, 1987). Limit behavior of

\[ a_n^{-1} \bigvee_{t=1}^{n} X_t \]

• Weak convergence of point processes with iid points.

• Solution to stochastic recurrence equations, \( Y_t = A_t Y_{t-1} + B_t \)

• Weak convergence of sample autocovariance.
Operations on regularly varying vectors — products

**Products** (Breiman 1965). Suppose $X, Y > 0$ are independent with $X \sim \text{RV}(\alpha)$ and $E Y^{\alpha+\varepsilon} < \infty$ for some $\varepsilon > 0$. Then $XY \sim \text{RV}(\alpha)$ with

$$P(XY > x) \sim E Y^{\alpha} P(X > x).$$

**Multivariate version.** Suppose the random vector $X$ is regularly varying and $A$ is a matrix independent of $X$ with

$$0 < E \|A\|^{\alpha+\varepsilon} < \infty.$$

Then

$AX$ is regularly varying with index $\alpha$. 
Example: SV model $X_t = \sigma_t Z_t$

Suppose $Z_t \sim RV(\alpha)$ and

$$\log \sigma_t^2 = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}, \quad \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty, \{\varepsilon_t\} \sim \text{IID } N(0, \sigma^2).$$

Then $Z_n = (Z_1, \ldots, Z_n)'$ is regular varying with index $\alpha$ and so is $X_n = (X_1, \ldots, X_n)' = \text{diag} (\sigma_1, \ldots, \sigma_n) Z_n$ with spectral distribution concentrated on $(\pm 1, 0), (0, \pm 1)$.

**Figure:** plot of $(X_t, X_{t+1})$ for realization of 10,000.
Operations on regularly varying vectors — linear combinations

Linear combinations:

\( X \sim RV(\alpha) \Rightarrow \) all linear combinations of \( X \) are regularly varying

i.e., there exist \( \alpha \) and slowly varying fcn \( L(.) \), s.t.

\[
P(c^T X > t)/(t^{-\alpha}L(t)) \to w(c), \text{ exists for all real-valued } c,
\]

where

\[
w(tc) = t^{-\alpha}w(c).
\]

Use vague convergence with \( A_c = \{ y: c^T y > 1 \} \), i.e.,

\[
P(X \in tA_c) = \frac{P(c^T X > t)}{t^{-\alpha}L(t)} \to \mu(A_c) =: w(c),
\]

where \( t^{-\alpha}L(t) = P(|X| > t) \).
Operations on regularly-varying vectors-linear combinations

Converse?

\( X \sim \text{RV}(\alpha) \iff \) all linear combinations of \( X \) are regularly varying?

There exist \( \alpha \) and slowly varying fcn \( L(.) \), s.t.

\[
\text{(LC)} \quad P(c^T X > t)/(t^{-\alpha} L(t)) \to w(c), \text{ exists for all real-valued } c.
\]

**Theorem.** Let \( X \) be a random vector.

1. If \( X \) satisfies (LC) with \( \alpha \) non-integer, then \( X \) is \( \text{RV}(\alpha) \).

2. If \( X > 0 \) satisfies (LC) for non-negative \( c \) and \( \alpha \) is non-integer, then \( X \) is \( \text{RV}(\alpha) \).

3. If \( X > 0 \) satisfies (LC) with \( \alpha \) an odd integer, then \( X \) is \( \text{RV}(\alpha) \).
Operations on regularly-varying vectors-linear combinations

Idea of argument: Define the measures

\[ m_t(\cdot) = \frac{P(X \in t\cdot)}{t^{-\alpha}L(t)} \]

- By assumption we know that for fixed \( c \), \( m_t(A_c) \rightarrow \mu(A_c) \).
- \( \{m_t\} \) is tight: For \( B \) bded away from 0, \( \sup_t m_t(B) < \infty \).
- Do subsequential limits of \( \{m_t\} \) coincide?

  If \( m_{t'} \rightarrow_v \mu_1 \) and \( m_{t''} \rightarrow_v \mu_2 \), then
  \[ \mu_1(A_c) = \mu_2(A_c) \quad \text{for all } c \neq 0. \]

Problem: Need \( \mu_1 = \mu_2 \) but only have equality on \( A_c \) not a \( \pi \)-system. In general, equality need not hold (see Ex 6.1.35 in Meerschaert & Scheffler (2001)).
Operations on regularly-varying vectors-linear combinations

Solution: Need to show agreement on a nice class of fcns, eg. 
\[ f(y) = \exp\{i(x,y)\} \].

Integrability problem. \( \mu_j(tB) \approx t^{-\alpha} \) for \( t \) around 0 and \( \infty \).

Consider the measures for \( \alpha \in (2n-2,2n) \) defined by

\[ \nu_j(B) = (-1)^n \int_B (e^{i(1,y)} - e^{-i(1,y)})^{2n} \, d\mu_j(y) \]

These are finite measures satisfying:

\[ \int_{\mathbb{R}^d} e^{i(x,y)} \, d\nu_j(y) = (-1)^n \sum \left( -1 \right)^k \binom{2n}{k} e^{i(x-2n1+2k1,y)} \, d\mu_j(y) \]

However, the summands are not integrable wrt \( \mu_j \).
Using the identity, 

\[ \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} k^m = 0, \quad \text{for } m = 0, \ldots, 2n-1, \]

and setting

\[ e_m(z) = e^{iz} - 1 - iz - \cdots - \frac{i^m}{m!} z^m, \]

The above integral, for \( \alpha \in (2n-1, 2n) \), can be written as

\[
\int_{\mathbb{R}^d} e^{i(x,y)} d\nu_j(y) = (-1)^n \int_{\mathbb{R}^d} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} e_{2n-1}(x - 2n1 + 2k1, y) d\mu_j(y)
\]

\[ = (-1)^n \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \int_{\mathbb{R}^d} e_{2n-1}(x - 2n1 + 2k1, y) d\mu_j(y) \]

Integrals on the right-hand side are finite and coincide.
Operations on regularly-varying vectors-linear combinations

For $X > 0$ and $c > 0$, use Laplace transforms.

**Problem:** For integer $\alpha$ this argument does not work.

**Argument for $\alpha$ odd:**

Let $(N_1, \ldots, N_d)$ be a vector of iid $N(0,1)$ rvs indep of $X$. Then

$$N_1^2 (c^2, X^2) := N_1^2 (c_1^2 X_1^2 + \cdots + c_d^2 X_d^2) = (c_1 X_1 N_1 + \cdots + c_d X_d N_d)^2$$

Since $(c, X)$ is $RV(\alpha)$ for all $c \neq 0$, the rhs can be shown to be $RV$ with index $\alpha/2$, a **non-integer**.

A Tauberian argument shows that if $N_1 Y$ is $RV$, then so is $Y$. It follows, with $Y = \sqrt{\{(c^2, X^2)\}}$, that $X^2$ is regularly varying with index $\alpha/2$. Hence $X$ is $RV(\alpha)$. 
Applications of theorem

1. Kesten (1973). Under general conditions, (LC) holds with \( L(t) = 1 \) for stochastic recurrence equations of the form

\[
Y_t = A_t Y_{t-1} + B_t, \quad (A_t, B_t) \sim \text{IID},
\]

\( A_t \) \( d \times d \) random matrices, \( B_t \) random \( d \)-vectors.

It follows that the distr of \( Y_t \), and in fact all of the finite dim’l distrs of \( Y_t \) are regularly varying.

2. GARCH processes. Since GARCH processes can be embedded in a SRE, the finite dim’l distributions of GARCH are regularly varying.
Applications of theorem

Example of ARCH(1): \( X_t = (\alpha_0 + \alpha_1 X_{t-1}^2)^{1/2} Z_t, \) \( \{Z_t\} \sim \text{IID}. \)

\( \alpha \) found by solving \( E|\alpha_1 Z^2|^\alpha = 1. \)

<table>
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<th>( \alpha_1 )</th>
<th>.312</th>
<th>.577</th>
<th>1.00</th>
<th>1.57</th>
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<td>( \alpha )</td>
<td>8.00</td>
<td>4.00</td>
<td>2.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Distr of \( \theta \):

\[
P(\theta \in \bullet) = E\left\{ \| (B,Z) \|^\alpha I(\text{arg}((B,Z)) \in \bullet) \right\} / E\| (B,Z) \|^\alpha
\]

where \( P(B = 1) = P(B = -1) = .5 \)
Example: ARCH(1) model $X_t = (\alpha_0 + \alpha_1 X_{t-1}^2)^{1/2} Z_t$

Example of ARCH(1): $\alpha_0 = 1$, $\alpha_1 = 1$, $\alpha = 2$

Figures: plots of $(X_t, X_{t+1})$ and estimated distribution of $\theta$ for realization of 10,000.
**Theorem** Let \( \{X_t\} \) be an iid sequence of random vectors satisfying 1 of the 3 conditions in the theorem. Then

\[
N_n := \sum_{t=1}^{n} \varepsilon_{X_t/a_n} \xrightarrow{d} N := \sum_{j=1}^{\infty} \varepsilon_{P_i\theta_i},
\]

if and only if for every \( c \neq 0 \)

\[
N_{n,c} := \sum_{t=1}^{n} \varepsilon_{c'X_t/a_n} \xrightarrow{d} N_c := \sum_{j=1}^{\infty} \varepsilon_{c'P_i\theta_i},
\]

where \( \{a_n\} \) satisfies \( n\mathbb{P}(|X_t| > a_n) \rightarrow 1 \), and \( N \) is a Poisson process with intensity measure \( \mu \).

- \( \{P_i\} \) are Poisson pts corresponding to the radial part (intensity measure \( \alpha x^{-\alpha-1} \, dx \)).

- \( \{\theta_i\} \) are iid with the spectral distribution given by the RV
Point process convergence

**Theorem** (Davis & Hsing `95, Davis & Mikosch `97). Let \( \{X_t\} \) be a stationary sequence of random \( m \)-vectors. Suppose

(i) finite dimensional distributions are jointly regularly varying (let \((\theta_{-k}, \ldots, \theta_k)\) be the vector in \( S^{(2k+1)m-1} \) in the definition).

(ii) mixing condition \( A(n) \) or strong mixing.

(iii) \[ \lim_{k \to \infty} \limsup_{n \to \infty} P\left( \bigvee_{k \leq |t| \leq r_n} |X_t| > a_n y \bigg| |X_0| > a_n y \right) = 0. \]

Then

\[ \gamma = \lim_{k \to \infty} E\left( |\theta_0^{(k)}|^\alpha - \bigvee_{j=1}^k |\theta_j^{(k)}| \right) + E \left( |\theta_0^{(k)}|^\alpha \right) \]

exists. If \( \gamma > 0 \), then

\[ N_n := \sum_{t=1}^n \mathcal{E}_{X_t/a_n} \overset{d}{\rightarrow} N := \sum_{i=1}^\infty \sum_{j=1}^\infty \mathcal{E}_{P_{iQ_{ij}}}, \]
\( (P_i) \) are points of a Poisson process on \((0, \infty)\) with intensity function
\[
\nu(dy) = \gamma \alpha y^{\alpha-1} dy.
\]
\( \sum_{j=1}^{\infty} \epsilon_{Q_{ij}}, \ i \geq 1, \) are iid point process with distribution \(Q\), and \(Q\) is the weak limit of
\[
\lim_{k \to \infty} E\left( \sum_{|t| \leq k} \epsilon_{\theta_i^{(k)}} \right) / E\left( \sum_{j=1}^{k} \left| \theta_0^{(k)} \right|^\alpha - \sum_{j=1}^{k} \left| \theta_j^{(k)} \right| \right).
\]

Remarks:

1. GARCH and SV processes satisfy the conditions of the theorem.
2. Limit distribution for sample ACF follows from this theorem.
Summary for GARCH(p,q)

\(\alpha \in (0,2)\):

\[
\left(\hat{\rho}_X(h)\right)_{h=1,\ldots,m} \xrightarrow{d} \left(V_h / V_0\right)_{h=1,\ldots,m},
\]

\(\alpha \in (2,4)\):

\[
\left(n^{1-2/\alpha}\hat{\rho}_X(h)\right)_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1}(0)\left(V_h\right)_{h=1,\ldots,m}.
\]

\(\alpha \in (4,\infty)\):

\[
\left(n^{1/2}\hat{\rho}_X(h)\right)_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1}(0)\left(G_h\right)_{h=1,\ldots,m}.
\]

Remark: Similar results hold for the sample ACF based on \(|X_t|\) and \(X_t^2\).
Summary for SV

$\alpha \in (0, 2)$:

$$(n / \ln n)^{1/\alpha} \hat{\rho}_X(h) \xrightarrow{d} \frac{\|\sigma_1 \sigma_{h+1}\|_\alpha S_h}{\|\sigma_1\|_\alpha^2 S_0}. \quad (1)$$

$\alpha \in (2, \infty)$:

$$\left(n^{1/2} \hat{\rho}_X(h)\right)_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1}(0) \left(G_h\right)_{h=1,\ldots,m}. \quad (2)$$
Sample ACF for GARCH and SV Models (1000 reps)

(a) GARCH(1,1) Model, n=10000

(b) SV Model, n=10000
Sample ACF for Squares of GARCH (1000 reps)

(a) GARCH(1,1) Model, n=10000

(b) GARCH(1,1) Model, n=100000
Sample ACF for Squares of SV (1000 reps)

(c) SV Model, n=10000

(d) SV Model, n=100000
Wrap-up

- Regular variation is a flexible tool for modeling both dependence and tail heaviness.
- Useful for establishing point process convergence of heavy-tailed time series.
- Point process theory plays a key role in establishing convergence for a variety of statistics such as sample ACVF and ACF.

Unresolved issues related to RV ⇔ (LC)

- $\alpha = 2n$?
- There is an example for which $X_1, X_2 > 0$, and $(c, X_1)$ and $(c, X_2)$ have the same limits for all $c > 0$.
- $\alpha = 2n-1$ and $X \not\sim 0$ (not true in general).