Application of the Innovations Algorithm to Nonlinear State-Space Models

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Two CSU Statistics Department Projects

**STARMAP**
Space-Time Aquatic Resources Modeling and Analysis Program (funded by the EPA-STAR program)

**PRIMES**
Program for Interdisciplinary Mathematics, Ecology, and Statistics (recommended for funding by NSF IGERT)

- Student fellowships
- Postdoctoral fellowships
- Short-term visitors
- Workshops/seminars and more.
Generalized state-space models

- Observation driven
- Parameter driven

Innovations algorithm (recursive one-step ahead prediction algorithm)

- Applications
  - Gaussian likelihood calculations
  - simulation
  - generalized least squares estimation

Time series of counts

- Examples (asthma data, polio data)
- Generalized linear models (GLM)
- Estimating equations (Zeger)
- MCEM (Chan and Ledolter)
- Importance sampling
  - Durbin and Koopman
- Approximation to the likelihood (Davis, Dunsmuir, and Wang)
- Simulation results

Examples
Generalized State-Space Models (parameter driven)

Observations: \( y^{(t)} = (y_1, \ldots, y_t) \)

States: \( \alpha^{(t)} = (\alpha_1, \ldots, \alpha_t) \)

Observation equation:
\[
p(y_t \mid \alpha_t) := p(y_t \mid \alpha_t, \alpha^{(t-1)}, y^{(t-1)})
\]

State equation:
\[
p(\alpha_{t+1} \mid \alpha_t) := p(\alpha_{t+1} \mid \alpha_t, \alpha^{(t-1)}, y^{(t)})
\]

Joint density:
\[
p(y_1, \ldots, y_n, \alpha_1, \ldots, \alpha_n)
= p(y_n \mid \alpha_n, \alpha^{(n-1)}, y^{(n-1)}) p(\alpha_n, \alpha^{(n-1)}, y^{(n-1)})
= p(y_n \mid \alpha_n) p(\alpha_n \mid \alpha^{(n-1)}, y^{(n-1)}) p(\alpha^{(n-1)}, y^{(n-1)})
= \cdots \cdots
= \left( \prod_{j=1}^n p(y_j \mid \alpha_j) \right) \left( \prod_{j=2}^n p(\alpha_j \mid \alpha_{j-1}) \right) p(\alpha_1)
\]
Conditional independence:
\[
p(y_1, \ldots, y_n \mid \alpha_1, \ldots, \alpha_n) = \prod_{j=1}^{n} p(y_j \mid \alpha_j)
\]
Filtering or posterior density:
\[
p(\alpha_t \mid y^{(t)}) = p(y_t \mid \alpha_t)p(\alpha_t \mid y^{(t-1)})/p(y_t \mid y^{(t-1)})
\]
Predictive densities:
\[
p(\alpha_{t+1} \mid y^{(t)}) = \int p(\alpha_t \mid y^{(t)}) \ p(\alpha_{t+1} \mid \alpha_t) \ d\mu(\alpha_t)
\]
\[
p(y_{t+1} \mid y^{(t)}) = \int p(y_{t+1} \mid \alpha_{t+1}) \ p(\alpha_{t+1} \mid y^{(t)}) \ d\mu(\alpha_{t+1})
\]
Examples of parameter driven models

Poisson model for time series of counts

Observation equation:
\[ p(y_t | \alpha_t) = \frac{e^{\alpha_t y_t} e^{-\alpha_t}}{y_t!}, \quad y_t = 0, 1, \ldots, \]

State equation: State variables follow a regression model with Gaussian AR(1) noise
\[ \alpha_t = \beta^T x_t + W_t, \quad W_t = \phi W_{t-1} + Z_t, \quad \{Z_t\} \sim \text{WN}(0, \sigma^2) \]

The resulting transition density of the state variables is
\[ p(\alpha_{t+1} | \alpha_t) = n(\alpha_{t+1} ; \beta^T x_{t+1} + \phi (\alpha_t - \beta^T x_t), \sigma^2) \]

Remark: The case \( \sigma^2 = 0 \) corresponds to a log-linear model with Poisson noise.
Examples of parameter driven models

A stochastic volatility model for financial data (Taylor `86):

Model:

\[ Y_t = \sigma_t Z_t , \{ Z_t \} \sim \text{IID } N(0,1) \]

\[ \alpha_t = \phi \alpha_{t-1} + W_t , \{ W_t \} \sim \text{IID } N(0, \sigma^2), \]

where \( \alpha_t = \log \sigma_t \).

The resulting observation and state transition densities are

\[ p(y_t | \alpha_t) = n(y_t ; 0, \exp(2\alpha_t)) \]

\[ p(\alpha_{t+1} | \alpha_t) = n(\alpha_{t+1} ; \phi \alpha_t , \sigma^2 ) \]

Properties:

- Martingale difference sequence.
- Stationary.
- Strongly mixing at a geometric rate.
The Innovations Algorithm

Innovations Algorithm (Brockwell and Davis `87): \( \{X_t\} \) is a zero-mean time series with ACVF \( \kappa(i,j) \), then

\[
\hat{X}_{t+1} = P_{sp\{1,x_1,\ldots,x_t\}} X_{t+1} = \theta_{t1}(X_t - \hat{X}_t) + \cdots + \theta_{tt}(X_1 - \hat{X}_1)
\]

The coefficients \( \theta_{t1}, \ldots, \theta_{tt} \) and prediction errors \( v_{t-1} \) can be computed recursively from the equations,

\[
v_0 = \kappa(1,1)
\]

\[
\theta_{t,t-k} = \left[ \kappa(t+1,k+1) - \sum_{j=0}^{k-1} \theta_{k,k-j} \theta_{t,t-j} v_j \right] v_{k-1}^{-1}, \ k = 0,\ldots,t-1,
\]

and

\[
v_t = \kappa(t+1,t+1) - \sum_{j=0}^{t-1} \theta_{t,t-j}^2 v_j.
\]
Remarks:

- Innovations algorithm expresses one-step predictor in terms of previous innovations, \(X_1 - \hat{X}_1, \ldots, X_t - \hat{X}_t\), that are uncorrelated.

- If \(\{X_t\}\) is an MA(q) process

\[
X_{t+1} = Z_{t+1} + \theta_1 Z_t + \cdots + \theta_q Z_{t-q}, \quad \{Z_t\} \sim WN(0, \sigma^2)
\]

then \((\theta_{t1}, \ldots, \theta_{tt}) = (\theta_{t1}, \ldots, \theta_{tq}, 0, \ldots, 0)\) for all \(t\).

- Innovations algorithm is well adapted for ARMA(p,q) models—only need to apply to MA(q) piece (see B&D `96).
The Innovations Algorithm—Applications

Likelihood calculation:

Using the IA representation,

\[ \hat{X}_t = \theta_{t-1,1}(X_{t-1} - \hat{X}_{t-1}) + \cdots + \theta_{t-1,t-1}(X_1 - \hat{X}_1) \]

we have

\[
\begin{bmatrix}
X_1 \\
X_2 \\
X_3 \\
\vdots \\
X_n
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
\theta_{1,1} & 1 & 0 & \cdots & 0 \\
\theta_{2,2} & \theta_{2,1} & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\theta_{n-1,n-1} & \theta_{n-1,n-2} & \theta_{n-1,n-3} & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
X_1 - \hat{X}_1 \\
X_2 - \hat{X}_2 \\
X_3 - \hat{X}_3 \\
\vdots \\
X_n - \hat{X}_n
\end{bmatrix}
\]

\[ X_n = C_n (X_n - \hat{X}_n) \]

By taking covariances of both sides it follows that

\[ \Gamma_n = E(X_n X_n') = C_n D_n C_n', \quad D_n = \text{diag}(v_0, \ldots, v_{n-1}) \]
The Innovations Algorithm—Applications

Quadratic form:

\[ X'_n \Gamma^{-1}_n X_n = (X_n - \hat{X}_n)' C'_n (C'^{-1}_n D_n^{-1} C'_n) C_n (X_n - \hat{X}_n) \]
\[ = (X_n - \hat{X}_n)' D_n^{-1} (X_n - \hat{X}_n) \]
\[ = \sum_{t=1}^{n} (X_t - \hat{X}_t)^2 / \nu_{t-1} \]

Determinant:

\[ \text{det}(\Gamma_n) = \text{det}(C_n D_n C'_n) = \nu_0 \cdots \nu_{n-1} \]

Gaussian likelihood:

\[ L(\Gamma_n) = (2\pi)^{-n/2} (\nu_0 \cdots \nu_{n-1})^{-1/2} \exp\left\{ -1/2 \sum_{t=1}^{n} (X_t - \hat{X}_t)^2 / \nu_{t-1} \right\} \]

Simulation: If \{Z_t\} \sim iid N(0,1), put \[ X_t = \nu_{t-1}^{-1/2} Z_t + \theta_{t-1,1} \nu_{t-2}^{-1/2} Z_{t-1} + \cdots + \theta_{t-1,t-1} \nu_0^{-1/2} Z_1. \]

Then \[ X_n = (X_1, \cdots, X_n)' = C'_n D_n^{-1/2} Z_n \]

has covariance matrix \( \Gamma_n \).
Count data: \( Y_1, \ldots, Y_n \)

Regression (explanatory) variable: \( x_t \)

Model: Distribution of the \( Y_t \) given \( x_t \) and a stochastic process \( \alpha_t \) are indep Poisson distributed with mean

\[
\mu_t = \exp(x_t^T \beta + \alpha_t).
\]

The distribution of the stochastic process \( \alpha_t \) may depend on a vector of parameters \( \gamma \).

Note: \( \alpha_t = 0 \) corresponds to standard Poisson regression model.

Primary objective: Inference about \( \beta \).
Polio Data With Estimated Regression Function
Parameter-Driven Model for the Mean Function $\mu_t$

**Parameter-driven specification:** (Assume $Y_t \mid \mu_t$ is Poisson($\mu_t$))

$$\log \mu_t = x_t^T \beta + \alpha_t,$$

where \{\alpha_t\} is a stationary Gaussian process.

**e.g.** (AR(1) process)

$$(\alpha_t + \sigma^2/2) = \phi(\alpha_{t-1} + \sigma^2/2) + \varepsilon_t, \quad \{\varepsilon_t\} \sim \text{IID N}(0, \sigma^2(1-\phi^2)).$$

**Advantages of this model specification:**

- properties of model (ergodicity and mixing) easy to derive.
- interpretability of regression parameters

$$E(Y_t) = \exp(x_t^T \beta) \exp(\alpha_t) = \exp(x_t^T \beta), \quad \text{if } \exp(\alpha_t) = 1.$$

**Disadvantages:**

- estimation is difficult-likelihood function not easily calculated (MCEM, importance sampling, estimating eqns).
- model building can be laborious

**Remark:** See Davis, Dunsmuir, and Wang (1999) for testing of the existence of a latent process and estimating its ACF.
Model:

\[ Y_t \mid \alpha_t, x_t \sim \text{Pois}(\exp(x_t^T \beta + \alpha_t)) \]

\[ \alpha_t = \phi \alpha_{t-1} + \varepsilon_t, \quad \{\varepsilon_t\} \sim \text{IID N}(0, \sigma^2) \]

Relative Likelihood: Let \( \psi = (\beta, \phi, \sigma^2) \) and suppose \( g(y_n, \alpha_n; \psi_0) \) is an approximating joint density for \( Y_n = (Y_1, \ldots, Y_n)' \) and \( \alpha_n = (\alpha_1, \ldots, \alpha_n)' \).

\[
L(\psi) = \int p(y_n \mid \alpha_n) p(\alpha_n) d\alpha_n
\]

\[
= \int p(y_n \mid \alpha_n) p(\alpha_n) \frac{g(y_n, \alpha_n; \psi_0)}{g(y_n, \alpha_n; \psi_0)} g(y_n, \alpha_n; \psi_0) d\alpha_n
\]

\[
= \int p(y_n \mid \alpha_n) p(\alpha_n) \frac{g(\alpha_n \mid y_n; \psi_0) g(y_n; \psi_0)}{g(y_n, \alpha_n; \psi_0)} g(y_n, \alpha_n; \psi_0) d\alpha_n
\]

\[
\frac{L(\psi)}{L_g(\psi_0)} = \int p(y_n \mid \alpha_n) p(\alpha_n) \frac{g(\alpha_n \mid y_n; \psi_0)}{g(y_n, \alpha_n; \psi_0)} g(y_n, \alpha_n; \psi_0) d\alpha_n
\]
\[
\frac{L(\psi)}{L_g(\psi_0)} = \int \frac{p(y_n | \alpha_n) p(\alpha_n)}{g(y_n, \alpha_n; \psi_0)} g(\alpha_n | y_n; \psi_0) d\alpha_n
\]

\[= E_g \left[ \frac{p(y_n | \alpha_n) p(\alpha_n)}{g(y_n, \alpha_n; \psi_0)} | y_n; \psi_0 \right] \]

\[\sim \frac{1}{N} \sum_{j=1}^{N} \frac{p(y_n | \alpha_{n}^{(j)}) p(\alpha_{n}^{(j)})}{g(y_n, \alpha_{n}^{(j)}; \psi_0)} , \]

where \( \{ \alpha_{n}^{(j)} ; j = 1, \ldots, N \} \sim \text{iid } g(\alpha_n | y_n; \psi_0) \).

Notes:

- This is a “one-sample” approximation to the relative likelihood. That is, for one realization of the \( \alpha \)'s, we have, in principle, an approximation to the whole likelihood function.

- Approximation is only good in a neighborhood of \( \psi_0 \). Geyer suggests maximizing ratio wrt \( \psi \) and iterate replacing \( \psi_0 \) with \( \hat{\psi} \).
Importance Sampling — example

Simulation example: \( Y_t \mid \alpha_t \sim \text{Pois}(\exp(0.7 + \alpha_t)) \),

\[ \alpha_t = 0.5 \alpha_{t-1} + \epsilon_t, \quad \{\epsilon_t\} \sim \text{IID } N(0, 0.3), \quad n = 200, \quad N = 1000 \]
Simulation example: \( Y_t \mid \alpha_t \sim \text{Pois}(\exp(0.7 + \alpha_t)) \), \( \phi = 0.5 \), \( \sigma^2 = 0.3 \), \( n = 200 \), \( N = 1000 \)
Importance Sampling (cont)

Choice of *importance density* $g$:

Durbin and Koopman suggest a linear state-space approximating model

$$Y_t = \mu_t + x_t^T \beta + \alpha_t + Z_t, \quad Z_t \sim N(0, H_t),$$

with

$$\mu_t = y_t - \hat{\alpha}_t - x_t' y_t e^{-(\hat{\alpha}_t + x_t' \beta)} + 1,$$

$$H_t = e^{-(\hat{\alpha}_t + x_t' \beta)},$$

where the $\hat{\alpha}_t = E_g (\alpha_t | y_n)$ are calculated recursively under the approximating model until convergence.

With this choice of approximating model, it turns out that

$$g(\alpha_n | y_n; \psi_0) \sim N(\Gamma_n^{-1} \tilde{y}_n, \Gamma_n^{-1}),$$

where

$$\tilde{y}_n = y_n - e^{X\beta + \hat{\alpha}_n} + e^{X\beta + \hat{\alpha}_n} \hat{\alpha}_n,$$

$$\Gamma_n = \text{diag}(e^{X\beta + \hat{\alpha}_n}) + (E(\alpha_n \alpha_n'))^{-1}.$$
Importance Sampling (cont)

Components required in the calculation.

- $g(y_n, \alpha_n)$
  - $\tilde{y}'_n \Gamma^{-1}_n \tilde{y}_n$
  - $\text{det}(\Gamma_n)$

- simulate from $N(\Gamma^{-1}_n \tilde{y}_n, \Gamma^{-1}_n)$
  - compute $\Gamma^{-1}_n \tilde{y}_n$
  - simulate from $N(0, \Gamma^{-1}_n)$
Importance Sampling (cont)

Details.

\[
(\Sigma_n^{-1}) = \begin{pmatrix}
1 & -\phi & 0 & \cdots & 0 \\
-\phi & 1+\phi^2 & -\phi & \cdots & 0 \\
0 & -\phi & 1+\phi^2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1+\phi^2
\end{pmatrix}
\]

\[
\Gamma_n = \text{diag}(e^{\hat{\alpha} + \beta}) + \sigma^{-2} = \begin{pmatrix}
1 & -\phi & 0 & \cdots & 0 \\
-\phi & 1+\phi^2 & -\phi & \cdots & 0 \\
0 & -\phi & 1+\phi^2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1+\phi^2
\end{pmatrix}
\]

This is the covariance function of a 1-dependent sequence, so that \( \Gamma_n = C_n D_n C_n' \), where

\[
C_n = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
\theta_{1,1} & 1 & 0 & \cdots & 0 \\
0 & \theta_{2,1} & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}
\]
Importance Sampling (cont)

It follows that
\[ \tilde{y}_n' \Gamma_n^{-1} \tilde{y}_n = \sum_{t=1}^{n} (\tilde{y}_t - \hat{y}_t)^2 / \nu_{t-1} \]
and
\[ \Gamma_n^{-1} \tilde{y}_n = C_n^{-1} D_n^{-1} C_n (\tilde{y}_n - \hat{y}_n) \]
\[ = C_n^{-1} (D_n^{-1} (\tilde{y}_n - \hat{y}_n)) \]
which can be solved for the vector \( \Gamma_n^{-1} \tilde{y}_n \) via the recursion
\[ C_n' \Gamma_n^{-1} \tilde{y}_n = D_n^{-1} (\tilde{y}_n - \hat{y}_n). \]

All of these calculations can be carried out quickly using the innovations algorithm.

To simulate from \( N(0, \Gamma_n^{-1}) \) note that
\[ U_n = C_n'^{-1} D_n^{-1} Z_n, \]
where \( Z_n \sim N(0,1) \), has covariance matrix \( \Gamma_n^{-1} \).
Importance Sampling — example

Simulation example: $\beta = .7$, $\phi = .5$, $\sigma^2 = .3$, $n = 200$, $N = 1000$, 50 realizations plotted
Estimation Methods — Approximation to the likelihood

Joint density function:

\[ p(y_n, \alpha_n) \propto \frac{\text{det}(G)^{1/2}}{\prod_{t=1}^{n} y_t!} \exp\{-(y_n^T (\alpha_n + X\beta) - e^{1^T (\alpha_n + X\beta)} - \alpha_n^T G_n \alpha_n / 2}\}, \]

where \( G_n^{-1} = E(\alpha_n^T \alpha_n) \).

Conditional density function:

\[ p(\alpha_n \mid y_n) \propto \exp\{-y_n^T \alpha_n - e^{1^T (\alpha_n + X\beta)} - \alpha_n^T G_n \alpha_n / 2\}, \]

which, by expanding the term, \( e^{1^T (\alpha_n + X\beta)} \) in a neighborhood of \( \alpha_n^* \), and ignoring third-order + terms yields the approximation

\[ p_{\alpha}(\alpha_n \mid y_n) \propto \exp\{-y_n^T (\alpha_n + X\beta) - e^{1^T (\alpha_n^* + X\beta)} + (\alpha_n - \alpha_n^*)^T e^{\alpha_n^* + X\beta} \]

\[ + \frac{1}{2} (\alpha_n - \alpha_n^*)^T \text{diag}(e^{\alpha_n^* + X\beta})(\alpha_n - \alpha_n^*) - \alpha_n^T G_n \alpha_n / 2\}. \]
Estimation Methods — Approximation to the likelihood

After simplification, we find

\[
p_a(\alpha_n | y_n) \propto \exp\left\{- (y_n^T (\alpha_n + X\beta)) - e^{T (\alpha_n^* + X\beta)} + (\alpha_n - \alpha_n^*)^T e^{\alpha_n^* + X\beta} \right\} \\
+ \frac{1}{2} (\alpha_n - \alpha_n^*)^T \text{diag}(e^{\alpha_n^* + X\beta})(\alpha_n - \alpha_n^*) - \alpha_n^T G_n \alpha_n / 2 \right\}.
\]

\[
\sim N(\Gamma_n^{-1} \tilde{y}_n, \Gamma_n^{-1})
\]

Approximate likelihood:

\[
p_a(y_n; \psi) = \frac{p(y_n, \alpha_n)}{p_a(\alpha_n | y_n)} \propto \frac{\det(G_n)^{1/2}}{\det(\Gamma_n)^{1/2}} \exp\{y_n^T X\beta + .5 \tilde{y}_n^T \Gamma_n^{-1} \tilde{y}_n \},
\]

\[
\tilde{y}_n = y_n - \exp\{X\beta\} \exp\{\alpha_n^*\} + \exp\{\alpha_n^*\} \exp\{X\beta\} \alpha_n^*
\]

(component-wise multiplication for vectors)

Note: We actually expand the joint density for \( Y_n \) and \( \alpha_n \) in a neighborhood of \( \alpha^* \).
Implementation:

1. Let $\alpha^* = \alpha^*(\psi)$ be the converged value of $\alpha^{(j)}(\psi)$, where

   $$\alpha^{(j+1)}(\psi) = \Gamma_n^{-1} \tilde{y}_n(\psi)$$

2. Maximize $p_a(y_n; \psi)$ with respect to $\psi$. 
Model: $Y_t | \alpha_t \sim Pois(\exp(0.7 + \alpha_t))$, $\alpha_t = 0.5 \alpha_{t-1} + \varepsilon_t$, $\{\varepsilon_t\} \sim$ IID $N(0, 0.3)$, $n = 200$

Estimation methods:

- Importance sampling ($N=1000$, $\psi_0$ updated a maximum of 10 times)

<table>
<thead>
<tr>
<th>beta</th>
<th>phi</th>
<th>sigma2</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean</td>
<td>0.6982</td>
<td>0.4718</td>
</tr>
<tr>
<td>std</td>
<td>0.1059</td>
<td>0.1476</td>
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- Approximation to likelihood

<table>
<thead>
<tr>
<th>beta</th>
<th>phi</th>
<th>sigma2</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean</td>
<td>0.7036</td>
<td>0.4579</td>
</tr>
<tr>
<td>std</td>
<td>0.0951</td>
<td>0.1365</td>
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</tbody>
</table>
Model:  \( Y_t \mid \alpha_t \sim Pois(\exp(0.7 + \alpha_t)) \),  
\( \alpha_t = 0.5 \alpha_{t-1} + \varepsilon_t \),  
\( \{\varepsilon_t\} \sim IID N(0, 0.3) \),  
\( n = 200 \)

Approx likelihood

Importance Sampling
Application to Model Fitting for the Polio Data

Model for \{\alpha_t\}:
\[ \alpha_t = \phi \alpha_{t-1} + \epsilon_t, \quad \{\epsilon_t\} \sim \text{IID } N(0, \sigma^2). \]

- Importance sampling (\psi_0\text{ updated 5 times for each } N=100, 500, 1000, )
- Simulation based on 1000 replications and the fitted AL model.

<table>
<thead>
<tr>
<th></th>
<th>Import Sampling</th>
<th></th>
<th>Approx Like</th>
<th></th>
<th>GLM</th>
<th></th>
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<tbody>
<tr>
<td></td>
<td>(\hat{\beta}_{IS})</td>
<td>Simulation</td>
<td>(\hat{\beta}_{AL})</td>
<td>Simulation</td>
<td>(\hat{\beta}_{GLM})</td>
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<tr>
<td><strong>Intercept</strong></td>
<td><strong>0.203</strong></td>
<td>0.223</td>
<td>0.381</td>
<td><strong>0.202</strong></td>
<td>0.210</td>
<td>0.343</td>
</tr>
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<td><strong>Trend</strong> (×10^-3)</td>
<td><strong>-2.675</strong></td>
<td>-2.778</td>
<td>3.979</td>
<td><strong>-2.690</strong></td>
<td>-2.720</td>
<td>3.415</td>
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<td><strong>cos(2\pi t/12)</strong></td>
<td><strong>0.110</strong></td>
<td>0.103</td>
<td>0.124</td>
<td><strong>0.113</strong></td>
<td>0.111</td>
<td>0.123</td>
</tr>
<tr>
<td><strong>sin(2\pi t/12)</strong></td>
<td><strong>-0.456</strong></td>
<td>-0.456</td>
<td>0.151</td>
<td><strong>-0.454</strong></td>
<td>-0.454</td>
<td>0.143</td>
</tr>
<tr>
<td><strong>cos(2\pi t/6)</strong></td>
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<td>0.401</td>
<td>0.123</td>
<td><strong>0.396</strong></td>
<td>0.400</td>
<td>0.114</td>
</tr>
<tr>
<td><strong>sin(2\pi t/6)</strong></td>
<td><strong>0.015</strong></td>
<td>0.024</td>
<td>0.118</td>
<td><strong>0.016</strong></td>
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<td>(\phi)</td>
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<td>0.198</td>
<td><strong>0.845</strong></td>
<td>0.764</td>
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<tr>
<td>(\sigma^2)</td>
<td><strong>0.088</strong></td>
<td>0.100</td>
<td>0.068</td>
<td><strong>0.104</strong></td>
<td>0.114</td>
<td>0.075</td>
</tr>
</tbody>
</table>
Application to Model Fitting for the Polio Data (cont)

Approx Likelihood

Importance Sampling
Polio Data: observed and conditional mean (approx like)
Application to Sydney Asthma Count Data

Data: \(Y_1, \ldots, Y_{1461}\) daily asthma presentations in a Campbelltown hospital.

Preliminary analysis identified.

- no upward or downward trend
- annual cycle modeled by \(\cos(2\pi t/365), \sin(2\pi t/365)\)
- seasonal effect modeled by

\[
P_{ij}(t) = \frac{1}{B(2.5,5)} \left( \frac{t-T_{ij}}{100} \right)^{2.5} \left( 1 - \frac{t-T_{ij}}{100} \right)^5
\]

where \(B(2.5,5)\) is the beta function and \(T_{ij}\) is the start of the \(j^{th}\) school term in year \(i\).

- day of the week effect modeled by separate indicator variables for Sunday and Monday (increase in admittance on these days compared to Tues-Sat).

- Of the meteorological variables (max/min temp, humidity) and pollution variables (ozone, NO, NO₂), only humidity at lags of 12-20 days and NO₂(max) appear to have an association.
## Results for Asthma Data—(IS & AL)

<table>
<thead>
<tr>
<th>Term</th>
<th>IS</th>
<th>AL</th>
<th>Mean</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>0.590</td>
<td>0.591</td>
<td>0.593</td>
<td>.0658</td>
</tr>
<tr>
<td>Sunday effect</td>
<td>0.138</td>
<td>0.138</td>
<td>0.139</td>
<td>.0531</td>
</tr>
<tr>
<td>Monday effect</td>
<td>0.229</td>
<td>0.231</td>
<td>0.230</td>
<td>.0495</td>
</tr>
<tr>
<td>$\cos(2\pi t/365)$</td>
<td>-0.218</td>
<td>-0.218</td>
<td>-0.217</td>
<td>.0415</td>
</tr>
<tr>
<td>$\sin(2\pi t/365)$</td>
<td>0.200</td>
<td>0.179</td>
<td>0.181</td>
<td>.0437</td>
</tr>
<tr>
<td>Term 1, 1990</td>
<td>0.188</td>
<td>0.198</td>
<td>0.194</td>
<td>.0638</td>
</tr>
<tr>
<td>Term 2, 1990</td>
<td>0.183</td>
<td>0.130</td>
<td>0.129</td>
<td>.0664</td>
</tr>
<tr>
<td>Term 1, 1991</td>
<td>0.080</td>
<td>0.075</td>
<td>0.070</td>
<td>.0733</td>
</tr>
<tr>
<td>Term 2, 1991</td>
<td>0.177</td>
<td>0.164</td>
<td>0.157</td>
<td>.0665</td>
</tr>
<tr>
<td>Term 1, 1992</td>
<td>0.223</td>
<td>0.221</td>
<td>0.214</td>
<td>.0667</td>
</tr>
<tr>
<td>Term 2, 1992</td>
<td>0.243</td>
<td>0.239</td>
<td>0.237</td>
<td>.0620</td>
</tr>
<tr>
<td>Term 1, 1993</td>
<td>0.379</td>
<td>0.397</td>
<td>0.394</td>
<td>.0625</td>
</tr>
<tr>
<td>Term 2, 1993</td>
<td>0.127</td>
<td>0.111</td>
<td>0.108</td>
<td>.0682</td>
</tr>
<tr>
<td>Humidity $H_t/20$</td>
<td>0.009</td>
<td>0.010</td>
<td>0.007</td>
<td>.0032</td>
</tr>
<tr>
<td>$\text{NO}_2\text{ max}$</td>
<td>-0.125</td>
<td>-0.107</td>
<td>-0.108</td>
<td>.0347</td>
</tr>
<tr>
<td>AR(1), $\phi$</td>
<td>0.385</td>
<td>0.788</td>
<td>0.468</td>
<td>.3790</td>
</tr>
<tr>
<td>$\sigma^2$</td>
<td>0.053</td>
<td>0.010</td>
<td>0.018</td>
<td>.0153</td>
</tr>
</tbody>
</table>
Asthma Data: observed and conditional mean

1990

1991

1992

1993
Summary Remarks

1. Importance sampling offers a nice clean method for estimation in parameter driven models.

2. The innovations algorithm allows for quick implementation of importance sampling. Extends easily to higher-order AR structure.

3. Relative likelihood approach is a one-sample based procedure.

4. Approximation to the likelihood is a non-simulation based procedure which may have great potential especially with large sample sizes and/or large number of explanatory variables.