Asymptotic Theory for Some Nonlinear Time Series Models

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Outline

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Characteristics of Some Financial Data

- heavy tailed
  \[ P(|X_1| > x) \sim C x^{-\alpha}, \quad 0 < \alpha < 4. \]

- uncorrelated
  \[ \hat{\rho}_x(h) \text{ near 0 for all lags } h > 0 \text{ (MGD sequence)} \]

- \( |X_t| \) and \( X_t^2 \) have slowly decaying autocorrelations
  \[ \hat{\rho}_{|X|}(h) \text{ and } \hat{\rho}_{X^2}(h) \text{ converge to 0 slowly as } h \text{ increases.} \]

- process exhibits ‘stochastic volatility’.
500-daily log-returns of NZ/US exchange rate
ACF of $X(t)=\log$-returns of NZ/US exchange rate
ACF of $X^2(t)$
ACF of $|X(t)|$
Plot of $M_t(4)/S_t(4)$
Plot of $M_t(k)/S_t(k)$

- $k = 4$
- $k = 3$
- $k = 2$
- $k = 1$
Hill’s plot of tail index
Linear Processes

Model: \( X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_t \) \( \{Z_t\} \sim \text{IID}, \ P(|Z_t|>x) \sim C x^{-\alpha}, \ 0<\alpha<2. \)

Properties:

- \( P(|X_t|>x) \sim C_2 x^{-\alpha} \)
- Define \( \rho(h) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} / \sum_{j=-\infty}^{\infty} \psi_j^2 \).

If \( \alpha > 2 \)

\( n^{1/2}(\rho(h) - \rho(h)) \overset{d}{\longrightarrow} \sum_{j=1}^{\infty} (\rho(h+j)+\rho(h-j)-2\rho(j)\rho(h)) N_j, \ \{N_t\} \sim \text{IIDN} \)

If \( 0 < \alpha < 2 \)

\( (n / \ln n)^{1/\alpha} (\rho(h) - \rho(h)) \overset{d}{\longrightarrow} \sum_{j=1}^{\infty} (\rho(h+j)+\rho(h-j)-2\rho(j)\rho(h)) S_j / S_0, \)

\( \{S_t\} \sim \text{IID stable (\alpha), } S_0 \text{ stable (\alpha/2)} \)
Background Results

**Joint regular variation of** $X=(X_1, \ldots, X_m)$: There exist constants $x_n$ and a random vector $\theta \in S^{m-1}$ such that

$$nP(|X|>tx_n, X/|X| \in \bullet) \to t^{-\alpha} P(\theta \in \bullet)$$

(vague convergence on $S^{m-1}$).

**Mixing condition $A_{(a_n)}$ for a stationary sequence $\{X_t\}$**: Let $a_n$ be such that $nP(|X_t|>a_n) \to 1$. Then $A_{(a_n)}$ holds if there exists a sequence of integers $r_n$ such that $r_n \to \infty$, $k_n=[n/r_n] \to \infty$ and

$$E \exp\left\{-\sum_{t=1}^{n} f(X_t/a_n)\right\} - \left(E \exp\left\{-\sum_{t=1}^{r_n} f(X_t/a_n)\right\}\right)^{k_n} \to 0,$$

for every bounded, non-negative step function $f$ on $\mathbb{R}^m/\{0\}$ with bounded support.
Point Process Convergence

**Theorem** (Davis & Hsing `95, Davis & Mikosch `97). Let \( \{X_t\} \) be a stationary sequence of random vectors. Suppose

(i) finite dimensional distributions are jointly regularly varying (let \((\theta_{-k}, \ldots, \theta_k)\) be the vector in \(S^{(2k+1)m-1}\) in the definition).

(ii) mixing condition \( A(a_n) \) or strong mixing.

(iii) \( \lim_{k \to \infty} \limsup_{n \to \infty} P(\bigvee_{k \leq |t| \leq r_n} |X_t| > a_n y | |X_0| > a_n y) = 0. \)

Then

\[
\gamma = \lim_{k \to \infty} E(\left| \theta_0^{(k)} \right|^\alpha - \bigvee_{j=1}^k \left| \theta_j^{(k)} \right|)_+ / E \left| \theta_0^{(k)} \right|^\alpha
\]

exists. If \( \gamma > 0 \), then

\[
N_n := \sum_{t=1}^n \epsilon_{X_t / a_n} \xrightarrow{d} N := \sum_{i=1}^\infty \sum_{j=1}^\infty \epsilon_{P_i Q_{ij}},
\]
where

- \((P_i)\) are points of a Poisson process on \((0,\infty)\) with intensity function \(v(dy) = \gamma \alpha y^{-\alpha - 1} dy\).

- \(\sum_{j=1}^{\infty} \varepsilon_{Q_{ij}}, i \geq 1,\) are iid point process with distribution \(Q,\) and \(Q\) is the weak limit of

\[
\lim_{k \to \infty} E(|\theta_0^{(k)}|^{\alpha} - \bigvee_{j=1}^{k} |\theta_j^{(k)}|) + I_e \left( \sum_{|\ell| \leq k} \varepsilon_{\theta_{\ell}^{(k)}} \right) / E(|\theta_0^{(k)}|^{\alpha} - \bigvee_{j=1}^{k} |\theta_j^{(k)}|) +
\]
Application of Point Process Convergence

Set-up: Let \( \{X_t\} \) be a stationary sequence and set

\[
X_t = X_t(m) = (X_t, \ldots, X_{t+m}).
\]

Suppose \( X_t \) satisfies the conditions of previous theorem. Then

\[
N_n := \sum_{t=1}^{n} \varepsilon_{X_t/a_n} \xrightarrow{d} N := \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varepsilon_{P_iQ_{ij}},
\]

Sample ACVF and ACF:

\[
\hat{\gamma}_X(h) = n^{-1} \sum_{t=1}^{n-h} X_t X_{t+h}, \quad h \geq 0, \quad \text{ACVF}
\]

\[
\hat{\rho}_X(h) = \hat{\gamma}_X(h) / \hat{\gamma}_X(0), \quad h \geq 1, \quad \text{ACF}
\]

If \( EX_0^2 < \infty \), then define \( \gamma_X(h) = EX_0 X_h \) and \( \rho_X(h) = \gamma_X(h) / \gamma_X(0) \).
Limit Behavior of Sample ACVF and ACF

(i) If $\alpha \in (0,2)$, then

\[
(na_n^{-2} \hat{\gamma}_X (h))_{h=0,\ldots,m} \xrightarrow{d} (V_h)_{h=0,\ldots,m}
\]

\[
(\hat{\rho}_X (h))_{h=1,\ldots,m} \xrightarrow{d} (V_h / V_0)_{h=1,\ldots,m},
\]

where

\[
V_h = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P_i^2 Q_{ij}^{(0)} Q_{ij}^{(h)}, \quad h = 0,\ldots,m.
\]

(ii) If $\alpha \in (2,4) + \text{additional condition}$, then

\[
(na_n^{-2} (\hat{\gamma}_X (h) - \gamma_X (h)))_{h=0,\ldots,m} \xrightarrow{d} (V_h)_{h=0,\ldots,m}
\]

\[
(na_n^{-2} (\hat{\rho}_X (h) - \rho_X (h)))_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1} (0) (V_h - \rho_X (h) V_0)_{h=1,\ldots,m}.
\]
## Stochastic Recurrence Equations

\[ Y_t = A_t Y_{t-1} + B_t, \quad (A_t, B_t) \sim \text{IID}, \]

\[ A_t \text{ } d \times d \text{ random matrices, } B_t \text{ random } d \text{-vectors} \]

### Examples

**ARCH(1):** \( X_t = (\alpha_0 + \alpha_1 X_{t-1}^2)^{1/2} Z_t, \quad \{Z_t\} \sim \text{IID} \). Then the squares follow an SRE with \( Y_t = X_t^2, \quad A_t = \alpha_1 Z_t^2, \quad B_t = \alpha_0 Z_t^2 \).

**GARCH(2,1):** \( X_t = \sigma_t Z_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \alpha_2 X_{t-2}^2 + \beta_1 \sigma_{t-1}^2 \). Then \( Y_t = (X_t^2, X_{t-1}^2, \sigma_t^2)' \) follows the SRE given by

\[
\begin{bmatrix}
X_t^2 \\
X_{t-1}^2 \\
\sigma_t^2
\end{bmatrix}
= \begin{bmatrix}
\alpha_1 Z_t^2 & \alpha_2 Z_t^2 & \beta_1 Z_t^2 \\
1 & 0 & 0 \\
\alpha_1 & \alpha_2 & \beta_1
\end{bmatrix}
\begin{bmatrix}
X_{t-1}^2 \\
X_{t-2}^2 \\
\sigma_{t-1}^2
\end{bmatrix}
+ \begin{bmatrix}
\alpha_0 Z_t^2 \\
0 \\
0
\end{bmatrix}
\]
Examples (tricks)

GARCH(1,1): \( X_t = \sigma_t Z_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2. \)
Although this process does not have a 1-dimensional SRE representation, the process \( \sigma_t^2 \) does. Iterating, we have

\[
\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 Z_{t-1}^2 + \beta_1 \sigma_{t-1}^2
\]

\[
= (\alpha_1 Z_{t-1}^2 + \beta_1) \sigma_{t-1}^2 + \alpha_0.
\]

Bilinear(1): \( X_t = aX_{t-1} + bX_{t-1} Z_{t-1} + Z_t, \quad \{Z_t\} \sim \text{IID} \)

\[
= Y_{t-1} + Z_t,
\]

\[
Y_t = A_t Y_{t-1} + B_t, \quad A_t = a + bZ_t, \quad B_t = A_t Z_t
\]
Stochastic Recurrence Equations (cont)

\[ Y_t = A_t Y_{t-1} + B_t, \quad (A_t, B_t) \sim \text{IID} \]

**Existence of stationary solution**

- \( E \ln^+ \| A_1 \| < \infty \)
- \( E \ln^+ \| B_1 \| < \infty \)
- \( \inf n^{-1} E \ln \| A_1 \ldots A_n \| =: \gamma < 0 \) (\( \gamma \) – top Lyapunov exponent)

Ex. (d=1) \( E \ln |A_1| < 0 \).

**Strong mixing**

If \( E \| A_1 \|^\varepsilon < \infty, E |B_1|^\varepsilon < \infty \) for some \( \varepsilon > 0 \), then the SRE \( (Y_t) \) is *geometrically ergodic* \( \Rightarrow \) *strong mixing* with geometric rate (Meyn and Tweedie `93).
Stochastic Recurrence Equations (cont)

Regular variation of the marginal distribution (Kesten)

Assume \( \mathbf{A} \) and \( \mathbf{B} \) have non-negative entries and

- \( \mathbb{E} \| \mathbf{A}_1 \|^\varepsilon < 1 \) for some \( \varepsilon > 0 \)
- \( \mathbf{A}_1 \) has no zero rows a.s.
- W.P. 1, \( \{ \ln \rho(\mathbf{A}_1 \ldots \mathbf{A}_n) \text{ is dense in } \mathbb{R} \text{ for some } n, \mathbf{A}_1 \ldots \mathbf{A}_n > 0 \} \)
- There exists a \( \kappa_0 > 0 \) such that \( \mathbb{E} \| \mathbf{A} \|^\kappa_0 \ln^+ \| \mathbf{A} \| < \infty \) and
  \[
  \mathbb{E} \left( \min_{i=1,\ldots,d} \sum_{j=1}^d A_{ij} \right)^{\kappa_0} \geq d^{\kappa_0/2}
  \]

Then there exists a \( \kappa_1 \in (0, \kappa_0] \) such that \( \mathbf{Y} \) is regularly varying with index \( \kappa_1 \). (Also need \( \mathbb{E} |\mathbf{B}|^{\kappa_1} < \infty \).)
**Application to GARCH**

**Proposition:** Let \((Y_t)\) be the soln to the SRE based on the squares of a GARCH model. Assume

- Top Lyapunov exponent \(\gamma < 0\). (See Bougerol and Picard`92)
- \(Z\) has a positive density on \((-\infty, \infty)\) with all moments finite or
  \[E|Z|^h = \infty, \text{ for all } h \geq h_0 \text{ and } E|Z|^h < \infty \text{ for all } h < h_0.\]
- Not all the GARCH parameters vanish.

Then \((Y_t)\) is *strongly mixing* with geometric rate and all finite dimensional distributions are *regularly varying* with index \(\kappa_1\).

**Corollary:** The corresponding GARCH process is strongly mixing and has all finite dimensional distributions that are regularly varying with index \(\kappa = 2\kappa_1\).
Application to GARCH (cont)

Remarks:
1. Kesten’s result applied to an iterate of $Y_t$, i.e., $Y_{tm} = \tilde{A}_t Y_{(t-1)m} + \tilde{B}_t$

2. Determination of $\kappa$ is difficult. Explicit expressions only known in two(?) cases.

- ARCH(1): $E|\alpha Z^2|^{\kappa/2} = 1$.

| $\alpha$ | .312 | .577 | 1.00 | 1.57 |
| $\kappa$ | 8.00 | 4.00 | 2.00 | 1.00 |

- GARCH(1,1): $E|\alpha Z^2 + \beta|^{\kappa/2} = 1$ (Mikosch and Størbekk)

- For IGARCH ($\alpha + \beta = 1$), then $\kappa = 2 \Rightarrow$ infinite variance.

- Can estimate $\kappa$ empirically by replacing expectations with sample moments.
Summary for GARCH(p,q)

\( \kappa \in (0,2) \):

\[
(\hat{\rho}_X(h))_{h=1,\ldots,m} \xrightarrow{d} (V_h / V_0)_{h=1,\ldots,m},
\]

\( \kappa \in (2,4) \):

\[
\left( n^{1-2/\alpha} \hat{\rho}_X(h) \right)_{h=1,\ldots,m} \xrightarrow{d} \gamma^{-1}_X(0)(V_h)_{h=1,\ldots,m}.
\]

\( \kappa \in (4,\infty) \):

\[
\left( n^{1/2} \hat{\rho}_X(h) \right)_{h=1,\ldots,m} \xrightarrow{d} \gamma^{-1}_X(0)(G_h)_{h=1,\ldots,m}.
\]

Remark: Similar results hold for the sample ACF based on \( |X_t| \) and \( X_t^2 \).
Realization of fitted GARCH

Fitted GARCH(1,1) model for NZ-USA exchange:

\[ X_t = \sigma_t Z_t, \quad \sigma_t^2 = (6.70)10^{-7} + 0.1519 X_{t-1}^2 + 0.772 \sigma_{t-1}^2 \]

\( (Z_t) \sim \text{IID t-distr with 5 df.} \quad \kappa \text{ is approximately } 3.8 \)
ACF of Fitted GARCH(1,1) Process

ACF of squares of realization 1

ACF of squares of realization 2
ACF of 2 realizations of an \((ARCH)^2\): \(X_t = (0.001 + 0.7 X_{t-1})^{1/2} Z_t\)
ACF of 2 realizations of an |ARCH|: $X_t = (0.001 + X_{t-1})^{1/2} Z_t$
Stochastic Volatility Models

SVM: \( X_t = \sigma_t Z_t \)

- \((Z_t) \sim \text{IID with mean 0 (if it exists)}\)
- \((\sigma_t)\) is a stationary process (\(2 \log \sigma_t\) is a linear process) given by

\[
\log \sigma_t^2 = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}, \quad \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty, \quad (\varepsilon_t) \sim \text{IID } \mathcal{N}(0, \sigma^2)
\]

Heavy tailedness: Assume \(Z_t\) has Pareto tails with index \(\alpha\), i.e.,

\[
P(|Z_t| > z) \sim C z^{-\alpha} \Rightarrow P(|X_t| > z) \sim C E\sigma^\alpha z^{-\alpha}.
\]

Then if \(\alpha \in (0,2)\),

\[
(n / \ln n)^{1/\alpha} \hat{\rho}_X(h) \overset{d}{\to} \frac{\|\sigma_1 \sigma_{h+1}\|_\alpha}{\|\sigma_1\|_\alpha^2} \frac{S_h}{S_0}.
\]
Other powers:

1. Absolute values: $\alpha \in (1,2)$,

\[
E|X_t| = E|\sigma_t|E|Z_t|, \quad E|X_t X_{t+h}| = (E|\sigma_t \sigma_{t+h}|)(E|Z_t|E|Z_{t+h}|)
\]

\[
\text{Cov}(X_t, X_{t+h}) = \text{Cov}(\sigma_t, \sigma_{t+h})(E|Z|^2)
\]

\[
\text{Cor}(X_t, X_{t+h}) = \text{Cor}(\sigma_t, \sigma_{t+h})(E|Z|^2)/ E|Z|^2
\]

\[= 0 \text{ (?).}\]

We obtain

\[n(n \ln n)^{-1/\alpha} (\hat{\gamma}_{|X|}(h) - \gamma_{|X|}(h)) \xrightarrow{d} \|\sigma_1 \sigma_{h+1}\|_\alpha S_h\]

and

\[(n / \ln n)^{1/\alpha} \hat{\rho}_X(h) \xrightarrow{d} \frac{\|\sigma_1 \sigma_{h+1}\|_\alpha}{\|\sigma_1\|_\alpha^2} S_h S_0.\]
2. Higher order: $\alpha \in (0,2)$

The squares are again a SV process and the results of the previous proposition apply. Namely,

\[
\left( \frac{n}{\ln n} \right)^{2/\alpha} \hat{\rho}_{X^2}(h) \overset{d}{\longrightarrow} \frac{\left\| \sigma_1^2 \sigma_{h+1}^2 \right\|_{\alpha/2}}{\left\| \sigma_1^2 \right\|_{\alpha/2}} \frac{S_h}{S_0}.
\]

In particular,

\[
\hat{\rho}_{X^2}(h) \overset{p}{\longrightarrow} 0.
\]
Stochastic Volatility Models (cont)

(log $X^2$) - mean for NZ-USA exchange rates
Stochastic Volatility Models (cont)

ACF/PACF for \((\log X^2)\) suggests ARMA (1,1) model:

\[
\mu = -11.5403, \quad Y_t = 0.9646 Y_{t-1} + \varepsilon_t - 0.8709 \varepsilon_{t-1}, \quad (\varepsilon_t) \sim WN(0, 4.6653)
\]
The ARMA (1,1) model for log $X^2$ leads to the SV model

$$X_t = \sigma_t Z_t$$

with

$$2 \ln \sigma_t = -11.5403 + v_t + \varepsilon_t$$

$$v_t = .9646 v_{t-1} + \gamma_t, \quad (\gamma_t) \sim WN(0,.07253)$$

$$\varepsilon_t \sim WN(0,4.2432).$$
Simulation of SVM model: Took $\varepsilon_t$ to be distributed according to log of a t random variable with 3 df (suitable normalized).

ACF: abs(realization)
Stochastic Volatility Models (cont)

ACF: $(\text{realization})^2$

ACF: $(\text{realization})^4$
Linear Processes With Nonlinear Behavior

Allpass ARMA

Causal AR polynomial: \( \phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p \), \( \phi(z) \neq 0 \) for \( |z| \leq 1 \).

Define MA polynomial:

\[
\theta(z) = -z^p \phi(z^{-1})/\phi_p = -(z^p - \phi_1 z^{p-1} - \ldots - \phi_p)/\phi_p
\]

\( \neq 0 \) for \( |z| \geq 1 \) (MA polynomial is non-invertible).

Model for data \( (X_t) \): \( \phi(B) X_t = \theta(B) Z_t \), \( (Z_t) \sim \text{IID (non-Gaussian)} \)

Properties:

- uncorrelated (flat spectrum) but data are dependent
- squares and absolute values are correlated
- \( X_t \) has heavy tails if noise is heavy-tailed.
Linear Processes With Nonlinear Behavior (cont)

Realization of an allpass model of order 2 (t3 noise )
Linear Processes With Nonlinear Behavior (cont)

ACF: (allpass)

ACF: (allpass)2

model
sample
Allpass model fitted to NZ-USA exchange rates:

Order = 6, $\phi_1 = .852$, $\phi_2 = .616$, $\phi_3 = .952$, $\phi_4 = .098$, $\phi_5 = -.158$, $\phi_6 = -.066$
Sample ACF of squares for S&P (a) 1961-1976, (b) 1977-1993

(a) ACF, Squares of S&P (1st half)

(b) ACF, Squares of S&P (2nd half)
Sample ACF for GARCH and SV Models (1000 reps)

(a) GARCH(1,1) Model, n=10000

(b) SV Model, n=10000
Sample ACF for Squares of GARCH and SV (1000 reps)

(a) GARCH(1,1) Model, n=10000

(b) SV Model, n=10000
Sample ACF for Squares of GARCH and SV (1000 reps)

(c) GARCH(1,1) Model, n=100000

(d) SV Model, n=100000