Estimation for State Space Models: an Approximate Likelihood Approach

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Joint work with:
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Ying Wang, Dept of Public Health, W. Virginia
Example: Pound-Dollar Exchange Rates
(Oct 1, 1981 - June 28, 1985; Koopman website)
Motivating Examples

- Time series of counts
- Stochastic volatility

Generalized state-space models

- Observation driven
- Parameter driven

Model setup and estimation

- Generalized linear models (GLM)
- Estimating equations (Zeger)
- MCEM (Chan and Ledolter)
- Importance sampling
  - Durbin and Koopman
- Approximation to the likelihood (Davis, Dunsmuir, and Wang)

Application

- Time series of counts
- Stochastic volatility
Generalized State-Space Models

Observations: \( y^{(t)} = (y_1, \ldots, y_t) \)

States: \( \alpha^{(t)} = (\alpha_1, \ldots, \alpha_t) \)

Observation equation:
\[
p(y_t \mid \alpha_t) := p(y_t \mid \alpha_t, \alpha^{(t-1)}, y^{(t-1)})
\]

State equation:
- observation driven
\[
p(\alpha_{t+1} \mid y^{(t)}) := p(\alpha_{t+1} \mid \alpha_t, \alpha^{(t-1)}, y^{(t)})
\]
- parameter driven
\[
p(\alpha_{t+1} \mid \alpha_t) := p(\alpha_{t+1} \mid \alpha_t, \alpha^{(t-1)}, y^{(t)})
\]
Examples of observation driven models

Poisson model for time series of counts

Observation equation:

\[ p(y_t \mid \alpha_t) = \frac{e^{\alpha_t y_t - e^{-\alpha_t}}}{y_t!}, \quad y_t = 0, 1, ..., \]

State equation:

\[ \alpha_{t+1} = \mu + \theta(Y_t - \exp\{\alpha_t\})/\exp\{\alpha_t/2\} \]

where the equation is defined recursively as a function of the past of the \( y_t \)'s.

Remarks:

- This is an example of a GLARMA(0,1) (see Davis, Dunsmuir, Streett (2003) for more details).
- Estimation is relatively straightforward (can calculate the likelihood in closed form).
- Stability behavior, such as stationarity and ergodicity, is difficult to derive.
An observation driven model for financial data:
Model (GARCH(p,q)):
\[ Y_t = \sigma_t Z_t, \{Z_t\} \sim \text{IID N}(0,1) \]
\[ \sigma_t^2 = \alpha_0 + \alpha_1 Y_{t-1}^2 + \cdots + \alpha_p Y_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_q \sigma_{t-q}^2 \]

Special case (ARCH(1)=GARCH(1,0)): The resulting observation and state transition density/equations are
\[ p(y_t \mid \sigma_t) = n(y_t ; 0, \sigma_t^2) \]
\[ \sigma_t^2 = \alpha_0 + \alpha_1 Y_{t-1}^2 \]

Properties:
• Martingale difference sequence.
• Stationary for \( \alpha_1 \in [0,2e^E), E = \text{Euler’s constant}. \)
• Strongly mixing at a geometric rate.
• For general ARCH (GARCH), properties are difficult to establish.
Examples of parameter driven models

Poisson model for time series of counts

Observation equation:
\[ p(y_t | \alpha_t) = \frac{e^{\alpha_t} e^{-\alpha_t}}{y_t!}, \quad y_t = 0, 1, \ldots, \]

State equation: State variables follow a regression model with Gaussian AR(1) noise
\[ \alpha_t = \beta^T x_t + W_t, \quad W_t = \phi W_{t-1} + Z_t, \quad \{Z_t\} \sim WN(0, \sigma^2) \]

The resulting transition density of the state variables is
\[ p(\alpha_{t+1} | \alpha_t) = n(\alpha_{t+1} ; \beta^T x_{t+1} + \phi (\alpha_t - \beta^T x_t), \sigma^2 ) \]

Remark: The case \( \sigma^2 = 0 \) corresponds to a log-linear model with Poisson noise.
A stochastic volatility model for financial data (Taylor `86):

Model:
\[ Y_t = \sigma_t Z_t, \{Z_t\} \sim \text{IID } N(0, 1) \]
\[ \alpha_t = \phi \alpha_{t-1} + \varepsilon_t, \{\varepsilon_t\} \sim \text{IID } N(0, \sigma^2), \]

where \( \alpha_t = 2 \log \sigma_t \).

The resulting observation and state transition densities are
\[ p(y_t | \alpha_t) = n(y_t; 0, \exp(2\alpha_t)) \]
\[ p(\alpha_{t+1} | \alpha_t) = n(\alpha_{t+1}; \phi \alpha_t, \sigma^2) \]

Properties:
- Martingale difference sequence.
- Stationary.
- Strongly mixing at a geometric rate.
Exponential Family Setup for Parameter-Driven Model

Time series data: $Y_1, \ldots, Y_n$

Regression (explanatory) variable: $x_t$

Observation equation:

$$p(y_t | \alpha_t) = \exp\{(\alpha_t + \beta^T x_t) y_t - b(\alpha_t + \beta^T x_t) + c(y_t)\}.$$  

State equation: \{\alpha_t\} follows an autoregressive process satisfying the recursions

$$\alpha_t = \gamma + \phi_1 \alpha_{t-1} + \phi_2 \alpha_{t-2} + \cdots + \phi_p \alpha_{t-p} + \varepsilon_t,$$

where \{\varepsilon_t\} ~ IID N(0,\sigma^2).

Note: $\alpha_t = 0$ corresponds to standard generalized linear model.

Original primary objective: Inference about $\beta$. 
GLM (ignores the presence of the latent process, i.e., $\alpha_t = 0.$)

- Estimating equations (Zeger ‘88): Let $\hat{\beta}$ be the solution to the equation

$$\frac{\partial \mu}{\partial \beta} \Gamma_n (y_n - \mu) = 0,$$

where $\mu = \exp(X \beta)$ and $\Gamma_n = \text{var}(Y_n)$.

- Monte Carlo EM (Chan and Ledolter ‘95)

- Importance sampling (Durbin & Koopman ‘01, Kuk ‘99, Kuk & Chen ‘97):

- Approximate likelihood (Davis, Dunsmuir & Wang ’98)
Estimation Methods Specialized to Poisson Example— GLM estimation

Model: \( Y_t \mid \alpha_t, x_t \sim \text{Pois}(\exp(x_t^T \beta + \alpha_t)) \).

GLM log-likelihood:

\[
l(\beta) = -\sum_{t=1}^{n} e^{x_t^T \beta} + \sum_{t=1}^{n} y_t x_t^T \beta - \log \left( \prod_{t=1}^{n} y_t! \right)
\]

(This \textit{likelihood} ignores presence of the latent process.)

Assumptions on regressors:

\[
\Omega_{I,n} = n^{-1} \sum_{t=1}^{n} x_t x_t^T \mu_t \rightarrow \Omega_I(\beta),
\]

\[
\Omega_{II,n} = n^{-1} \sum_{t=1}^{n} \sum_{s=1}^{n} x_t x_s^T \mu_t \mu_s \gamma_{(s-t)} \rightarrow \Omega_{II}(\beta),
\]
Theory of GLM Estimation in Presence of Latent Process

Theorem (Davis, Dunsmuir, Wang `00). Let \( \hat{\beta} \) be the GLM estimate of \( \beta \) obtained by maximizing \( l(\beta) \) for the Poisson regression model with a stationary lognormal latent process. Then

\[
n^{1/2} (\hat{\beta} - \beta) \xrightarrow{d} N(0, \Omega_I^{-1} + \Omega_I^{-1} \Omega_{II} \Omega_I^{-1}).
\]

Notes:

1. \( n^{-1} \Omega_I^{-1} \) is the asymptotic cov matrix from a std GLM analysis.
2. \( n^{-1} \Omega_I^{-1} \Omega_{II} \Omega_I^{-1} \) is the additional contribution due to the presence of the latent process.
3. Result also valid for more general latent processes (mixing, etc),
4. The \( x_t \) can depend on the sample size \( n \).
Assume the \( \{\alpha_t\} \) follows a log-normal AR(1), where

\[
(\alpha_t + \sigma^2/2) = \phi(\alpha_{t-1} + \sigma^2/2) + \eta_t, \quad \{\eta_t\} \sim \text{IID } \mathcal{N}(0, \sigma^2(1-\phi^2)),
\]

with \( \phi = .82, \sigma^2 = .57. \)

## Application to Model for Polio Data

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Estimation Methods — Importance Sampling (Durbin and Koopman)

Model:

\[ Y_t \mid \alpha_t, x_t \sim \text{Pois}(\exp(x_t^T \beta + \alpha_t)) \]
\[ \alpha_t = \phi \alpha_{t-1} + \varepsilon_t, \quad \{\varepsilon_t\} \sim \text{IID } N(0, \sigma^2) \]

Relative Likelihood: Let \( \psi = (\beta, \phi, \sigma^2) \) and suppose \( g(y_n, \alpha_n; \psi_0) \) is an approximating joint density for \( Y_n = (Y_1, \ldots, Y_n)' \) and \( \alpha_n = (\alpha_1, \ldots, \alpha_n)' \).

\[
L(\psi) = \int p(y_n \mid \alpha_n) p(\alpha_n) d\alpha_n \\
= \int \frac{p(y_n \mid \alpha_n) p(\alpha_n)}{g(y_n, \alpha_n; \psi_0)} g(y_n, \alpha_n; \psi_0) d\alpha_n \\
= \int \frac{p(y_n \mid \alpha_n) p(\alpha_n)}{g(y_n, \alpha_n; \psi_0)} g(\alpha_n \mid y_n; \psi_0) g(y_n; \psi_0) d\alpha_n \\
\frac{L(\psi)}{L_g(\psi_0)} = \int \frac{p(y_n \mid \alpha_n) p(\alpha_n)}{g(y_n, \alpha_n; \psi_0)} g(\alpha_n \mid y_n; \psi_0) d\alpha_n
\]
Importance Sampling (cont)

\[
\frac{L(\psi)}{L_g(\psi_0)} = \int \frac{p(y_n | \alpha_n)p(\alpha_n)}{g(y_n, \alpha_n; \psi_0)} g(\alpha_n | y_n; \psi_0) d\alpha_n
\]

\[
= E_g \left[ \frac{p(y_n | \alpha_n)p(\alpha_n)}{g(y_n, \alpha_n; \psi_0)} | y_n; \psi_0 \right]
\]

\[
\approx \frac{1}{N} \sum_{j=1}^{N} \frac{p(y_n | \alpha_n^{(j)})p(\alpha_n^{(j)})}{g(y_n, \alpha_n^{(j)}; \psi_0)},
\]

where \( \{\alpha_n^{(j)}; j = 1, ..., N\} \sim \text{iid } g(\alpha_n | y_n; \psi_0). \)

Notes:

• This is a “one-sample” approximation to the relative likelihood. That is, for one realization of the \( \alpha \)'s, we have, in principle, an approximation to the whole likelihood function.

• Approximation is only good in a neighborhood of \( \psi_0 \). Geyer suggests maximizing ratio wrt \( \psi \) and iterate replacing \( \psi_0 \) with \( \hat{\psi} \).
Importance Sampling — example

Simulation example: \( Y_t | \alpha_t \sim \text{Pois}(\exp(.7 + \alpha_t)) \),

\[
\alpha_t = .5 \alpha_{t-1} + \varepsilon_t, \quad \{\varepsilon_t\} \sim \text{IID N}(0, .3), \quad n = 200, \quad N = 1000
\]
Importance Sampling — example

Simulation example: \( Y_t \mid \alpha_t \sim \text{Pois} \left( \exp(0.7 + \alpha_t) \right) \),

\[ \alpha_t = 0.5 \alpha_{t-1} + \epsilon_t, \quad \{\epsilon_t\} \sim \text{IID } N(0, 0.3), \quad n = 200, \quad N = 1000 \]
Importance Sampling — example

Simulation example: \( Y_t \mid \alpha_t \sim \text{Pois}(\exp(0.7 + \alpha_t)) \),

\[ \alpha_t = 0.5 \alpha_{t-1} + \epsilon_t, \quad \{\epsilon_t\} \sim \text{IID } N(0, 0.3), \quad n = 200, \quad N = 1000 \]
Choice of importance density $g$:

Durbin and Koopman suggest a linear state-space approximating model

$$Y_t = \mu_t + x_t^T \beta + \alpha_t + Z_t, \quad Z_t \sim N(0, H_t),$$

with

$$\mu_t = y_t - \hat{\alpha}_t - x'_t y_t e^{-(\hat{\alpha}_t + x'_t \beta)} + 1,$$

$$H_t = e^{-(\hat{\alpha}_t + x'_t \beta)},$$

where the $\hat{\alpha}_t = E_g(\alpha_t \mid y_n)$ are calculated recursively under the approximating model until convergence.

With this choice of approximating model, it turns out that

$$g(\alpha_n \mid y_n; \psi_0) \sim N(\Gamma_n^{-1} \tilde{y}_n, \Gamma_n^{-1}),$$

where

$$\tilde{y}_n = y_n - e^{x_n \beta + \hat{\alpha}_n} + e^{x_n \beta + \hat{\alpha}_n} \hat{\alpha}_n,$$

$$\Gamma_n = \text{diag}(e^{x_n \beta + \hat{\alpha}_n}) + (E(\alpha_n \alpha'_n))^{-1}.$$
Components required in the calculation.

- $g(y_n, \alpha_n)$
  - $\tilde{y}'_n \Gamma^{-1}_n \tilde{y}_n$
  - $\det(\Gamma_n)$
- simulate from $N(\Gamma^{-1}_n \tilde{y}_n, \Gamma^{-1}_n)$
  - compute $\Gamma^{-1}_n \tilde{y}_n$
  - simulate from $N(0, \Gamma^{-1}_n)$

Remark: These quantities can be computed quickly using a version of the innovations algorithm or the Kalman smoothing recursions.
Importance Sampling — example

Simulation example: $\beta = .7, \phi = .5, \sigma^2 = .3, n = 200, N = 1000, 50$ realizations plotted
Consider a Gaussian approximation $p_a(\alpha_n \mid y_n) = \phi(\alpha_n ; \mu_0 , \Sigma_0)$ to the posterior $p(\alpha_n \mid y_n) \propto p(\alpha_n \mid y_n) p(\alpha_n)$

where

$$G_n^{-1} = E(\alpha_n - \mu)^T (\alpha_n - \mu)$$

Likelihood:

$$L(\psi) = \int p(y_n \mid \alpha_n) p(\alpha_n) d\alpha_n$$

Consider a Gaussian approximation $p_a(\alpha_n \mid y_n) = \phi(\alpha_n ; \mu_0 , \Sigma_0)$ to the posterior $p(\alpha_n \mid y_n) \propto p(\alpha_n \mid y_n) p(\alpha_n)$

Setting equal the respective posterior modes $\alpha_a^*$ and $\alpha^*$ of $p_a(\alpha_n \mid y_n)$ and $p(\alpha_n \mid y_n)$, we have $\mu_0 = \alpha^*$, where $\alpha^*$ is the solution of the equation

$$\frac{\partial}{\partial \alpha_n} \log p(y_n \mid \alpha_n, \psi) - G_n (\alpha_n - \mu) = 0$$
Estimation Methods — Approximation to the likelihood (cont)

Matching Fisher information matrices:

\[
\Sigma_0 = \left( -\frac{\partial^2}{\partial \alpha \partial \alpha^T} \log p(y_n \mid \alpha_n, \psi) \bigg|_{\hat{\alpha}} + G_n \right)^{-1}
\]

Approximating posterior:

\[
p_a(\alpha_n \mid y_n, \psi) = \phi(\alpha_n, \alpha^*, \left( -\frac{\partial^2}{\partial \alpha \partial \alpha^T} \log p(y_n \mid \alpha_n, \psi) \bigg|_{\hat{\alpha}} + G_n \right)^{-1})
\]

Notes:

1. This approximating posterior is identical to the importance sampling density used by Durbin and Koopman.

2. In traditional Bayesian setting, posterior is approximately \( p_a \) for large \( n \) (see Bernardo and Smith, 1994).
Approximate likelihood: Note that

\[ p(\alpha_n \mid y_n) = \frac{p(y_n \mid \alpha_n) p(\alpha_n)}{L(\psi; y_n)}, \]

which by solving for L in the expression,

\[ p_a(\alpha_n^* \mid y_n, \psi) = p(\alpha_n^* \mid y_n, \psi), \]

we obtain

\[ L_a(\psi; y_n) = p(y_n \mid \alpha^*, \psi) p(\alpha^*, \psi) / p_a(\alpha^* \mid y_n, \psi) \]

\[ = \left| G_n \right|^{1/2} p(y_n \mid \alpha^*, \psi) \exp \left\{ - (\alpha^* - \mu)^T G_n (\alpha^* - \mu) / 2 \right\} \]

\[ \det \left( - \frac{\partial^2}{\partial \alpha \partial \alpha^T} \log p(y_n \mid \alpha_n, \psi) \right|_{\alpha^*} + G_n \right)^{1/2} \]
Case of exponential family:

\[ L_a(\psi; y_n) = \frac{|G_n|^{1/2}}{(K + G_n)^{1/2}} \exp \{ y_n^T \alpha^* - 1^T \{ b(\alpha^*) - c(y_n) \} - (\alpha^* - \mu)^T G_n (\alpha^* - \mu) / 2 \}, \]

where

\[ K = \text{diag}\left\{ \frac{\partial^2}{\partial \alpha_i^2} b_i(\alpha_i) \bigg|_{\alpha_i^*} \right\}, \]

and \( \alpha^* \) is the solution to the equation

\[ y_n - \frac{\partial}{\partial \alpha_n} b(\alpha_n) - G_n (\alpha_n - \mu) = 0. \]

Using a Taylor expansion, the latter equation can be solved iteratively.
Estimation Methods — Approximation to the likelihood

Implementation:

1. Let $\alpha^* = \alpha^*(\psi)$ be the converged value of $\alpha^{(j)}(\psi)$, where
   \[
   \alpha^{(j+1)}(\psi) = (\ddot{b}^j + G_n)^{-1} \tilde{y}_n^j(\psi),
   \]
   and
   \[
   \tilde{y}_n^j = y_n - \dot{b}^j + \ddot{b}^j \alpha^{(j)} + G_n \mu.
   \]

2. Maximize $p_a(y_n; \psi)$ with respect to $\psi$. 
Model: \( Y_t \mid \alpha_t \sim Pois(\exp(0.7 + \alpha_t)) \), \( \alpha_t = 0.5 \alpha_{t-1} + \varepsilon_t \), \( \{\varepsilon_t\} \sim \text{IID } N(0, 0.3) \), \( n = 200 \)

Estimation methods:

- Importance sampling (\( N=1000 \), \( \psi_0 \) updated a maximum of 10 times)

\[
\begin{array}{ccc}
\text{beta} & \text{phi} & \text{sigma2} \\
\text{mean} & 0.6982 & 0.4718 & 0.3008 \\
\text{std} & 0.1059 & 0.1476 & 0.0899 \\
\end{array}
\]

- Approximation to likelihood

\[
\begin{array}{ccc}
\text{beta} & \text{phi} & \text{sigma2} \\
\text{mean} & 0.7036 & 0.4579 & 0.2962 \\
\text{std} & 0.0951 & 0.1365 & 0.0784 \\
\end{array}
\]
Model: $Y_t \mid \alpha_t \sim Pois\left(\exp(0.7 + \alpha_t)\right)$, $\alpha_t = 0.5 \alpha_{t-1} + \varepsilon_t$, $\{\varepsilon_t\} \sim\text{iid } N(0, 0.3)$, $n = 200$

Approx likelihood

Importance Sampling
### Application to Model Fitting for the Polio Data

Model for \(\{\alpha_t\}\):

\[
\alpha_t = \phi \alpha_{t-1} + \varepsilon_t, \quad \{\varepsilon_t\} \sim \text{IID } N(0, \sigma^2).
\]

- Importance sampling (\(\psi_0\) updated 5 times for each \(N=100, 500, 1000,\))
- Simulation based on 1000 replications and the fitted AL model.

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<th>Import Sampling</th>
<th>Approx Like</th>
<th>GLM</th>
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<td>(\hat{\beta}_{AL})</td>
<td>(\hat{\beta}_{GLM})</td>
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<tr>
<td>Mean</td>
<td>SD</td>
<td>Mean</td>
<td>SD</td>
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<td>Intercept</td>
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Application to Model Fitting for the Polio Data (cont)

Approx Likelihood

Importance Sampling
Simulation Results

Stochastic volatility model:
\[ Y_t = \sigma_t Z_t, \{Z_t\} \sim \text{IID N}(0,1) \]
\[ \alpha_t = \gamma + \phi \alpha_{t-1} + \varepsilon_t , \{\varepsilon_t\} \sim \text{IID N}(0,\sigma^2) \], where \( \alpha_t = 2 \log \sigma_t \); n=1000, NR=500

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Application to Sydney Asthma Count Data

Data: \( Y_1, \ldots, Y_{1461} \) daily asthma presentations in a Campbelltown hospital.

Preliminary analysis identified.

- no upward or downward trend
- annual cycle modeled by \( \cos(2\pi t/365), \sin(2\pi t/365) \)
- seasonal effect modeled by
  \[
P_{ij}(t) = \frac{1}{B(2.5,5)} \left( \frac{t-T_{ij}}{100} \right)^{2.5} \left( 1 - \frac{t-T_{ij}}{100} \right)^{5}
\]
  where \( B(2.5,5) \) is the beta function and \( T_{ij} \) is the start of the \( j \)th school term in year \( i \).
- day of the week effect modeled by separate indicator variables for Sunday and Monday (increase in admittance on these days compared to Tues-Sat).
- Of the meteorological variables (max/min temp, humidity) and pollution variables (ozone, NO, NO\(_2\)), only humidity at lags of 12-20 days and NO\(_2\)(max) appear to have an association.
## Results for Asthma Data—(IS & AL)

<table>
<thead>
<tr>
<th>Term</th>
<th>IS</th>
<th>AL</th>
<th>Mean</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>0.590</td>
<td>0.591</td>
<td>0.593</td>
<td>0.0658</td>
</tr>
<tr>
<td>Sunday effect</td>
<td>0.138</td>
<td>0.138</td>
<td>0.139</td>
<td>0.0531</td>
</tr>
<tr>
<td>Monday effect</td>
<td>0.229</td>
<td>0.231</td>
<td>0.230</td>
<td>0.0495</td>
</tr>
<tr>
<td>cos(2πt/365)</td>
<td>-0.218</td>
<td>-0.218</td>
<td>-0.217</td>
<td>0.0415</td>
</tr>
<tr>
<td>sin(2πt/365)</td>
<td>0.200</td>
<td>0.179</td>
<td>0.181</td>
<td>0.0437</td>
</tr>
<tr>
<td>Term 1, 1990</td>
<td>0.188</td>
<td>0.198</td>
<td>0.194</td>
<td>0.0638</td>
</tr>
<tr>
<td>Term 2, 1990</td>
<td>0.183</td>
<td>0.130</td>
<td>0.129</td>
<td>0.0664</td>
</tr>
<tr>
<td>Term 1, 1991</td>
<td>0.080</td>
<td>0.075</td>
<td>0.070</td>
<td>0.0733</td>
</tr>
<tr>
<td>Term 2, 1991</td>
<td>0.177</td>
<td>0.164</td>
<td>0.157</td>
<td>0.0665</td>
</tr>
<tr>
<td>Term 1, 1992</td>
<td>0.223</td>
<td>0.221</td>
<td>0.214</td>
<td>0.0667</td>
</tr>
<tr>
<td>Term 2, 1992</td>
<td>0.243</td>
<td>0.239</td>
<td>0.237</td>
<td>0.0620</td>
</tr>
<tr>
<td>Term 1, 1993</td>
<td>0.379</td>
<td>0.397</td>
<td>0.394</td>
<td>0.0625</td>
</tr>
<tr>
<td>Term 2, 1993</td>
<td>0.127</td>
<td>0.111</td>
<td>0.108</td>
<td>0.0682</td>
</tr>
<tr>
<td>Humidity H_{t/20}</td>
<td>0.009</td>
<td>0.010</td>
<td>0.007</td>
<td>0.0032</td>
</tr>
<tr>
<td>NO_{2} max</td>
<td>-0.125</td>
<td>-0.107</td>
<td>-0.108</td>
<td>0.0347</td>
</tr>
<tr>
<td>AR(1), φ</td>
<td>0.385</td>
<td>0.788</td>
<td>0.468</td>
<td>0.3790</td>
</tr>
<tr>
<td>σ²</td>
<td>0.053</td>
<td>0.010</td>
<td>0.018</td>
<td>0.0153</td>
</tr>
</tbody>
</table>
Asthma Data: observed and conditional mean

1990

Counts

Day of Year

1991

Counts

Day of Year

1992

Counts

Day of Year

1993

Counts

Day of Year

cond mean
observed
Is the posterior distribution close to normal?

Compare posterior mean with posterior mode: Can compute the posterior mean using SIR (sampling importance-resampling)

Posterior mode: The mode of \( p(\alpha_n \mid y_n) \) is \( \alpha^* \) found at the last iteration.

Posterior mean: The mean of \( p(\alpha_n \mid y_n) \) can be found using SIR.

Let \( \alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(N)} \) be independent draws from the multivariate distr \( p_a(\alpha_n \mid y_n) \). For \( N \) large, an approximate iid sample from \( p(\alpha_n \mid y_n) \) can be obtained by drawing a random sample from \( \alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(N)} \) with probabilities

\[
p_i = \frac{w_i}{\sum_{i=1}^{N} w_i}, \quad w_i = \frac{p(\alpha^{(i)} \mid y_n)}{p_a(\alpha^{(i)} \mid y_n)} \propto \frac{L(\psi; y_n, \alpha^{(i)})}{p_a(\alpha^{(i)} \mid y_n)}, \quad i = 1, \ldots, N.
\]
Posterior mean vs posterior mode?

Polio data: blue = mean, red = mode
Summary Remarks

1. Importance sampling offers a nice clean method for estimation in parameter driven models.

2. The innovations algorithm allows for quick implementation of importance sampling. Extends easily to higher-order AR structure.

3. Relative likelihood approach is a one-sample based procedure.

4. Approximation to the likelihood is a non-simulation based procedure which may have great potential especially with large sample sizes and/or large number of explanatory variables.

5. Approximation likelihood approach is amenable to bootstrapping procedures for bias correction.