Asymptotic Theory for Some Nonlinear Time Series Models

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Outline

+ Characteristics of some financial data
    - NZ-USA exchange rate
+ Linear processes
+ Background results
+ Point process convergence
    - application to sample ACVF and ACF
+ Stochastic recurrence equations
    - GARCH(1,1)
+ Stochastic volatility processes
+ Linear processes with nonlinear behavior

Characteristics of Some Financial Data

- heavy tailed

$$P(|X_1| > x) \sim C\, x^{-\alpha}, \qquad 0 < \alpha < 4.$$

- uncorrelated

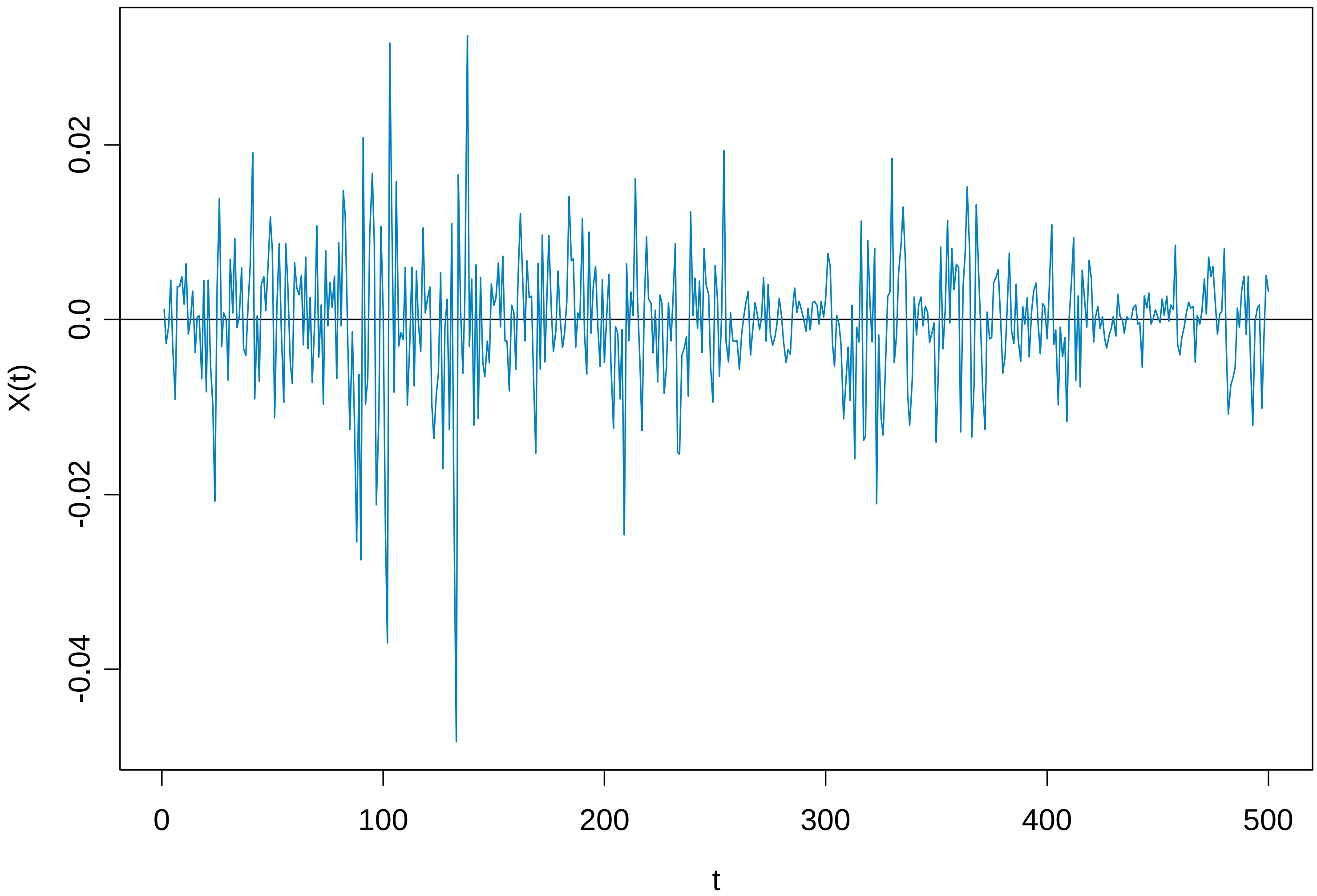
$\hat{\rho}_X(h)$ near 0 for all lags h > 0 (MGD sequence)

- $|X_t|$ and X_t^2 have slowly decaying autocorrelations

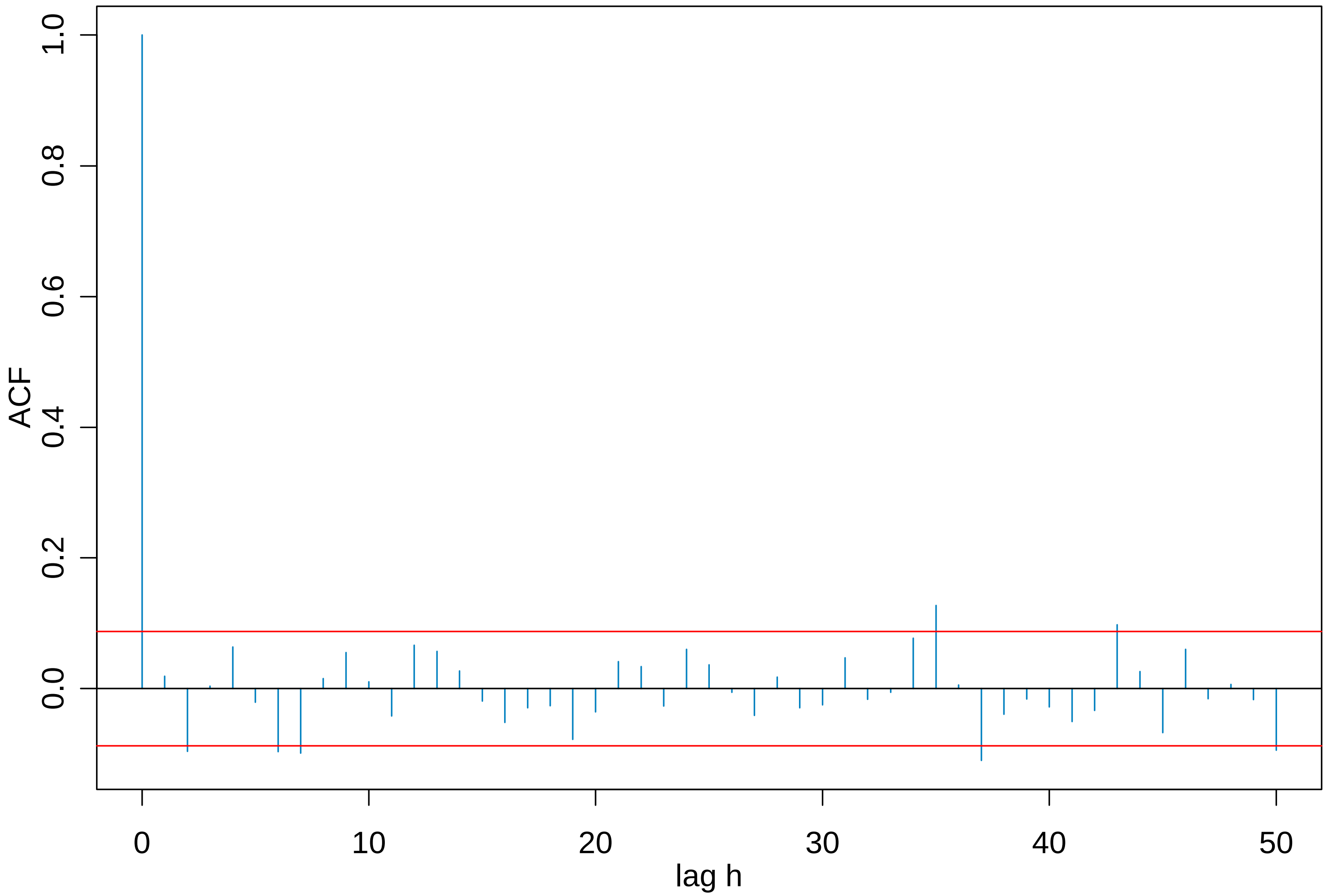
$\hat{\rho}_{|X|}(h)$ and $\hat{\rho}_{X^2}(h)$ converge to 0 *slowly* as h increases.

- process exhibits ‘stochastic volatility’.

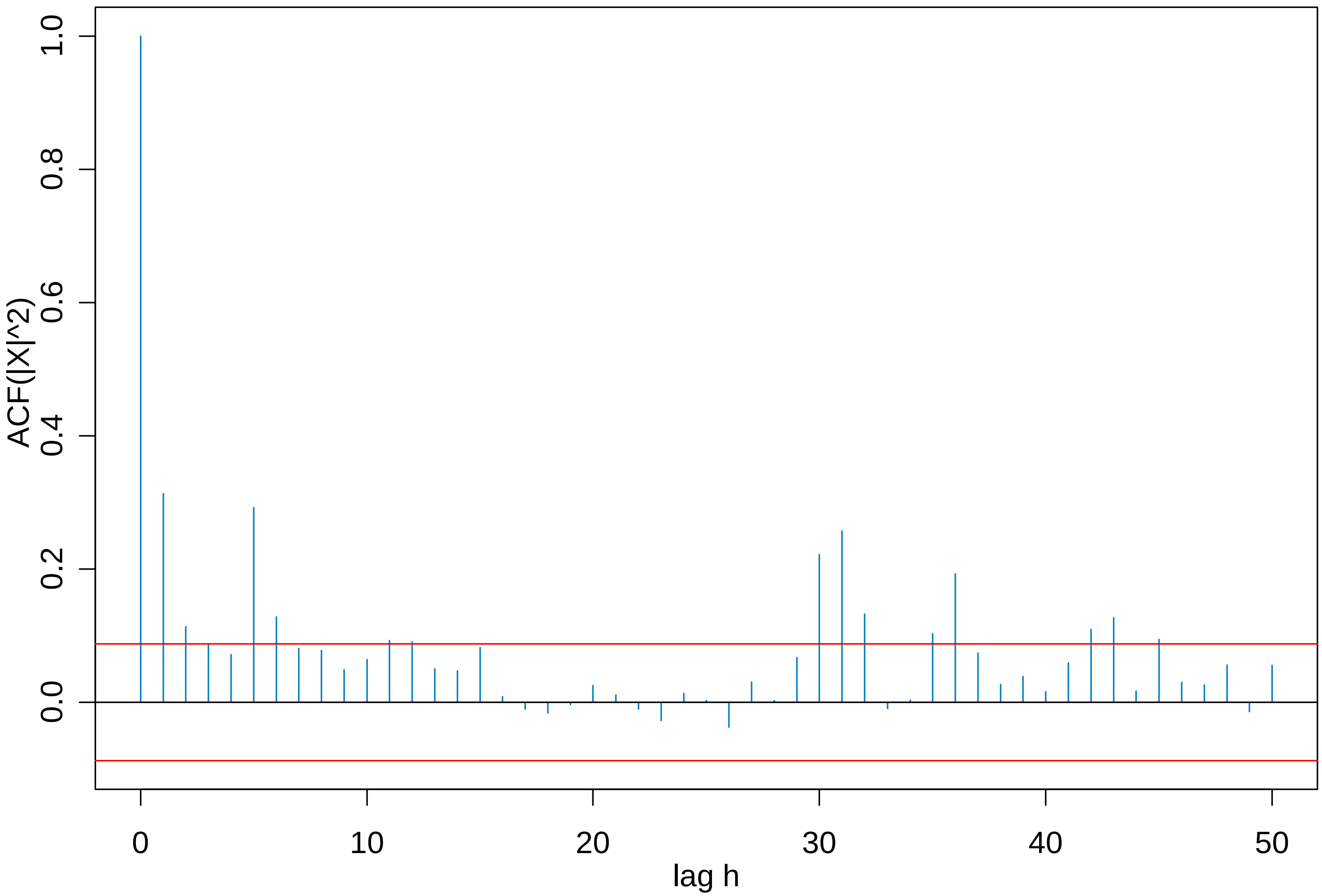
500-daily log-returns of NZ/US exchange rate



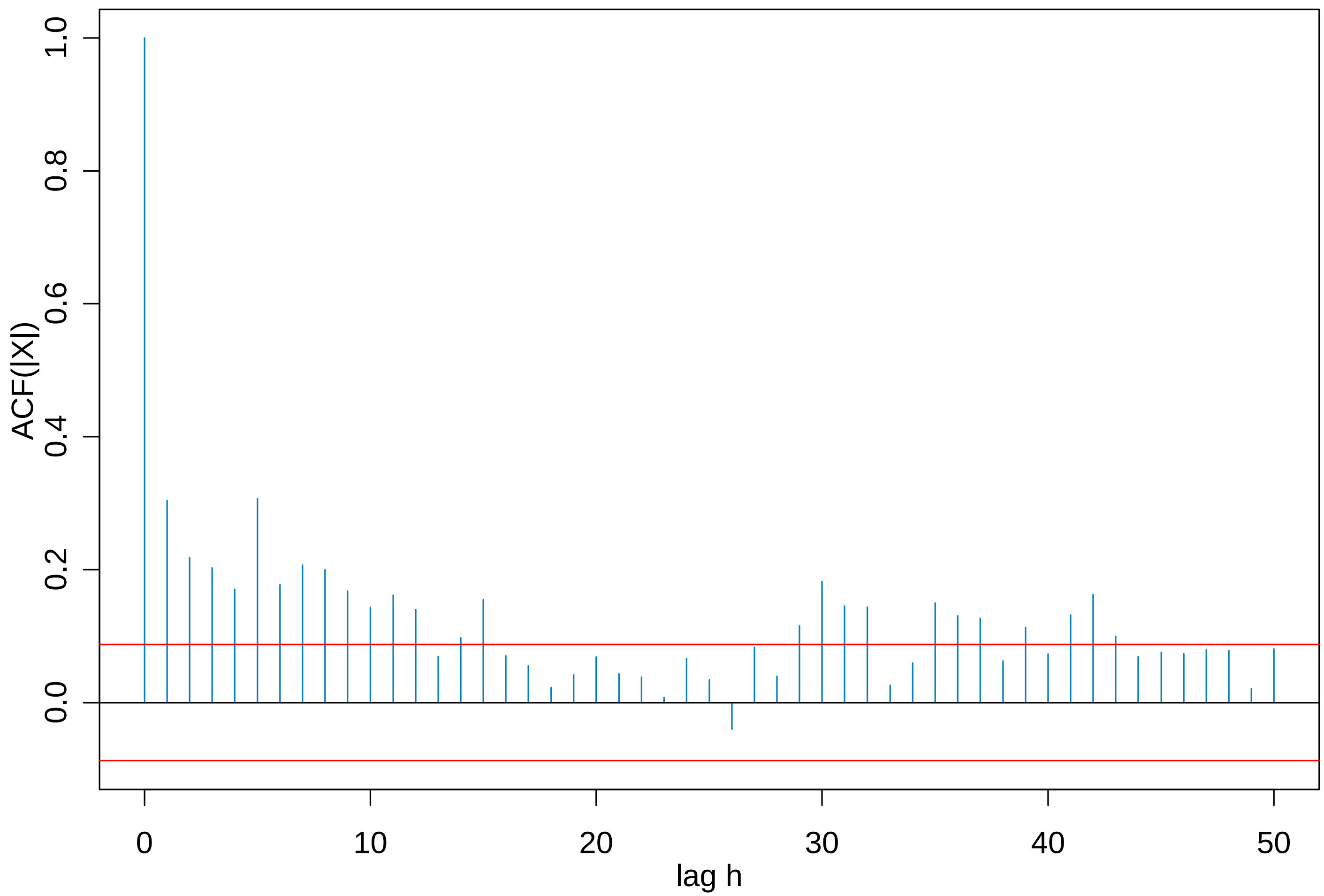
ACF of X(t)=log-returns of NZ/US exchange rate



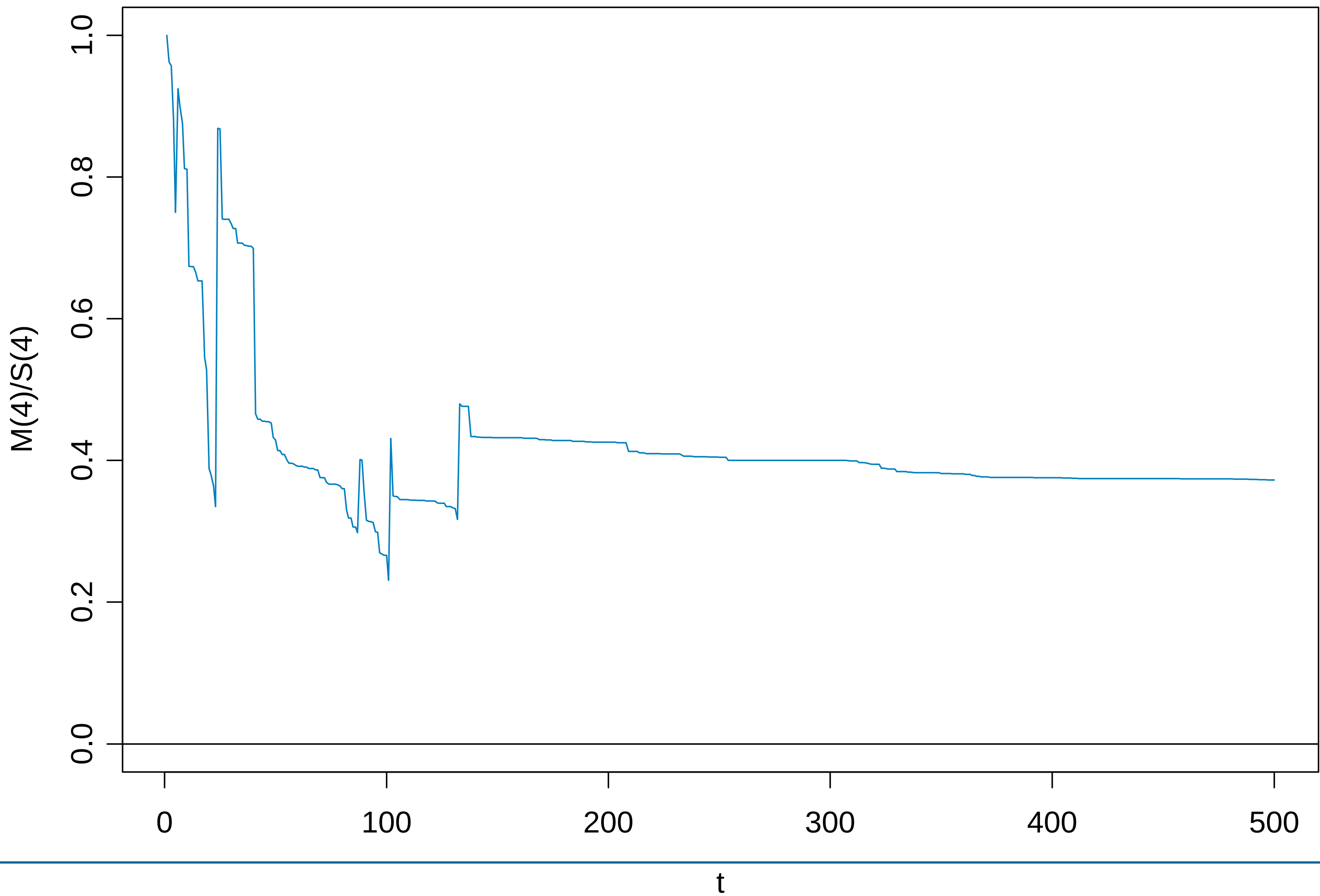
ACF of $X^2(t)$



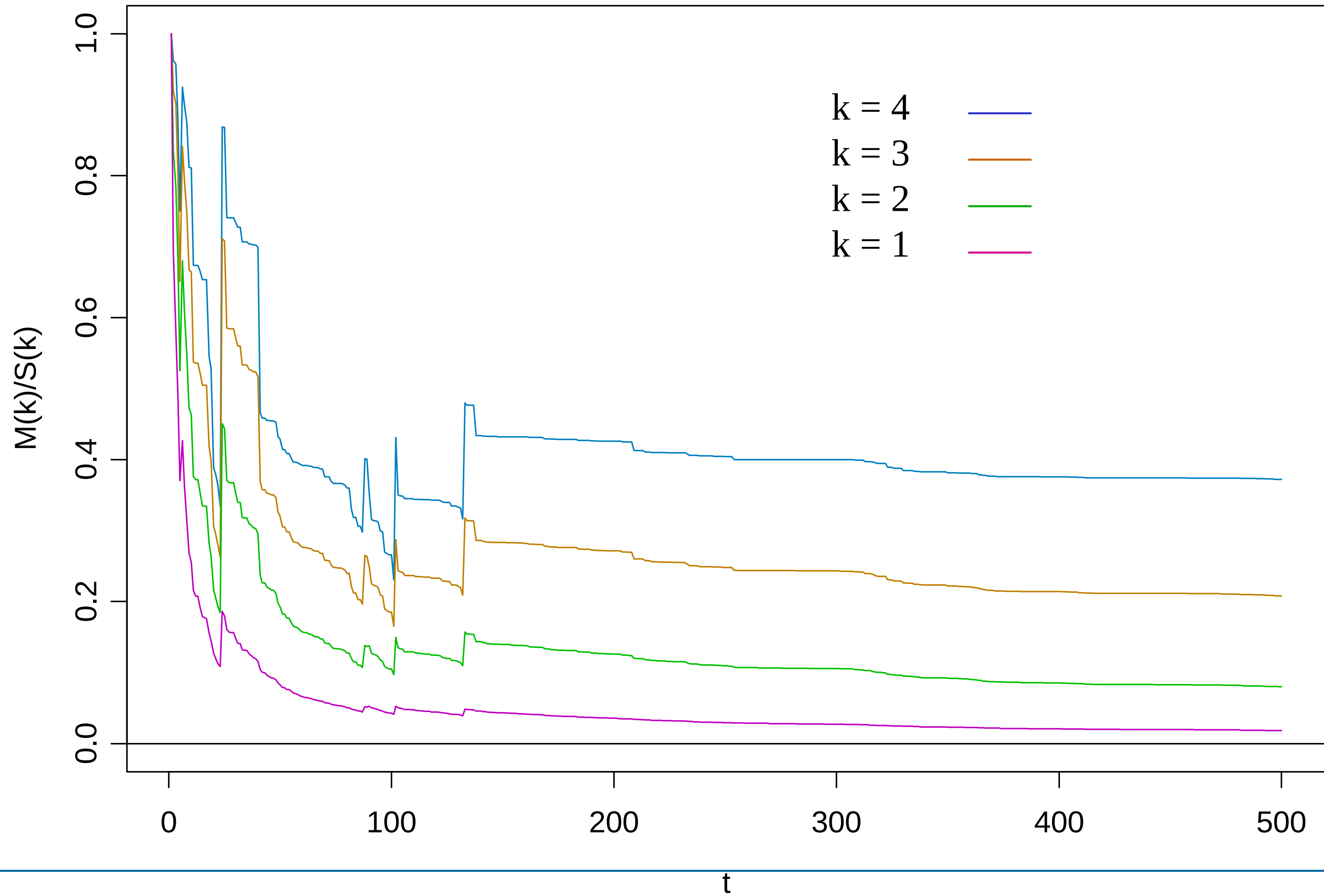
ACF of |X(t)|



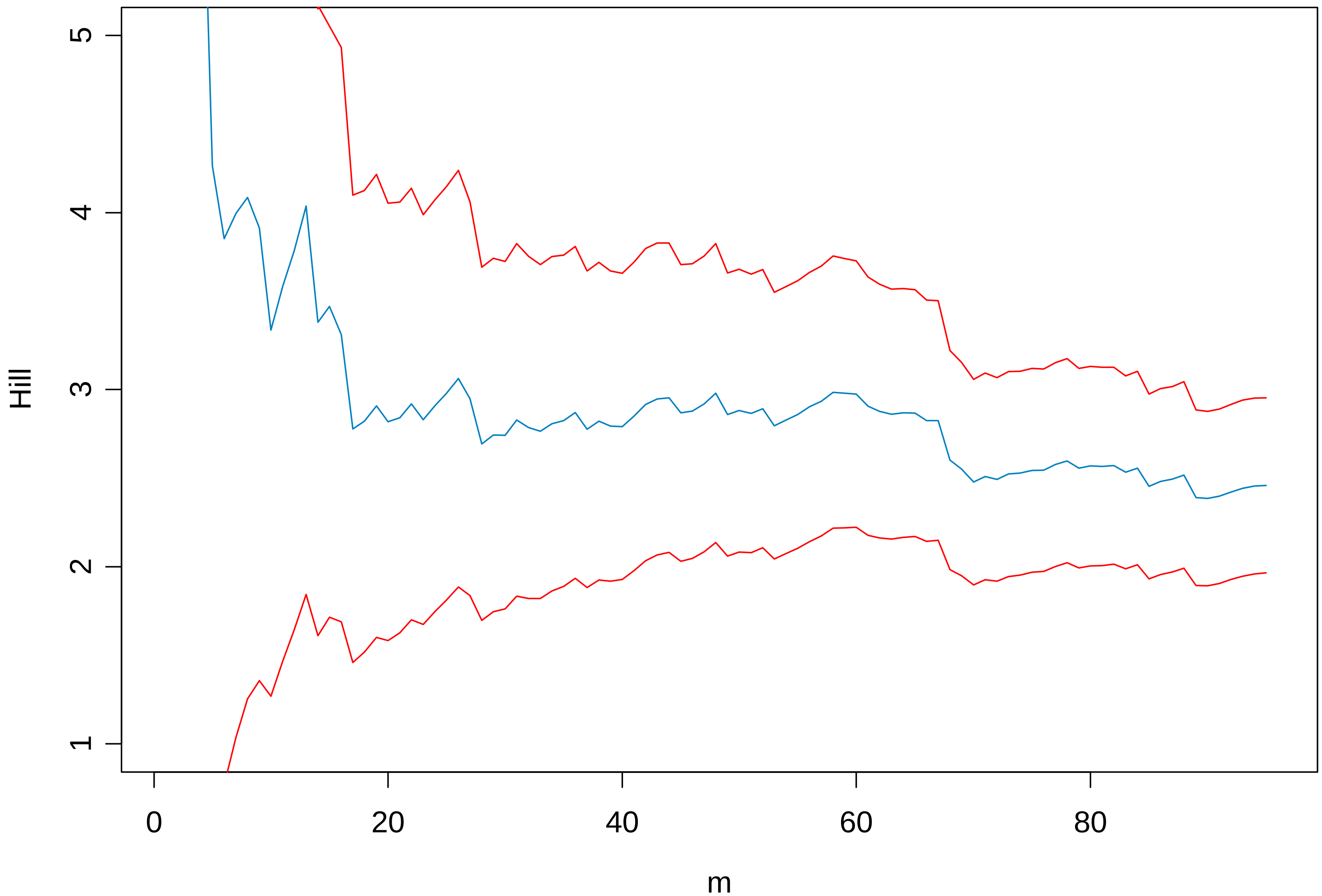
Plot of $M_t(4)/S_t(4)$



Plot of $M_t(k)/S_t(k)$



Hill's plot of tail index



Linear Processes

Model: $X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_t$ $\{Z_t\}$~IID, $P(|Z_t|>x) \sim Cx^{-\alpha}$, $0<\alpha<2$.

Properties:

- $P(|X_t|>x) \sim C_2\, x^{-\alpha}$
- Define $\rho(h) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} / \sum_{j=-\infty}^{\infty} \psi_j^2$.

If $\alpha > 2$

$$n^{1/2}(\rho(h) - \rho(h)) \xrightarrow{d} \sum_{j=1}^{\infty} (\rho(h+j)+\rho(h-j)-2\rho(j)\rho(h))\, N_j,$$ $\{N_t\}$~IIDN

If $0 < \alpha < 2$

$$(n / \ln n)^{1/\alpha} (\rho(h) - \rho(h)) \xrightarrow{d} \sum_{j=1}^{\infty} (\rho(h+j)+\rho(h-j)-2\rho(j)\rho(h))\, S_j / S_0,$$

$\{S_t\}$~IID stable (α), S_0 stable ($\alpha/2$)

Background Results

Joint regular variation of $\mathbf{X}=(X_1, \ldots, X_m)$: There exist constants x_n and a random vector $\theta \in S^{m-1}$ such that

$$nP(|\mathbf{X}|>tx_n\,,\ \mathbf{X}/|\mathbf{X}| \in \bullet) \to t^{-\alpha} P(\theta \in \bullet)$$

(vague convergence on S^{m-1}).

Mixing condition $\mathcal{A}(a_n)$ for a stationary sequence $\{\mathbf{X}_t\}$: Let a_n be such that $nP(|\mathbf{X}_t|> a_n)\to 1$. Then $\mathcal{A}(a_n)$ holds if there exists a sequence of integers r_n such that $r_n \to\infty$, $k_n=[n/r_n] \to\infty$ and

$$E\exp\left\{-\sum_{t=1}^{n} f(\mathbf{X}_t/a_n)\right\} - \left(E\exp\left\{-\sum_{t=1}^{r_n} f(\mathbf{X}_t/a_n)\right\}\right)^{k_n} \to 0,$$

for every bounded , non-negative step function f on $R^m/\{\mathbf{0}\}$ with bounded support.

Point Process Convergence

Theorem (Davis & Hsing `95, Davis & Mikosch `97). Let $\{\mathbf{X}_t\}$ be a stationary sequence of random vectors. Suppose

(i) finite dimensional distributions are jointly regularly varying (let $(\theta_{-k}, \ldots, \theta_k)$ be the vector in $S^{(2k+1)m-1}$ in the definition).

(ii) mixing condition $\mathcal{A}(a_n)$ or strong mixing.

(iii) $\lim_{k\to\infty} \limsup_{n\to\infty} P\left(\bigvee_{k \le |t| \le r_n} |\mathbf{X}_t| > a_n y \,\Big|\, |\mathbf{X}_0| > a_n y\right) = 0.$

Then

$$\gamma = \lim_{k\to\infty} E\left(|\theta_0^{(k)}|^\alpha - \bigvee_{j=1}^{k} |\theta_j^{(k)}|\right)_+ / E|\theta_0^{(k)}|^\alpha$$

exists. If $\gamma > 0$, then

$$N_n := \sum_{t=1}^{n} \varepsilon_{\mathbf{X}_t / a_n} \xrightarrow{d} N := \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varepsilon_{P_i \mathbf{Q}_{ij}},$$

Point Process Convergence (cont)

where

- (P_i) are points of a Poisson process on $(0,\infty)$ with intensity function $\nu(dy)=\gamma\alpha y^{-\alpha-1}dy$.
- $\sum_{j=1}^{\infty}\varepsilon_{\mathbf{Q}_{ij}}$, $i \geq 1$, are iid point process with distribution Q, and Q is the weak limit of

$$\lim_{k\to\infty} E(|\theta_0^{(k)}|^\alpha - \bigvee_{j=1}^{k}|\theta_j^{(k)}|)_+ \, I_\bullet(\sum_{|t|\le k}\varepsilon_{\theta_t^{(k)}}) / E(|\theta_0^{(k)}|^\alpha - \bigvee_{j=1}^{k}|\theta_j^{(k)}|)_+$$

Application of Point Process Convergence

Set-up: Let $\{X_t\}$ be a stationary sequence and set

$$\mathbf{X}_t=\mathbf{X}_t(m) = (X_t, \ldots , X_{t+m}).$$

Suppose $\mathbf{X}_t$ satisfies the conditions of previous theorem. Then

$$N_n := \sum_{t=1}^{n} \varepsilon_{\mathbf{X}_t / a_n} \xrightarrow{d} N := \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varepsilon_{P_i \mathbf{Q}_{ij}},$$

Sample ACVF and ACF:

$$\hat{\gamma}_X(h) = n^{-1} \sum_{t=1}^{n-h} X_t X_{t+h}, \; h \geq 0, \qquad \text{ACVF}$$

$$\hat{\rho}_X(h) = \hat{\gamma}_X(h) / \hat{\gamma}_X(h), \; h \geq 1, \qquad \text{ACF}$$

If $EX_0^2 < \infty$, then define $\gamma_X(h) = EX_0X_h$ and $\rho_X(h)=\gamma_X(h)/ \gamma_X(0)$.

Limit Behavior of Sample ACVF and ACF

(i) If $\alpha \in (0,2)$, then

$$(na_n^{-2}\hat{\gamma}_X(h))_{h=0,\ldots,m} \xrightarrow{d} (V_h)_{h=0,\ldots,m}$$

$$(\hat{\rho}_X(h))_{h=1,\ldots,m} \xrightarrow{d} (V_h/V_0)_{h=1,\ldots,m},$$

where

$$V_h = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P_i^2 Q_{ij}^{(0)} Q_{ij}^{(h)}, \quad h = 0,\ldots,m.$$

(ii) If $\alpha \in (2,4)$ + additional condition, then

$$\left(na_n^{-2}(\hat{\gamma}_X(h) - \gamma_X(h)\right)_{h=0,\ldots,m} \xrightarrow{d} \left(V_h\right)_{h=0,\ldots,m}$$

$$\left(na_n^{-2}(\hat{\rho}_X(h) - \rho_X(h))\right)_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1}(0)\left(V_h - \rho_X(h)V_0\right)_{h=1,\ldots,m}.$$

Stochastic Recurrence Equations

$$\mathbf{Y}_t = \mathbf{A}_t \mathbf{Y}_{t-1} + \mathbf{B}_t, \quad (\mathbf{A}_t, \mathbf{B}_t) \sim \text{IID},$$

$\mathbf{A}_t$ $d \times d$ random matrices, $\mathbf{B}_t$ random d-vectors

Examples

ARCH(1): $X_t = (\alpha_0 + \alpha_1 X^2_{t-1})^{1/2} Z_t, \quad \{Z_t\} \sim \text{IID}$. Then the squares follow an SRE with $Y_t = X_t^2, \ A_t = \alpha_1 Z_t^2, \ B_t = \alpha_0 Z_t^2$.

GARCH(2,1): $X_t = \sigma_t Z_t, \ \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \alpha_2 X_{t-2}^2 + \beta_1 \sigma_{t-1}^2$.

Then $\mathbf{Y}_t = (X_t^2, X_{t-1}^2, \sigma_t^2)'$ follows the SRE given by

$$\begin{bmatrix} X_t^2 \\ X_{t-1}^2 \\ \sigma_t^2 \end{bmatrix} = \begin{bmatrix} \alpha_1 Z_t^2 & \alpha_2 Z_t^2 & \beta_1 Z_t^2 \\ 1 & 0 & 0 \\ \alpha_1 & \alpha_2 & \beta_1 \end{bmatrix} \begin{bmatrix} X_{t-1}^2 \\ X_{t-2}^2 \\ \sigma_{t-1}^2 \end{bmatrix} + \begin{bmatrix} \alpha_0 Z_t^2 \\ 0 \\ 0 \end{bmatrix}$$

Stochastic Recurrence Equations (cont)

Examples (tricks)

GARCH(1,1): $X_t = \sigma_t Z_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2$.

Although this process does not have a 1-dimensional SRE representation, the process σ_t^2 does. Iterating, we have

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 Z_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$
$$= (\alpha_1 Z_{t-1}^2 + \beta_1)\sigma_{t-1}^2 + \alpha_0.$$

Bilinear(1): $X_t = aX_{t-1} + bX_{t-1}Z_{t-1} + Z_t, \quad \{Z_t\} \sim \text{IID}$

$$= Y_{t-1} + Z_t \,,$$

$$Y_t = A_t Y_{t-1} + B_t \quad A_t = a + bZ_t, \; B_t = A_t Z_t$$

Stochastic Recurrence Equations (cont)

$$\mathbf{Y}_t = \mathbf{A}_t \mathbf{Y}_{t-1} + \mathbf{B}_t, \quad (\mathbf{A}_t , \mathbf{B}_t) \sim \text{IID}$$

Existence of stationary solution

- $E \ln^+ \| \mathbf{A}_1 \| < \infty$
- $E \ln^+ \| \mathbf{B}_1 \| < \infty$
- $\inf n^{-1} E \ln \| \mathbf{A}_1 \dots \mathbf{A}_n \| =: \gamma < 0$ (γ –top Lyapunov exponent)

Ex. (d=1) $E \ln |\mathbf{A}_1| < 0$.

Strong mixing

If $E \|\mathbf{A}_1\|^\varepsilon < \infty$, $E |\mathbf{B}_1|^\varepsilon < \infty$ for some $\varepsilon > 0$, then the SRE $(\mathbf{Y}_t)$ is *geometrically ergodic* $\Rightarrow$ *strong mixing* with geometric rate (Meyn and Tweedie `93).

Stochastic Recurrence Equations (cont)

Regular variation of the marginal distribution (Kesten)

Assume **A** and **B** have non-negative entries and

- $E\,\|\mathbf{A}_1\|^{\varepsilon} < 1$ for some $\varepsilon > 0$
- $\mathbf{A}_1$ has no zero rows a.s.
- W.P. 1, $\{\ln \rho(\mathbf{A}_1 \ldots \mathbf{A}_n)$: is dense in $\mathbf{R}$ for some n, $\mathbf{A}_1 \ldots \mathbf{A}_n > 0\}$
- There exists a $\kappa_0 > 0$ such that $E\|A\|^{\kappa_0} \ln^+ \|A\| < \infty$ and

$$E\left(\min_{i=1,\ldots,d} \sum_{j=1}^{d} A_{ij}\right)^{\kappa_0} \geq d^{\kappa_0/2}$$

Then there exists a $\kappa_1 \in (0, \kappa_0]$ such that $\mathbf{Y}$ is regularly varying with index κ_1. (Also need $E\,|B|^{\kappa_1} < \infty$.)

Application to GARCH

Proposition: Let $(\mathbf{Y}_t)$ be the soln to the SRE based on the squares of a GARCH model. Assume

- Top Lyapunov exponent $\gamma < 0$. (See Bougerol and Picard`92)
- Z has a positive density on $(-\infty, \infty)$ with all moments finite or $E|Z|^h = \infty$, for all $h \geq h_0$ and $E|Z|^h < \infty$ for all $h < h_0$.
- Not all the GARCH parameters vanish.

Then $(\mathbf{Y}_t)$ is *strongly mixing* with geometric rate and all finite dimensional distributions are *regularly varying* with index κ_1.

Corollary: The corresponding GARCH process is strongly mixing and has all finite dimensional distributions that are regularly varying with index $\kappa = 2\kappa_1$.

Application to GARCH (cont)

Remarks:

1. Kesten's result applied to an iterate of $\mathbf{Y}_t$, i.e., $\mathbf{Y}_{tm} = \tilde{\mathbf{A}}_t \mathbf{Y}_{(t-1)m} + \tilde{\mathbf{B}}_t$

2. Determination of κ is difficult. Explicit expressions only known in two(?) cases.

- ARCH(1): $E|\alpha Z^2|^{\kappa/2} = 1$.

α	.312	.577	1.00	1.57
κ	8.00	4.00	2.00	1.00

- GARCH(1,1): $E|\alpha Z^2 + \beta|^{\kappa/2} = 1$ (Mikosch and St→ric→)
- For IGARCH ($\alpha + \beta = 1$), then $\kappa = 2 \Rightarrow$ infinite variance.
- Can estimate κ empirically by replacing expectations with sample moments.

Summary for GARCH(p,q)

$\kappa \in (0,2)$:

$$(\hat{\rho}_X(h))_{h=1,\ldots,m} \xrightarrow{d} (V_h / V_0)_{h=1,\ldots,m},$$

$\kappa \in (2,4)$:

$$\left(n^{1-2/\alpha}\hat{\rho}_X(h)\right)_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1}(0)\left(V_h\right)_{h=1,\ldots,m}.$$

$\kappa \in (4,\infty)$:

$$\left(n^{1/2}\hat{\rho}_X(h)\right)_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1}(0)\left(G_h\right)_{h=1,\ldots,m}.$$

Remark: Similar results hold for the sample ACF based on $|X_t|$ and X_t^2.

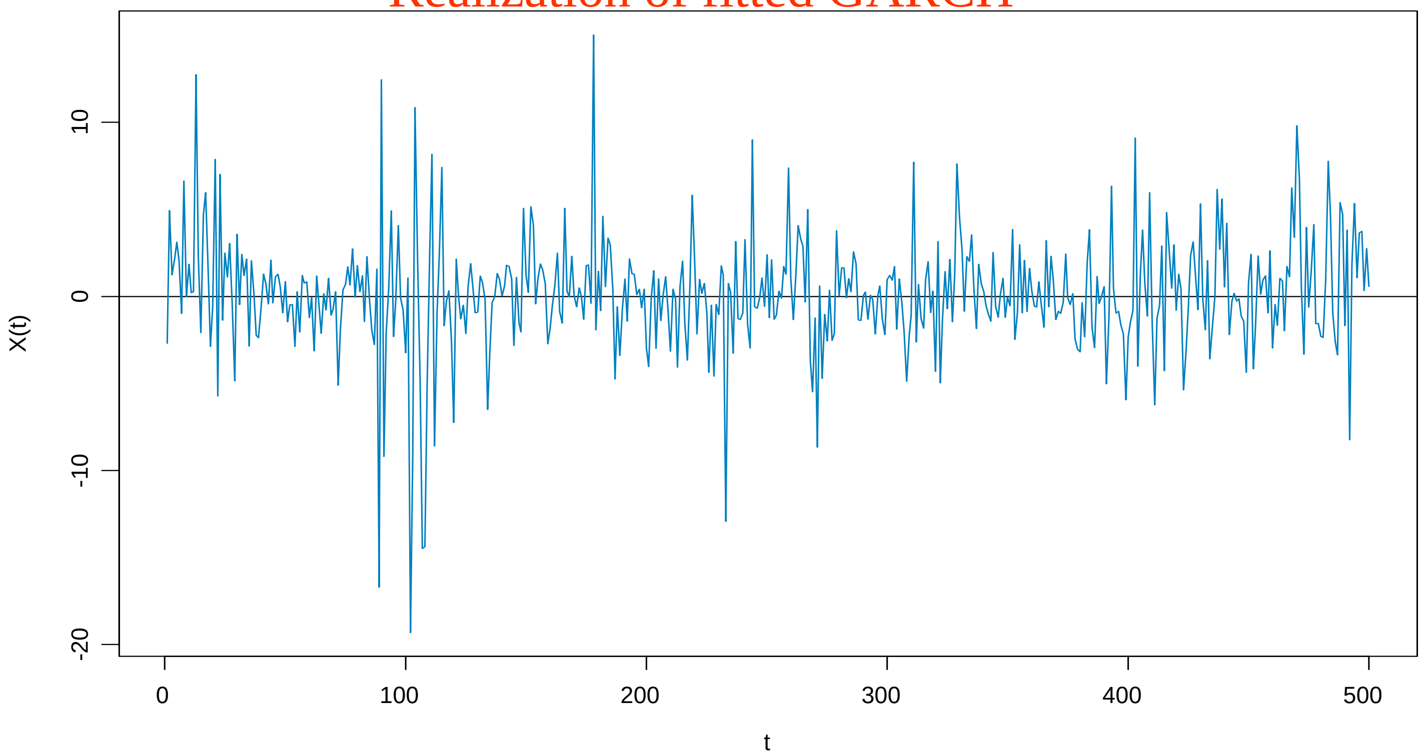
Realization of GARCH Process

Fitted GARCH(1,1) model for NZ-USA exchange:

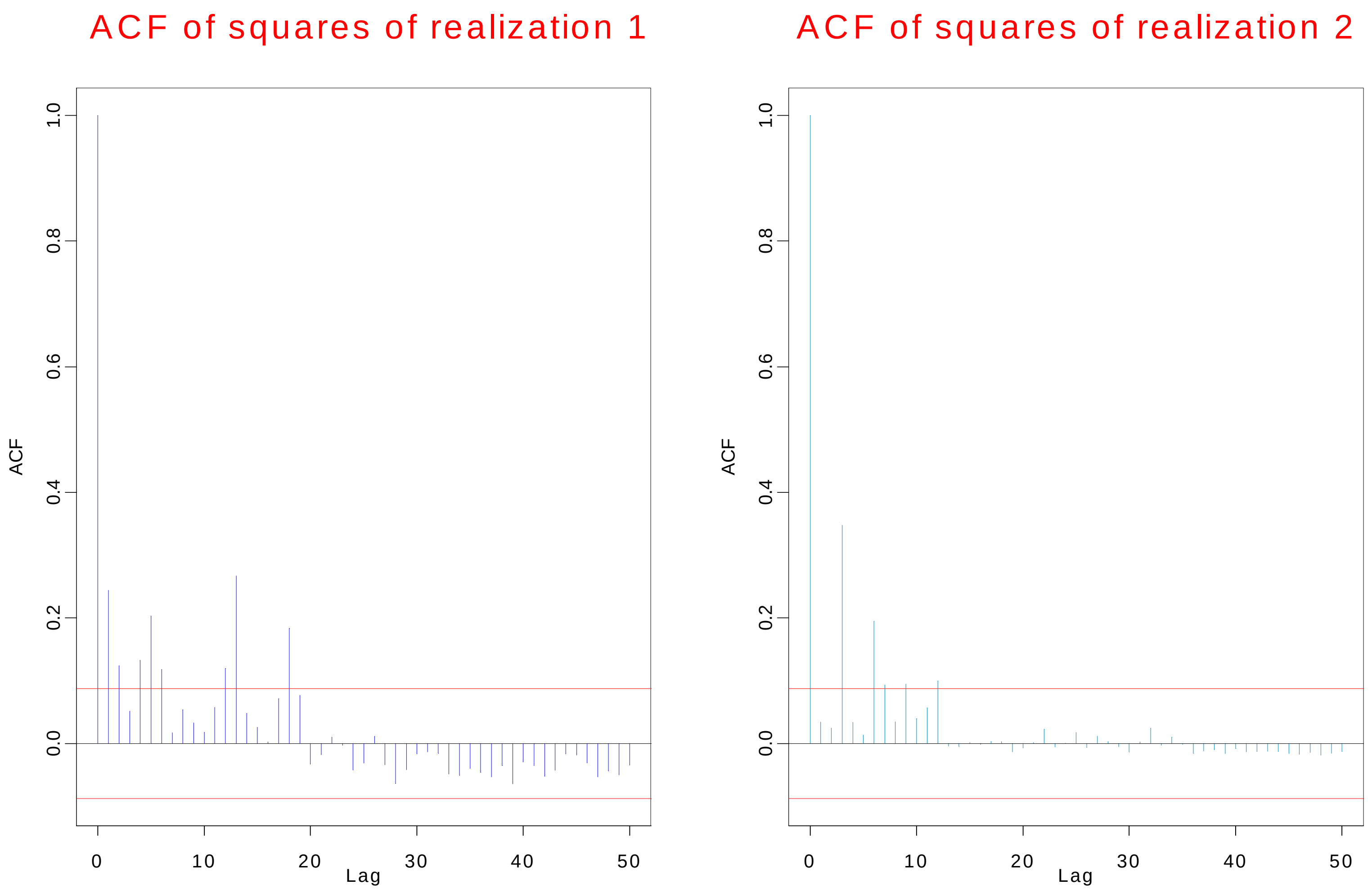
$$X_t = \sigma_t Z_t, \quad \sigma_t^2 = (6.70)10^{-7} + .1519 X_{t-1}^2 + .772\sigma_{t-1}^2$$

$(Z_t) \sim$ IID t-distr with 5 df. κ is approximately 3.8

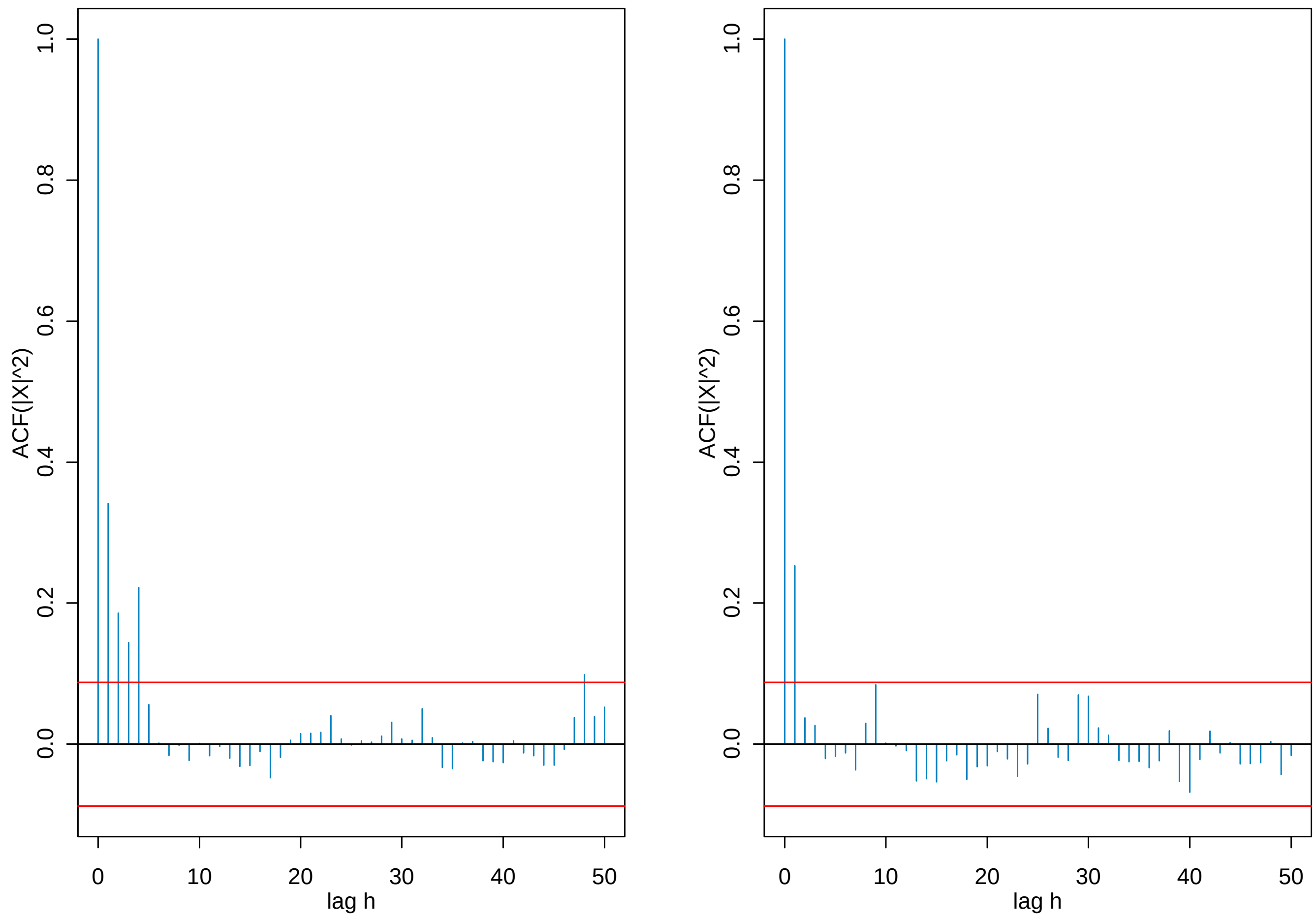
Realization of fitted GARCH



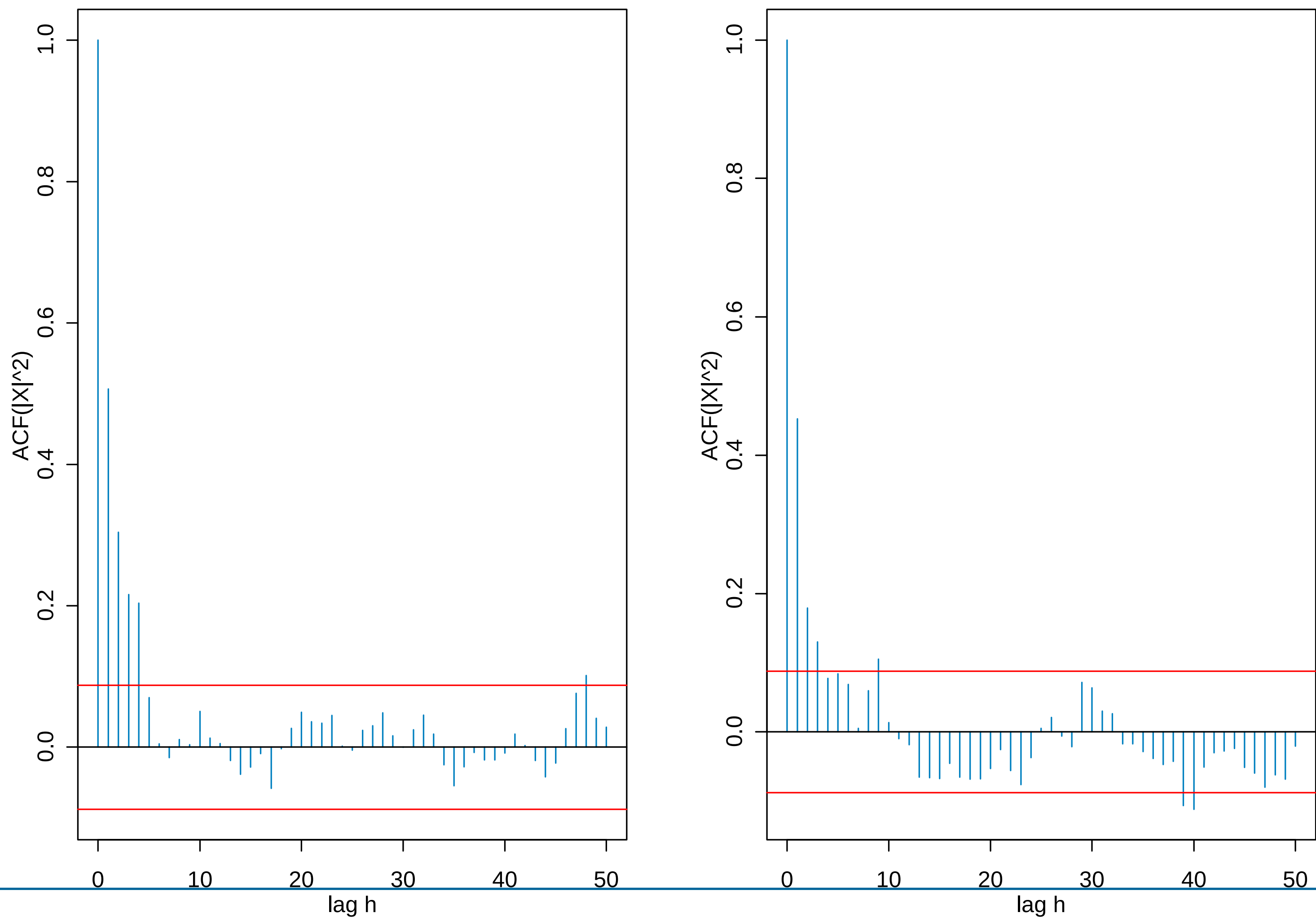
ACF of Fitted GARCH(1,1) Process



ACF of 2 realizations of an (ARCH)^2: $X_t=(.001+.7\ X_{t-1})^{1/2}\ Z_t$



ACF of 2 realizations of an |ARCH|: $X_t=(.001+ X_{t-1})^{1/2} Z_t$



Stochastic Volatility Models

SVM: $X_t = \sigma_t Z_t$

- $(Z_t) \sim$ IID with mean 0 (if it exists)
- (σ_t) is a stationary process ($2 \log \sigma_t$ is a linear process) given by

$$\log \sigma_t^2 = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}, \quad \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty, (\varepsilon_t) \sim \text{IID N}(0, \sigma^2)$$

Heavy tailedness: Assume Z_t has Pareto tails with index α, i.e.,

$$P(|Z_t| > z) \sim C z^{-\alpha} \Rightarrow P(|X_t| > z) \sim C E\sigma^\alpha z^{-\alpha}.$$

Then if $\alpha \in (0,2)$,

$$(n / \ln n)^{1/\alpha} \hat{\rho}_X(h) \xrightarrow{d} \frac{\|\sigma_1 \sigma_{h+1}\|_\alpha}{\|\sigma_1\|_\alpha^2} \frac{S_h}{S_0}.$$

Stochastic Volatility Models (cont)

Other powers:

1. Absolute values: $\alpha \in (1,2)$,

$$E|X_t| = E|\sigma_t| E|Z_t|, \quad E|X_t X_{t+h}| = (E|\sigma_t \sigma_{t+h}|)(E|Z_t| E|Z_{t+h}|)$$

$$\mathrm{Cov}(X_t, X_{t+h}) = \mathrm{Cov}(\sigma_t, \sigma_{t+h})(E|Z|)^2$$

$$\mathrm{Cor}(X_t, X_{t+h}) = \mathrm{Cor}(\sigma_t, \sigma_{t+h})(E|Z|)^2 / EZ^2$$

$$= 0 \ (?).$$

We obtain

$$n(n \ln n)^{-1/\alpha}\left(\hat{\gamma}_{|X|}(h) - \gamma_{|X|}(h)\right) \xrightarrow{d} \|\sigma_1 \sigma_{h+1}\|_\alpha S_h$$

and

$$(n / \ln n)^{1/\alpha} \hat{\rho}_X(h) \xrightarrow{d} \frac{\|\sigma_1 \sigma_{h+1}\|_\alpha}{\|\sigma_1\|_\alpha^2} \frac{S_h}{S_0}.$$

Stochastic Volatility Models (cont)

2. Higher order: $\alpha \in (0,2)$

The squares are again a SV process and the results of the previous proposition apply. Namely,

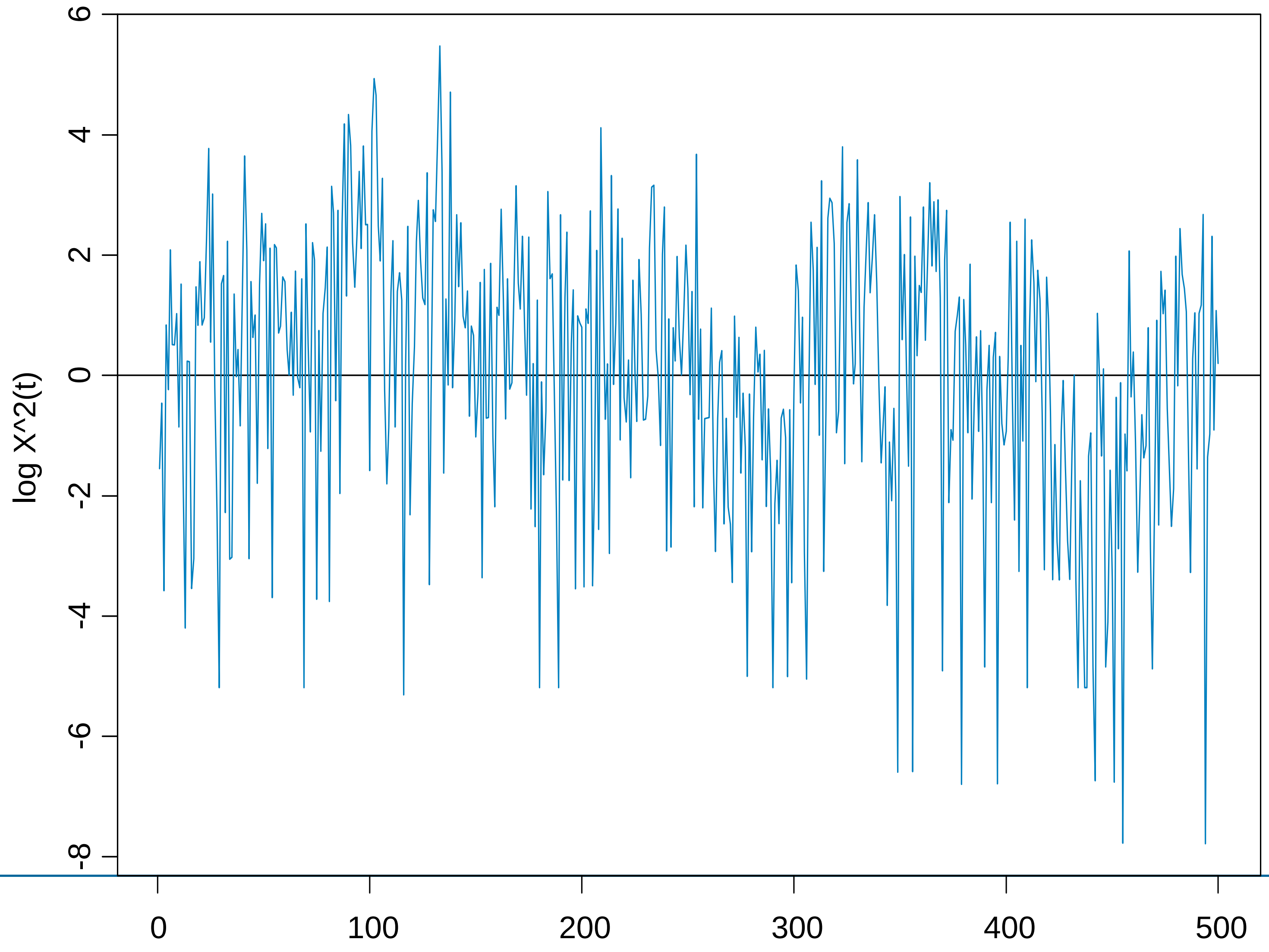
$$(n/\ln n)^{2/\alpha}\hat{\rho}_{X^2}(h) \xrightarrow{d} \frac{\left\|\sigma_1^2\sigma_{h+1}^2\right\|_{\alpha/2}}{\left\|\sigma_1^2\right\|_{\alpha/2}^2}\frac{S_h}{S_0}.$$

In particular,

$$\hat{\rho}_{X^2}(h) \xrightarrow{P} 0.$$

Stochastic Volatility Models (cont)

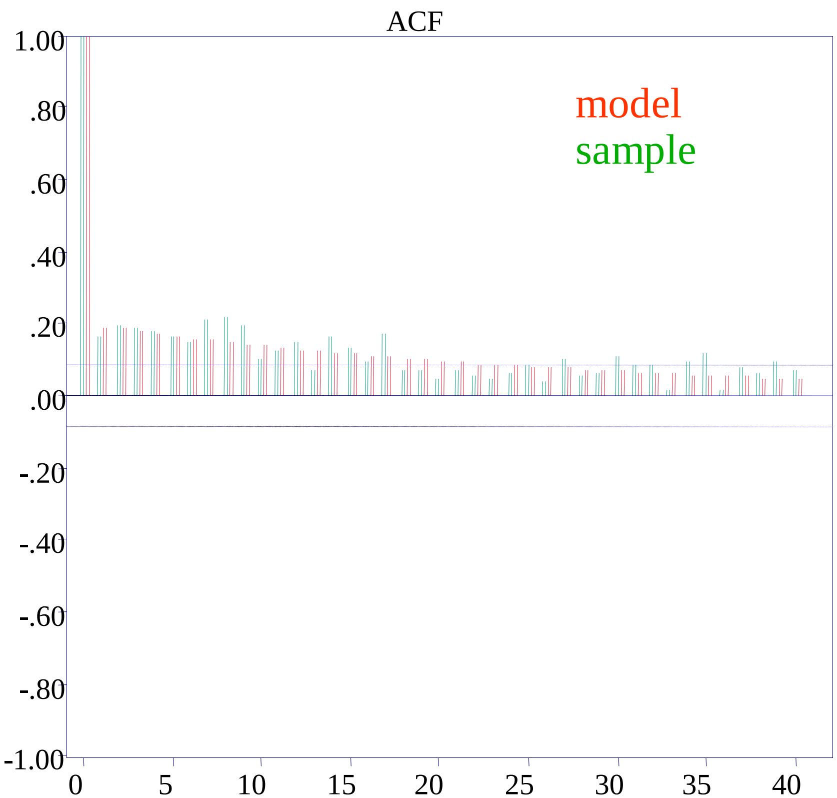
$(\log X^2)$ - mean for NZ-USA exchange rates

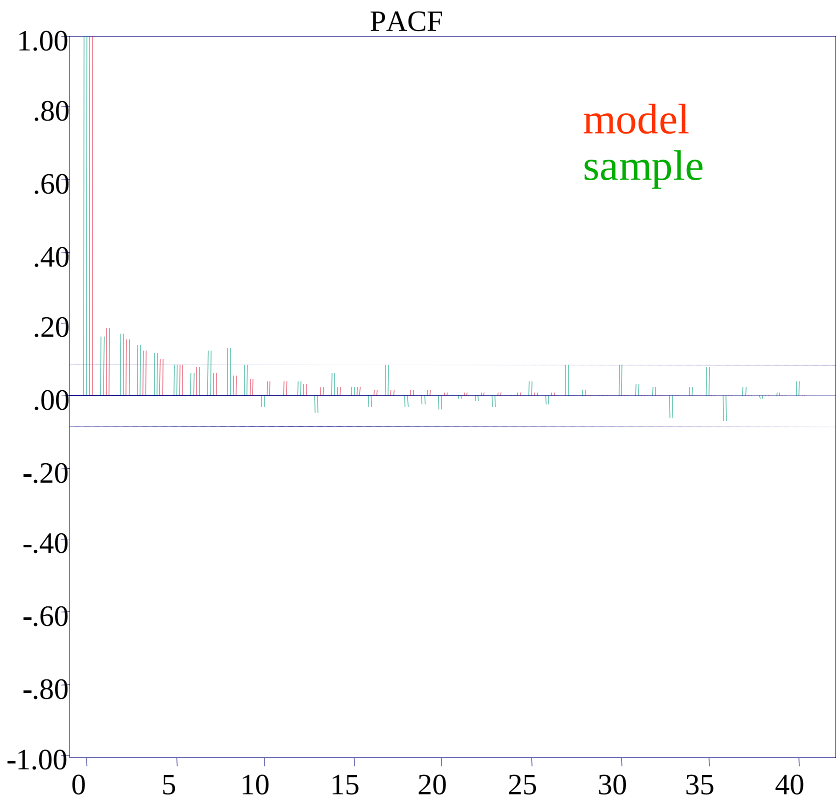


Stochastic Volatility Models (cont)

ACF/PACF for (log X^2) suggests ARMA (1,1) model:

μ= -11.5403, $Y_t = .9646Y_{t-1} + \varepsilon_t - .8709\,\varepsilon_{t-1}$, (ε_t)~WN(0,4.6653)





Stochastic Volatility Models (cont)

The ARMA (1,1) model for log X^2 leads to the SV model

$$X_t = \sigma_t Z_t$$

with

$$2 \ln \sigma_t = -11.5403 + v_t + \varepsilon_t$$

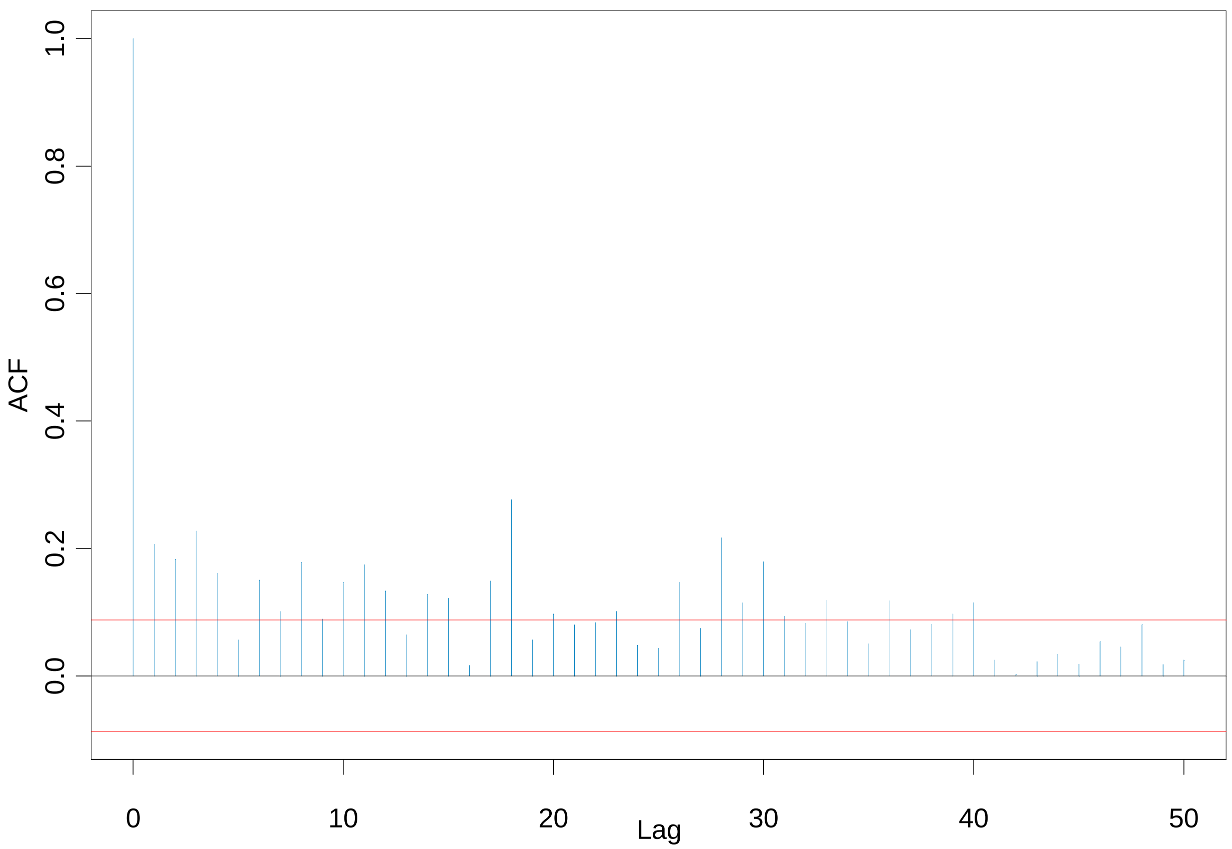
$$v_t = .9646\, v_{t-1} + \gamma_t\,, \quad (\gamma_t) \sim WN(0,.07253)$$

$$(\varepsilon_t) \sim WN(0,4.2432).$$

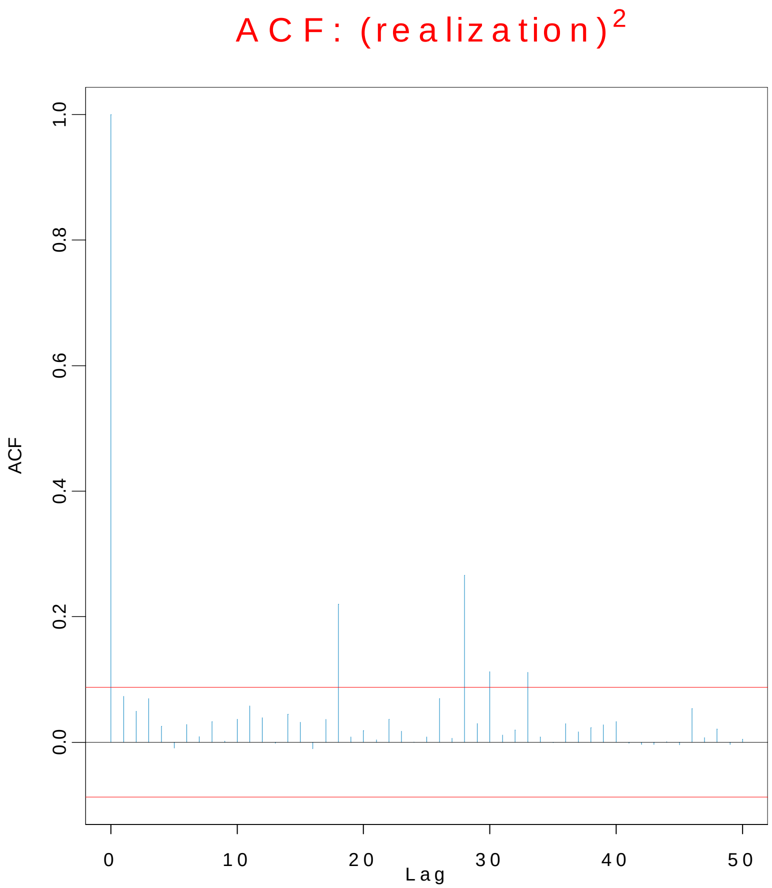
Stochastic Volatility Models (cont)

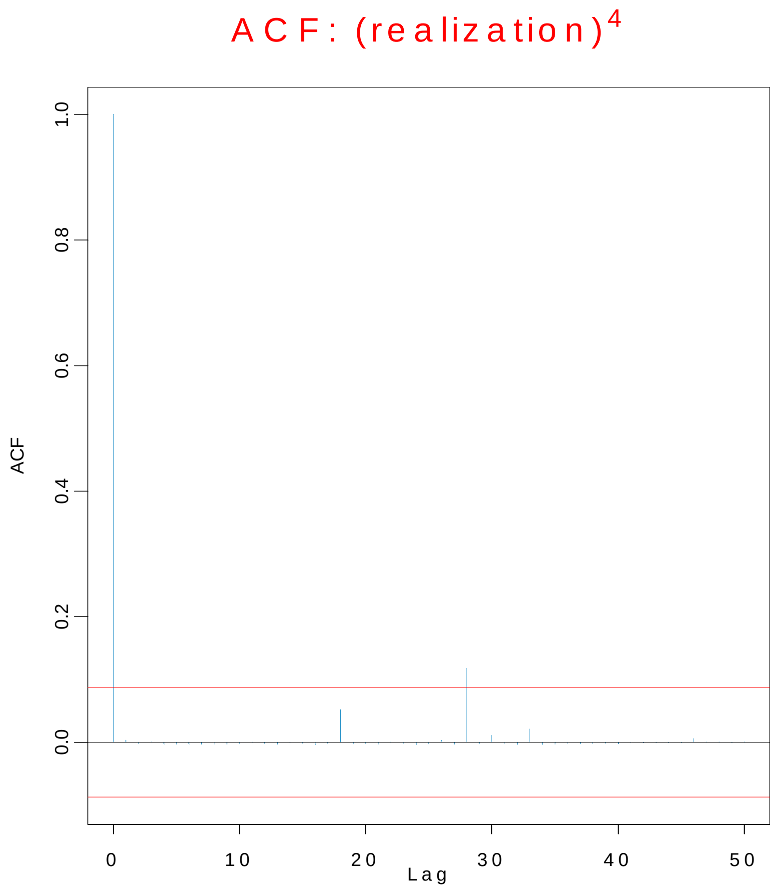
Simulation of SVM model: Took ε_t to be distributed according to log of a t random variable with 3 df (suitable normalized).

ACF: abs(realization)



Stochastic Volatility Models (cont)





Linear Processes With Nonlinear Behavior

Allpass ARMA

Causal AR polynomial: $\phi(z)=1-\phi_1 z - \ldots - \phi_p z^p$, $\phi(z) \neq 0$ for $|z|\leq 1$.

Define MA polynomial:

$$\theta(z) = -z^p \phi(z^{-1})/\phi_p = -(z^p - \phi_1 z^{p-1} - \ldots - \phi_p)/\phi_p$$

$$\neq 0 \text{ for } |z|\geq 1$$ (MA polynomial is non-invertible).

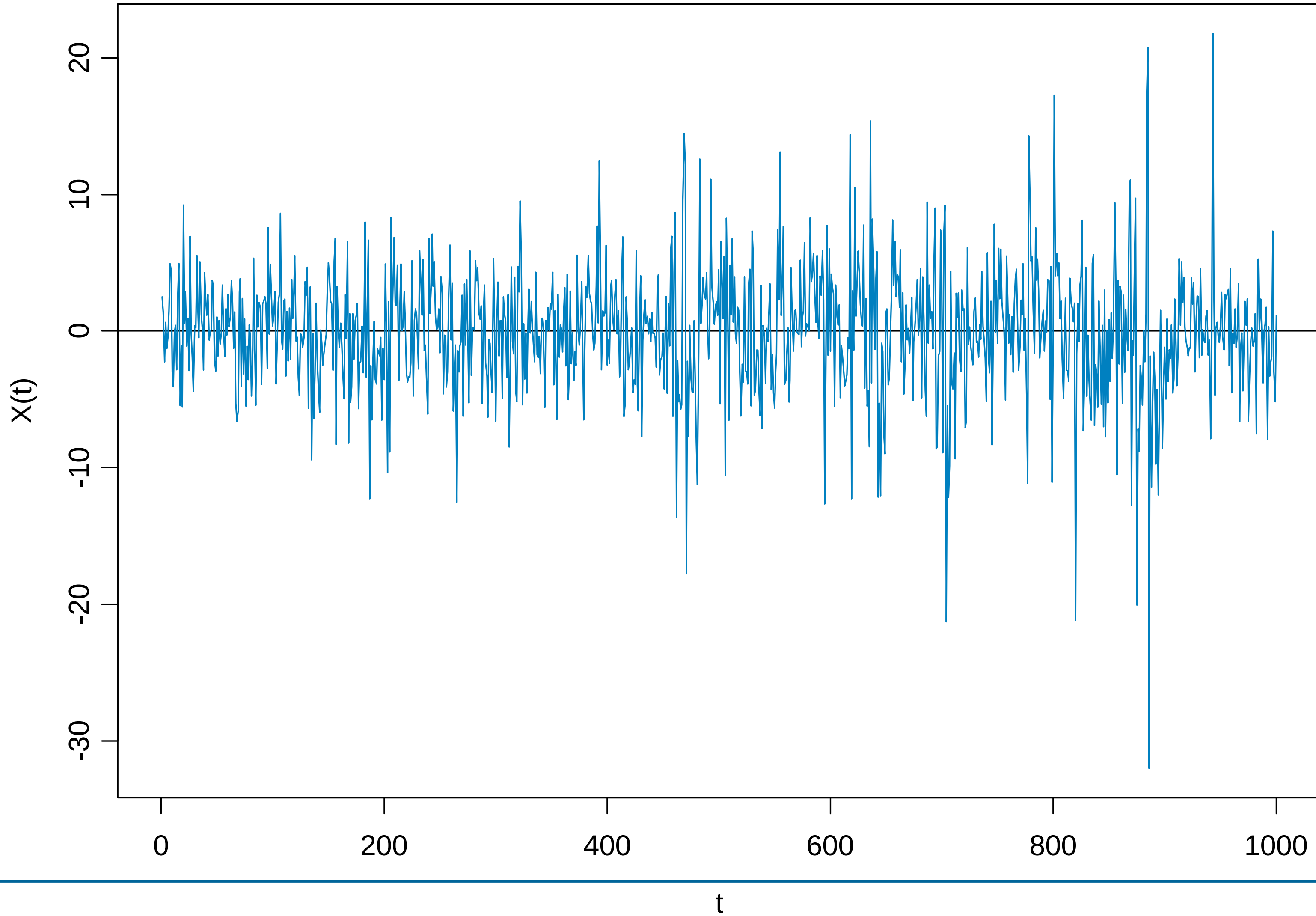
Model for data (X_t) : $\phi(B) X_t = \theta(B) Z_t$, $(Z_t) \sim$ IID (non-Gaussian)

Properties:

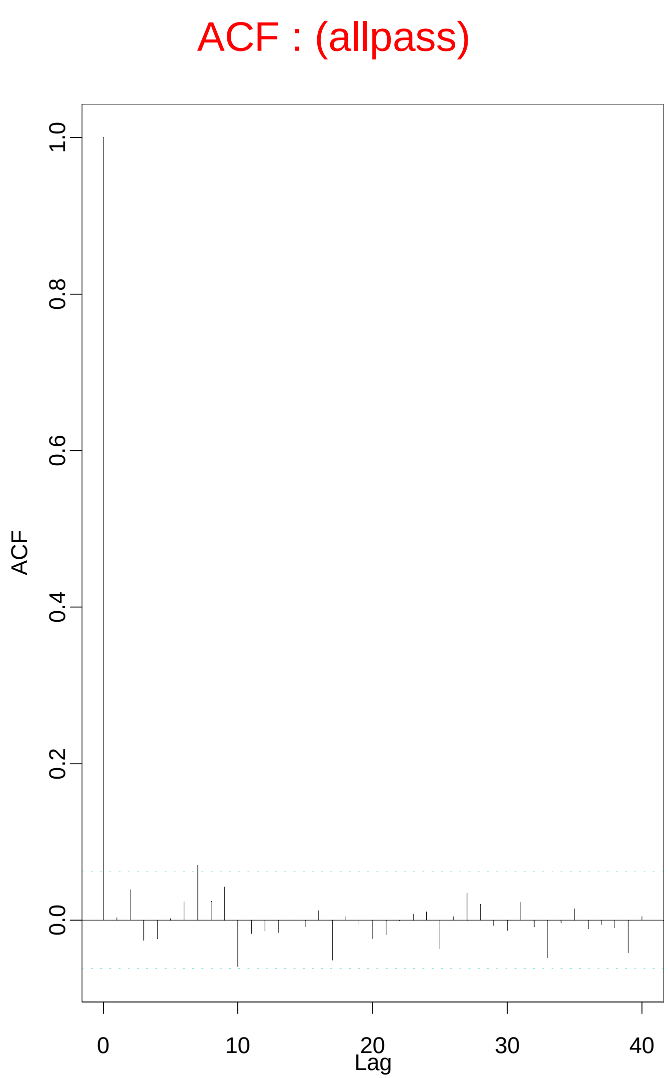
- uncorrelated (flat spectrum) but data are dependent
- squares and absolute values are correlated
- X_t has heavy tails if noise is heavy-tailed.

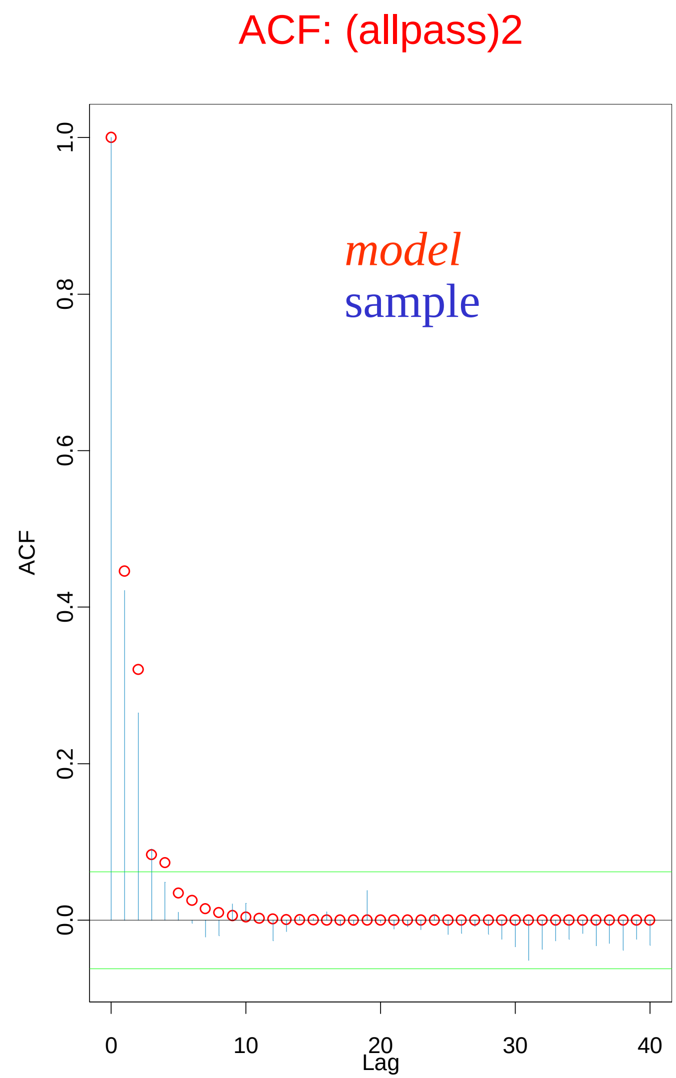
Linear Processes With Nonlinear Behavior (cont)

Realization of an allpass model of order 2 (t3 noise)



Linear Processes With Nonlinear Behavior (cont)



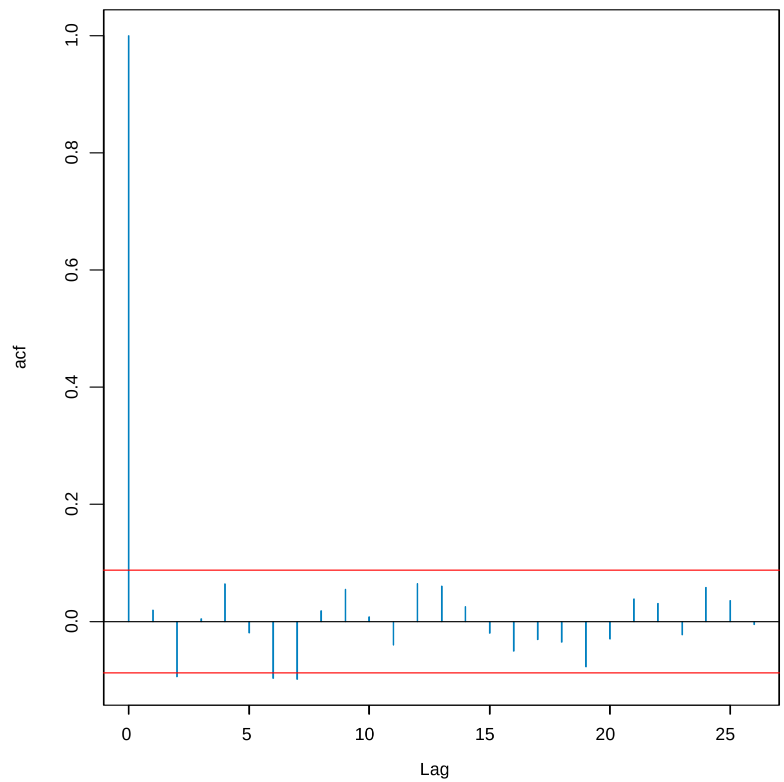


Linear Processes With Nonlinear Behavior (cont)

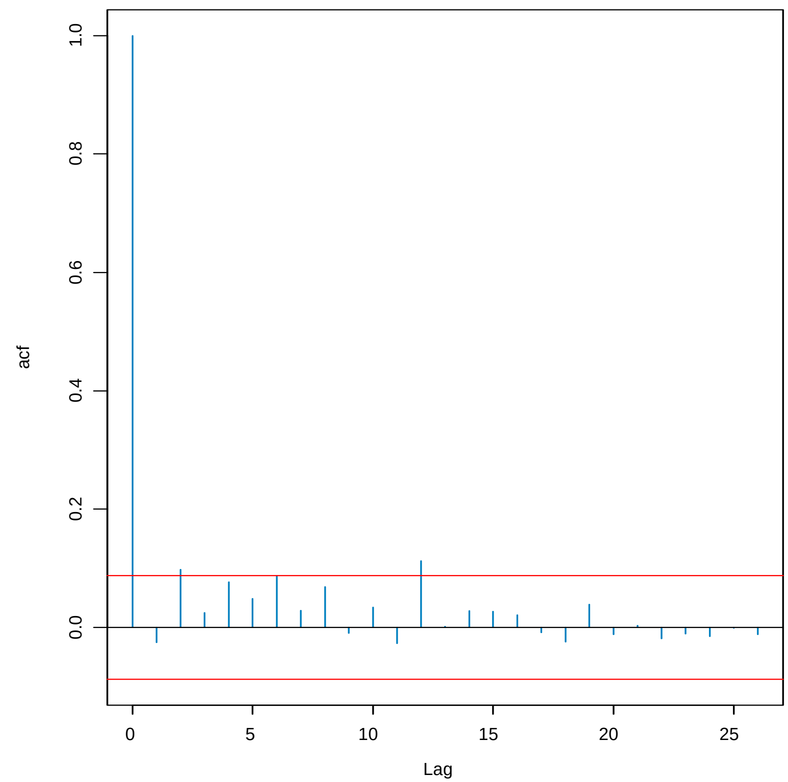
Allpass model fitted to NZ-USA exchange rates :

Order = 6, ϕ_1=.852, ϕ_2=.616, ϕ_3=. 952, ϕ_4=.098, ϕ_5= -.158, ϕ_6=-.066

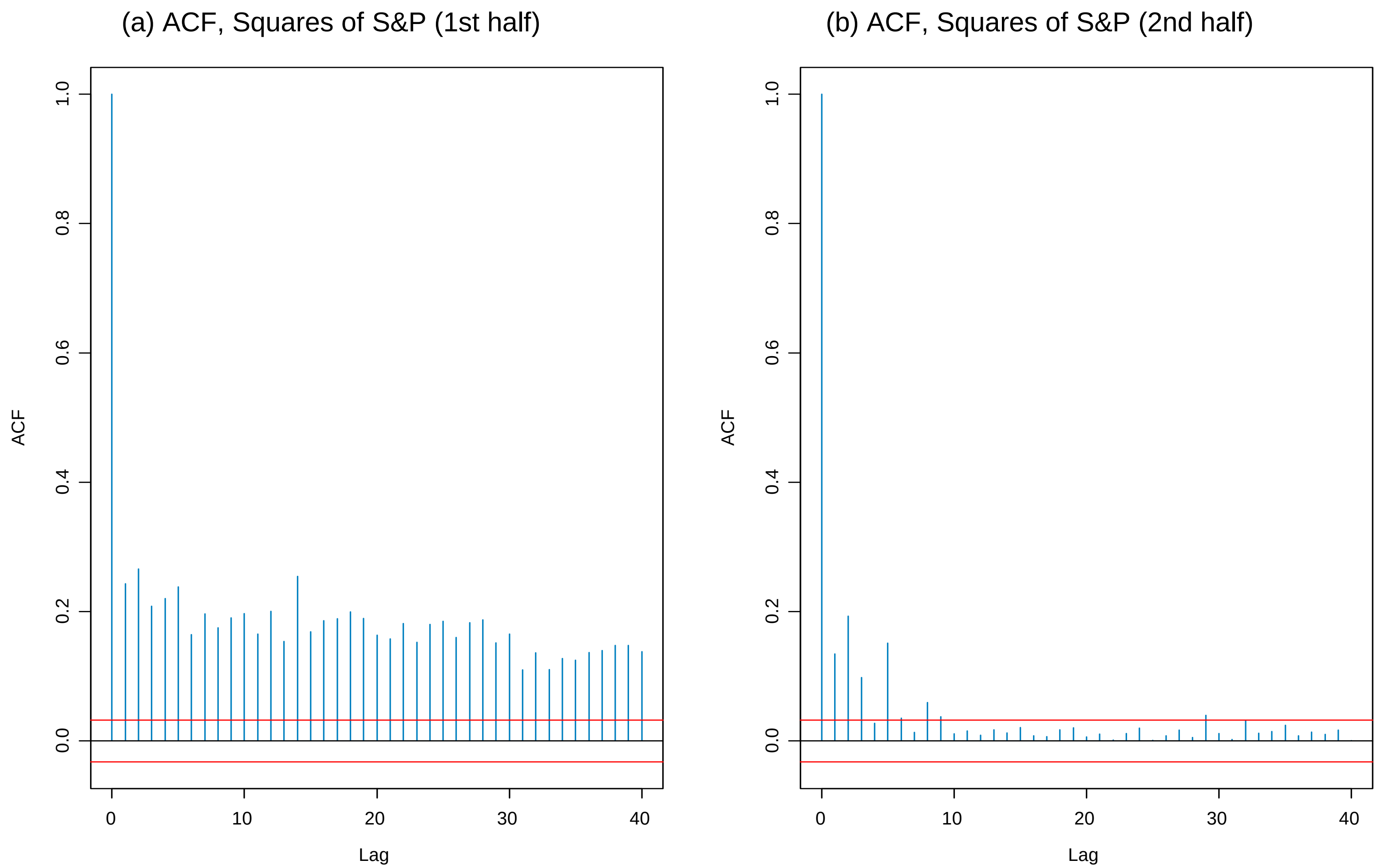
ACF: residuals



ACF: $(\text{residuals})^2$

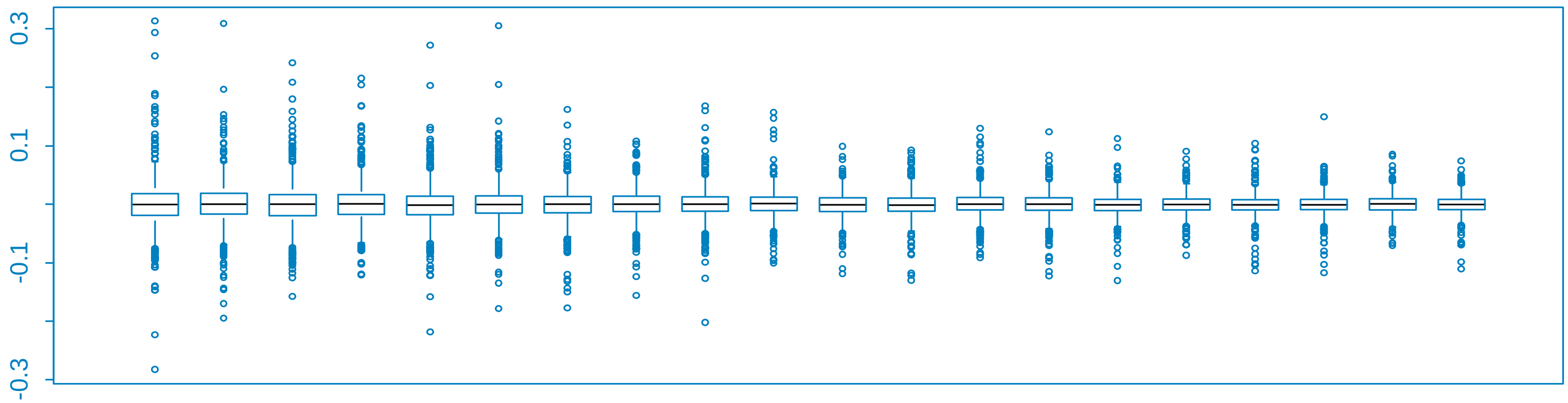


Sample ACF of squares for S&P (a) 1961-1976, (b) 1977-1993

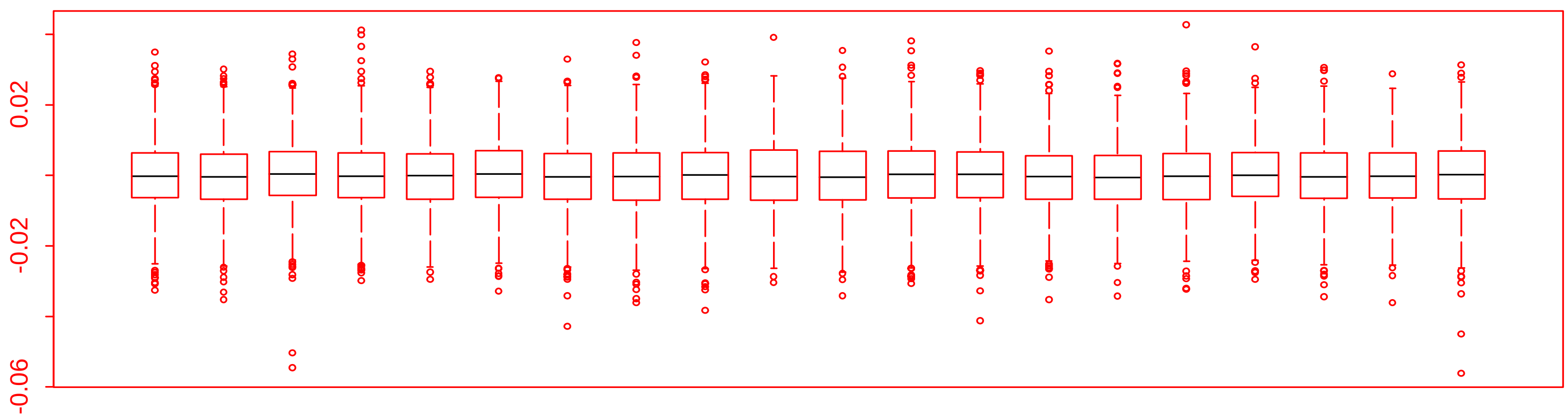


Sample ACF for GARCH and SV Models (1000 reps)

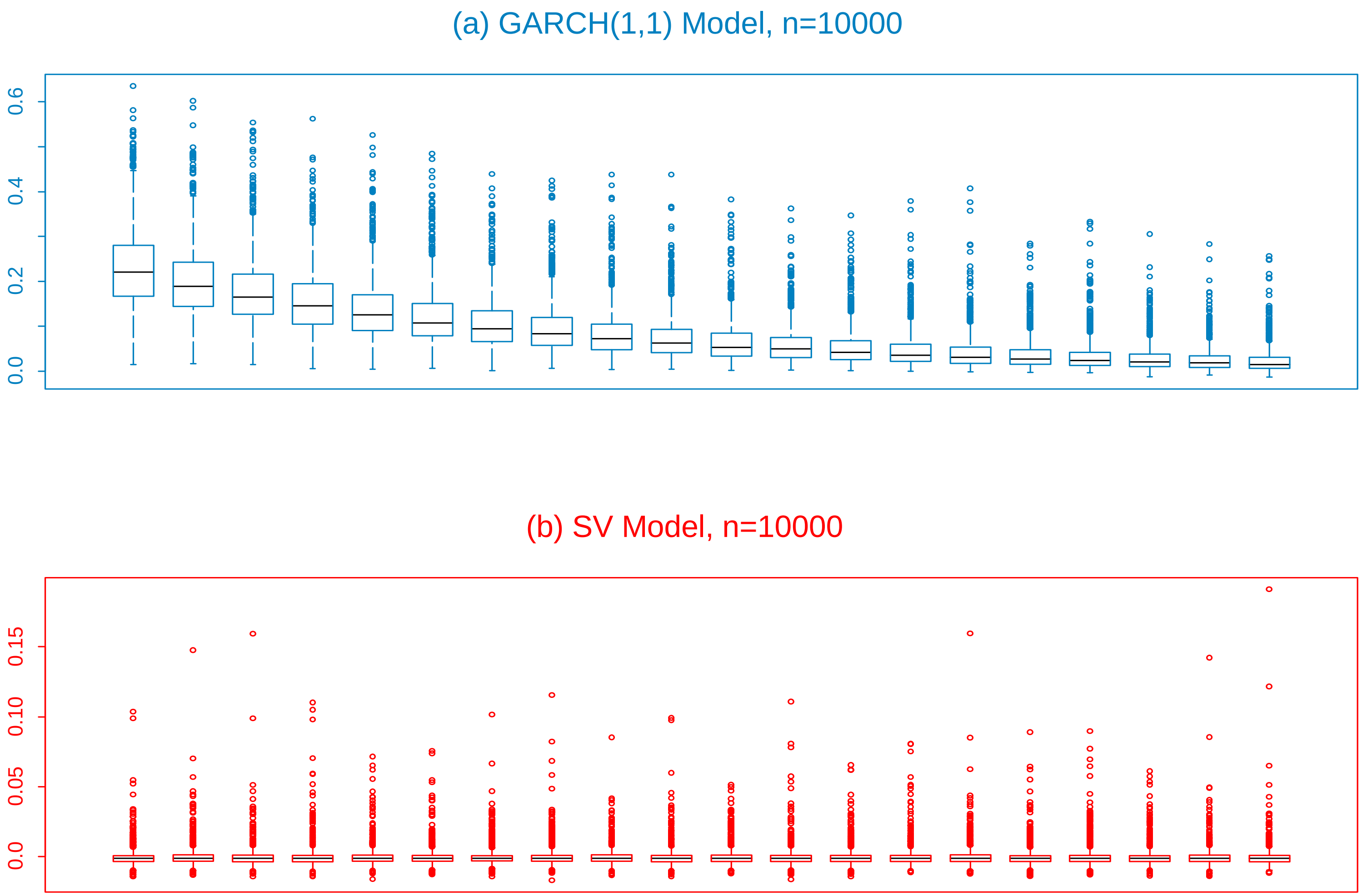
(a) GARCH(1,1) Model, n=10000



(b) SV Model, n=10000



Sample ACF for Squares of GARCH and SV (1000 reps)



Sample ACF for Squares of GARCH and SV (1000 reps)