Estimation for a Class of Generalized State-Space Models:

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Example: Pound-Dollar Exchange Rates
Motivating Examples
• Time series of counts
• Stochastic volatility

Generalized state-space models

Model setup and estimation
• Exponential family
• Estimation
  - Importance sampling
  - Approximation to the likelihood

Simulation and Application
• Time series of counts
• Stochastic volatility

How good is the posterior approximation?
• Posterior mode vs posterior mean

Application to estimating structural breaks
• Poisson model
• Stochastic volatility model
Generalized State-Space Models

Observations: \( y^{(t)} = (y_1, \ldots, y_t) \)

States: \( \alpha^{(t)} = (\alpha_1, \ldots, \alpha_t) \)

Observation equation:
\[
p(y_t | \alpha_t) := p(y_t | \alpha_t, \alpha^{(t-1)}, y^{(t-1)})
\]

State equation:
- observation driven
\[
p(\alpha_{t+1} | y^{(t)}) := p(\alpha_{t+1} | \alpha_t, \alpha^{(t-1)}, y^{(t)})
\]
- parameter driven
\[
p(\alpha_{t+1} | \alpha_t) := p(\alpha_{t+1} | \alpha_t, \alpha^{(t-1)}, y^{(t)})
\]
Exponential Family Setup for Parameter-Driven Model

Time series data: $Y_1, \ldots, Y_n$

Regression (explanatory) variable: $x_t$

Observation equation:

$$p(y_t | \alpha_t) = \exp\{((\alpha_t + \beta^T x_t)y_t - b(\alpha_t + \beta^T x_t) + c(y_t))\}.$$

State equation: \{\alpha_t\} follows an autoregressive process satisfying the recursions

$$\alpha_t = \gamma + \phi_1 \alpha_{t-1} + \phi_2 \alpha_{t-2} + \cdots + \phi_p \alpha_{t-p} + \varepsilon_t,$$

where \{\varepsilon_t\} \sim \text{IID } N(0, \sigma^2).

Note: $\alpha_t = 0$ corresponds to standard generalized linear model.

Original primary objective: Inference about $\beta$. 
Examples of parameter driven models

Poisson model for time series of counts

Observation equation:

\[ p(y_t | \alpha_t) = \frac{e^{(\beta^T x_t + \alpha_t) y_t} e^{-e^{(\beta^T x_t + \alpha_t)}}}{y_t!}, \quad y_t = 0, 1, ..., \]

State equation: State variables follow a Gaussian AR(1) process

\[ \alpha_t = \phi \alpha_{t-1} + \varepsilon_t , \quad \{\varepsilon_t\} \sim \text{IID N}(0, \sigma^2) \]

The resulting transition density of the state variables is

\[ p(\alpha_{t+1} | \alpha_t) = n(\alpha_{t+1} ; \phi \alpha_t, \sigma^2 ) \]

Remark: The case \( \sigma^2 = 0 \) corresponds to a log-linear model with Poisson noise.
A stochastic volatility model for financial data (Taylor `86):

Model:
\[ Y_t = \sigma_t Z_t, \{Z_t\} \sim \text{IID N}(0,1) \]
\[ \alpha_t = \gamma + \phi \alpha_{t-1} + \varepsilon_t, \{\varepsilon_t\} \sim \text{IID N}(0,\sigma^2), \]

where \( \alpha_t = 2 \log \sigma_t \).

The resulting observation and state transition densities are
\[ p(y_t | \alpha_t) = n(y_t; 0, \exp(\alpha_t)) \]
\[ p(\alpha_{t+1} | \alpha_t) = n(\alpha_{t+1}; \gamma + \phi \alpha_t, \sigma^2) \]
Model:

\[ Y_t \mid \alpha_t, x_t \sim Pois(\exp(x_t^T \beta + \alpha_t)) \]
\[ \alpha_t = \phi \alpha_{t-1} + \varepsilon_t, \quad \{\varepsilon_t\} \sim \text{IID } N(0, \sigma^2) \]

Relative Likelihood: Let \( \psi=(\beta, \phi, \sigma^2) \) and suppose \( g(y_n, \alpha_n; \psi_0) \) is an approximating joint density for \( Y_n = (Y_1, \ldots, Y_n)' \) and \( \alpha_n = (\alpha_1, \ldots, \alpha_n)' \).

\[
L(\psi) = \int p(y_n \mid \alpha_n) p(\alpha_n) d\alpha_n \\
= \int \frac{p(y_n \mid \alpha_n) p(\alpha_n)}{g(y_n, \alpha_n; \psi_0)} g(y_n, \alpha_n; \psi_0) d\alpha_n \\
= \int \frac{p(y_n \mid \alpha_n) p(\alpha_n)}{g(y_n, \alpha_n; \psi_0)} g(\alpha_n \mid y_n; \psi_0) g(y_n; \psi_0) d\alpha_n \\
\frac{L(\psi)}{L_g(\psi_0)} = \int \frac{p(y_n \mid \alpha_n) p(\alpha_n)}{g(y_n, \alpha_n; \psi_0)} g(\alpha_n \mid y_n; \psi_0) d\alpha_n
\]
Importance Sampling (cont)

\[
\frac{L(\psi)}{L_g(\psi_0)} = \int \frac{p(y_n | \alpha_n) p(\alpha_n)}{g(y_n, \alpha_n; \psi_0)} g(\alpha_n | y_n; \psi_0) d\alpha_n
\]

\[
= E_g \left[ \frac{p(y_n | \alpha_n) p(\alpha_n)}{g(y_n, \alpha_n; \psi_0)} | y_n; \psi_0 \right]
\]

\[
\sim \frac{1}{N} \sum_{j=1}^{N} \frac{p(y_n | \alpha_n^{(j)}) p(\alpha_n^{(j)})}{g(y_n, \alpha_n^{(j)}; \psi_0)},
\]

where \( \{\alpha_n^{(j)}; j = 1,\ldots,N\} \sim \text{iid } g(\alpha_n | y_n; \psi_0). \)
Importance Sampling — example

Simulation example: \( Y_t \mid \alpha_t \sim Pois(\exp(0.7 + \alpha_t)) \),

\[
\alpha_t = 0.5 \alpha_{t-1} + \epsilon_t, \ \{\epsilon_t\} \sim IID \ N(0, 0.3), n = 200, N = 1000
\]
Consider a Gaussian approximation $p_a(\alpha_n \mid y_n) = \phi(\alpha_n ; \mu_0, \Sigma_0)$ to the posterior

$$p(\alpha_n \mid y_n) \propto p(\alpha_n \mid y_n) p(\alpha_n)$$

Setting equal the respective posterior modes $\alpha^*_a$ and $\alpha^*$ of $p_a(\alpha_n \mid y_n)$ and $p(\alpha_n \mid y_n)$, we have $\mu_0 = \alpha^*$, where $\alpha^*$ is the solution of the equation

$$\frac{\partial}{\partial \alpha_n} \log p(y_n \mid \alpha_n, \psi) - G_n (\alpha_n - \mu) = 0$$
Matching Fisher information matrices:

$$\Sigma_0 = \left( - \frac{\partial^2}{\partial \alpha \partial \alpha^T} \log p(y_n | \alpha_n, \psi) \bigg|_{\alpha_n = \alpha^*} + G_n \right)^{-1}$$

Approximating posterior:

$$p_a (\alpha_n | y_n, \psi) = \phi(\alpha_n, \alpha^*, \left( - \frac{\partial^2}{\partial \alpha \partial \alpha^T} \log p(y_n | \alpha_n, \psi) \bigg|_{\alpha_n = \alpha^*} + G_n \right)^{-1}$$

Notes:

1. This approximating posterior is identical to the importance sampling density used by Durbin and Koopman.
2. In traditional Bayesian setting, posterior is approximately $p_a$ for large $n$ (see Bernardo and Smith, 1994).
3. Obtain same result if one applies a Taylor series expansion to the joint likelihood and ignore terms of order $> 2$. 
Approximate likelihood: Note that

\[ p(\alpha_n | y_n) = \frac{p(y_n | \alpha_n) p(\alpha_n)}{L(\psi; y_n)}, \]

which by solving for \( L \) in the expression,

\[ p_a(\alpha_n^* | y_n, \psi) = p(\alpha_n^* | y_n, \psi), \]

we obtain

\[
L_a(\psi; y_n) = p(y_n | \alpha^*, \psi) p(\alpha^*, \psi) / p_a(\alpha^* | y_n, \psi)
\]

\[
= |G_n|^{1/2} p(y_n | \alpha^*, \psi) \exp\left\{-(\alpha^* - \mu)^T G_n (\alpha^* - \mu) / 2\right\}
\]

\[
\det\left(-\frac{\partial^2}{\partial \alpha \partial \alpha^T} \log p(y_n | \alpha_n, \psi)\right)_{\alpha^*} + G_n \right)^{1/2}
\]
Model: \( Y_t | \alpha_t \sim Pois(\exp(0.7 + \alpha_t)) \), \( \alpha_t = 0.5 \alpha_{t-1} + \epsilon_t \), \( \{\epsilon_t\} \sim \text{IID } N(0, 0.3) \), \( n = 200 \)

Estimation methods:

- Importance sampling (\( N=1000 \), \( \psi_0 \) updated a maximum of 10 times)

\[
\begin{array}{ccc}
\text{beta} & \text{phi} & \text{sigma2} \\
\text{mean} & 0.6982 & 0.4718 & 0.3008 \\
\text{std} & 0.1059 & 0.1476 & 0.0899 \\
\end{array}
\]

- Approximation to likelihood

\[
\begin{array}{ccc}
\text{beta} & \text{phi} & \text{sigma2} \\
\text{mean} & 0.7036 & 0.4579 & 0.2962 \\
\text{std} & 0.0951 & 0.1365 & 0.0784 \\
\end{array}
\]
Is the posterior distribution close to normal?

Compare posterior mean with posterior mode: Can compute the posterior mean using SIR (sampling importance-resampling) or particle filtering.

Posterior mode: The mode of $p(\alpha_{n} | y_{n})$ is $\alpha^{*}$ found at the last iteration of AL.

Posterior mean: The mean of $p(\alpha_{n} | y_{n})$ can be found using SIR.

Let $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(N)}$ be independent draws from the multivariate distr $p_{a}(\alpha_{n} | y_{n})$. For $N$ large, an approximate iid sample from $p(\alpha_{n} | y_{n})$ can be obtained by drawing a random sample from $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(N)}$ with probabilities

$$p_{i} = \frac{w_{i}}{\sum_{i=1}^{N} w_{i}}, \quad w_{i} = \frac{p(\alpha^{(i)} | y_{n})}{p_{a}(\alpha^{(i)} | y_{n})} \propto \frac{L(\psi; y_{n}, \alpha^{(i)})}{p_{a}(\alpha^{(i)} | y_{n})}, \quad i = 1, \ldots, N.$$
Posterior mean vs posterior mode?

Pound/US exchange rate data: blue = mean, red = mode
State Space Model Setup:
Observation equation:

\[ p(y_t | \alpha_t) = \exp\{\alpha_t y_t - b(\alpha_t) + c(y_t)\}. \]

State equation: \( \{\alpha_t\} \) follows the piecewise AR(1) model given by

\[ \alpha_t = \gamma_k + \phi_k \alpha_{t-1} + \sigma_k \varepsilon_t, \quad \text{if} \quad \tau_{k-1} \leq t < \tau_k, \]

where \( 1 = \tau_0 < \tau_1 < \ldots < \tau_m < n \), and \( \{\varepsilon_t\} \sim \text{IID N}(0,1). \)

Parameters:

- \( m = \) number of break points
- \( \tau_k = \) location of break points
- \( \gamma_k = \) level in \( k^{th} \) epoch
- \( \phi_k = \) AR coefficients \( k^{th} \) epoch
- \( \sigma_k = \) scale in \( k^{th} \) epoch
Application to Structural Breaks—(cont)

Estimation: For \((m, \tau_1, \ldots, \tau_m)\) fixed, calculate the approximate likelihood evaluated at the “MLE”, i.e.,
\[
L_a(\hat{\psi}; y_n) = \frac{|G_n|^{1/2}}{(K + G_n)^{1/2}} \exp\{y_n^T \alpha^* - 1^T \{b(\alpha^*) - c(y_n)\} - (\alpha^* - \mu)^T G_n (\alpha^* - \mu) / 2\},
\]
where \(\hat{\psi} = (\hat{\gamma}_1, \ldots, \hat{\gamma}_m, \hat{\phi}_1, \ldots, \hat{\phi}_m, \hat{\sigma}_1^2, \ldots, \hat{\sigma}_m^2)\) is the MLE.

Goal: Optimize an objective function over \((m, \tau_1, \ldots, \tau_m)\).

Implementation problems:

• choice of objective function?
• how to optimize over \((m, \tau_1, \ldots, \tau_m)\)?

Solutions(?):

• use minimum description length (MDL) as an objective function
• use genetic algorithm for optimization
Minimum Description Length (MDL): Choose the model which maximizes the compression of the data or, equivalently, select the model that minimizes the code length of the data (i.e., amount of memory required to store the data).

\[
\text{Code Length(“data”)} = \text{CL(“fitted model”)} + \text{CL(“data | fitted model”)}
\]

\[
\sim \text{CL(“parameters”)} + \text{CL(“residuals”)}
\]

\[
\text{MDL}(m, \tau_1, \ldots, \tau_m) = \log(m) + m \log(n) + 1.5 \sum_{j=1}^{m} \log(\tau_j - \tau_{j-1}) - \sum_{j=1}^{m} \log(L_a(\hat{\psi}_j; y_{\tau_{j-1}:\tau_j}))
\]

Generalization: AR(p) segments can have unknown order.

\[
\text{MDL}(m,(\tau_1, p_1), \ldots, (\tau_m, p_m)) = \log(m) + m \log(n) + 0.5 \sum_{j=1}^{m} (p_j + 2) \log(\tau_j - \tau_{j-1}) - \sum_{j=1}^{m} \log(L_a(\hat{\psi}_j; y_{\tau_{j-1}:\tau_j}))
\]
Application to Structural Breaks—(cont)

Genetic Algorithm: Chromosome consists of $n$ genes, each taking the value of $-1$ (no break) or $p$ (order of AR process). Use natural selection to find a near optimal solution.

Map the break points with a chromosome $c$ via

$$(m, (\tau_1, p_1), \ldots, (\tau_m, p_m)) \leftrightarrow c = (\delta_1, \ldots, \delta_n),$$

Where

$$\delta_t = \begin{cases} -1, & \text{if no break point at } t, \\ p, & \text{if break point at time } t \text{ and AR order is } p. \end{cases}$$

For example,

$$c = (2, -1, -1, -1, 0, -1, -1, -1, 0, -1, -1, -1, 0, -1, 3, -1, -1, -1, -1)$$

$t$: 1 6 11 16

would correspond to a process as follows:

$$\text{AR}(2), t=1:5; \text{AR}(0), t=6:10; \text{AR}(0), t=11:15; \text{AR}(3), t=16:20$$
Implementation of Genetic Algorithm—(cont)

Generation 0: Start with \( L \) (200) random generated chromosomes, \( c_1, \ldots, c_L \) with associated MDL values, \( M(c_1), \ldots, M(c_L) \).

Generation 1: A new child in the next generation is formed from the chromosomes \( c_1, \ldots, c_L \) of the previous generation as follows:

- with probability \( \pi_c \), crossover occurs.
  - two parent chromosomes \( c_i \) and \( c_j \) are selected at random with probabilities proportional to the ranks of \( M(c_i) \).
  - \( k^{th} \) gene of child is \( \delta_k = \delta_{i,k} \) w.p. \( 1/2 \) and \( \delta_{j,k} \) w.p. \( 1/2 \)
- with probability \( 1 - \pi_c \), mutation occurs.
  - a parent chromosome \( c_i \) is selected
  - \( k^{th} \) gene of child is \( \delta_k = \delta_{i,k} \) w.p. \( \pi_1 \); \(-1\) w.p. \( \pi_2 \); and \( p \) w.p. \( \alpha_p \)
Implementation of Genetic Algorithm—(cont)

Execution of GA: Run GA for numerous generations or until convergence.

Various Strategies:

- include the top ten chromosomes from past generations in future generations.

- use multiple islands, in which populations run independently, and then allow migration after a fixed number of generations.
Count Data Example

Model: \( Y_t \mid \alpha_t \sim Pois(\exp\{\beta + \alpha_t \}) \), \( \alpha_t = \phi \alpha_{t-1} + \varepsilon_t \), \( \{\varepsilon_t\} \sim \text{IID } N(0, \sigma^2) \)

True model:
- \( Y_t \mid \alpha_t \sim Pois(\exp\{.7 + \alpha_t \}) \), \( \alpha_t = .5\alpha_{t-1} + \varepsilon_t \), \( \{\varepsilon_t\} \sim \text{IID } N(0, .3) \), \( t < 250 \)
- \( Y_t \mid \alpha_t \sim Pois(\exp\{.7 + \alpha_t \}) \), \( \alpha_t = - .5\alpha_{t-1} + \varepsilon_t \), \( \{\varepsilon_t\} \sim \text{IID } N(0, .3) \), \( t > 250 \).
- GA estimate 251, time 267secs
SV Process Example

Model: \( Y_t \mid \alpha_t \sim N(0, \exp\{\alpha_t\}) \), \( \alpha_t = \gamma + \phi \alpha_{t-1} + \varepsilon_t \), \( \{\varepsilon_t\} \sim \text{IID } N(0, \sigma^2) \)

True model:
- \( Y_t \mid \alpha_t \sim N(0, \exp\{\alpha_t\}) \), \( \alpha_t = -.05 + .975\alpha_{t-1} + \varepsilon_t \), \( \{\varepsilon_t\} \sim \text{IID } N(0, .05) \), \( t < 1500 \)
- \( Y_t \mid \alpha_t \sim N(0, \exp\{\alpha_t\}) \), \( \alpha_t = -.25 + .900\alpha_{t-1} + \varepsilon_t \), \( \{\varepsilon_t\} \sim \text{IID } N(0, .25) \), \( t > 1500 \).
- GA estimate 1502, time 1049 secs
SV Process Example

Model: \( Y_t \mid \alpha_t \sim N(0, \exp\{2\alpha_t\}) \), \( \alpha_t = \gamma + \phi \alpha_{t-1} + \varepsilon_t \), \( \{\varepsilon_t\} \sim \text{IID } N(0, \sigma^2) \)

True model:
- \( Y_t \mid \alpha_t \sim N(0, \exp\{2\alpha_t\}) \), \( \alpha_t = -.175 + .977\alpha_{t-1} + \varepsilon_t \), \( \{\varepsilon_t\} \sim \text{IID } N(0, .1810), \ t < 251 \)
- \( Y_t \mid \alpha_t \sim N(0, \exp\{2\alpha_t\}) \), \( \alpha_t = -.010 + .996\alpha_{t-1} + \varepsilon_t \), \( \{\varepsilon_t\} \sim \text{IID } N(0, .0089), \ t > 250 \)
- GA estimate 251, time 269s
SV Process Example-(cont)

True model:
- \( Y_t \mid \alpha_t \sim N(0, \exp\{2\alpha_t\}) \), \( \alpha_t = -.175 + .977\alpha_{t-1} + \varepsilon_t \), \( \{\varepsilon_t\} \sim \text{IID } N(0, .1810) \), \( t < 251 \)
- \( Y_t \mid \alpha_t \sim N(0, \exp\{2\alpha_t\}) \), \( \alpha_t = -.010 + .996\alpha_{t-1} + \varepsilon_t \), \( \{\varepsilon_t\} \sim \text{IID } N(0, .0089) \), \( t > 250 \).

Fitted model based on no structural break:
- \( Y_t \mid \alpha_t \sim N(0, \exp\{2\alpha_t\}) \), \( \alpha_t = -.0645 + .9889\alpha_{t-1} + \varepsilon_t \), \( \{\varepsilon_t\} \sim \text{IID } N(0, .0935) \)

**original series**

**simulated series**
Linear Process Example (Monthly Deaths & Serious Injuries, UK)

Data: $Y_t =$ number of monthly deaths and serious injuries in UK, Jan `75 – Dec `84, $(t = 1, \ldots, 120)$

Remark: Seat belt legislation introduced in Feb `83 $(t = 99)$. 
**Linear Process Example (Monthly Deaths & Serious Injuries, UK)**

Data: $Y_t = \text{number of monthly deaths and serious injuries in UK, Jan } 75 - \text{Dec } 84, (t = 1, \ldots, 120)$

Remark: Seat belt legislation introduced in Feb '83 ($t = 99$).

Results from GA: 3 pieces; time = 4.4secs

Piece 1: ($t=1,\ldots,98$) IID; Piece 2: ($t=99,\ldots108$) IID; Piece 3: $t=109,\ldots,120$ AR(1)
Summary Remarks

1. Importance sampling offers a nice clean method for estimation in parameter driven models.

2. Approximation to the likelihood is a non-simulation based procedure which may have great potential especially with large sample sizes and/or large number of explanatory variables.

3. Approximate likelihood approach is amenable to bootstrapping procedures for bias correction.

4. Posterior mode matches posterior mean reasonably well.

5. Approximate likelihood approach may be useful to the problem of structural break detection.