

# Estimation for State-Space Models: an Approximate Likelihood Approach

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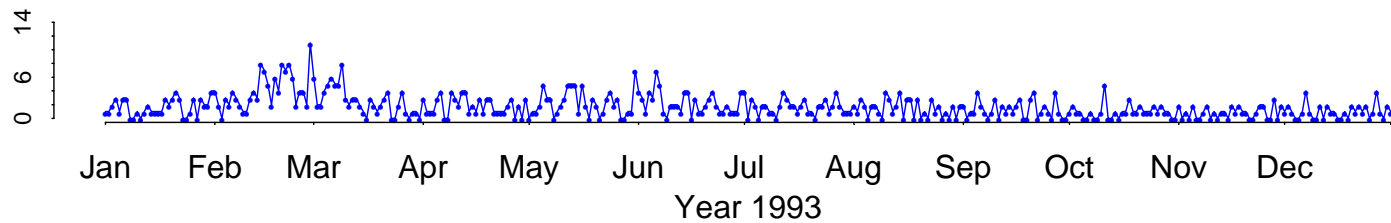
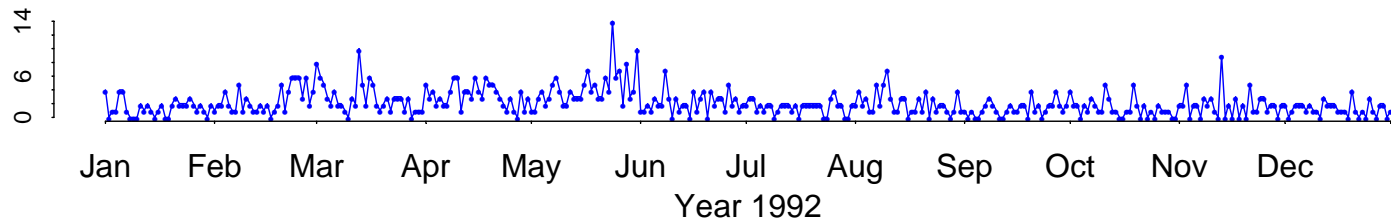
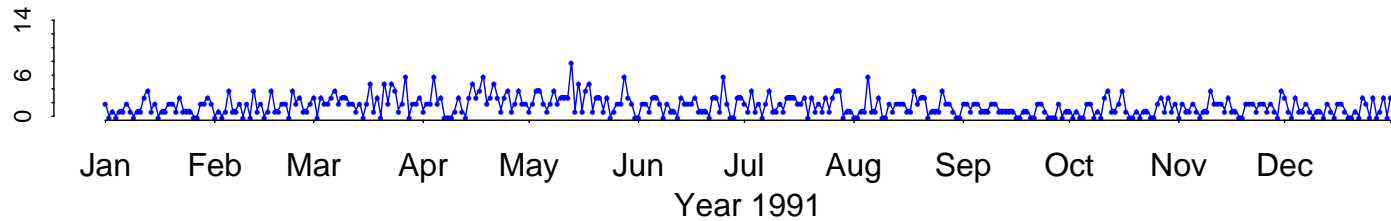
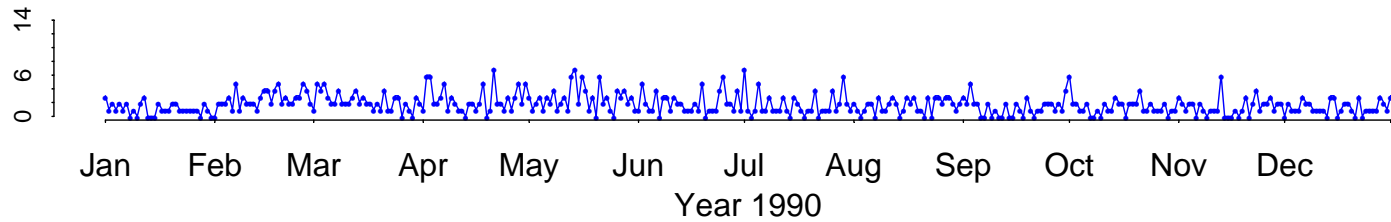
(<http://www.stat.colostate.edu/~rdavis/lectures>)

Joint work with:

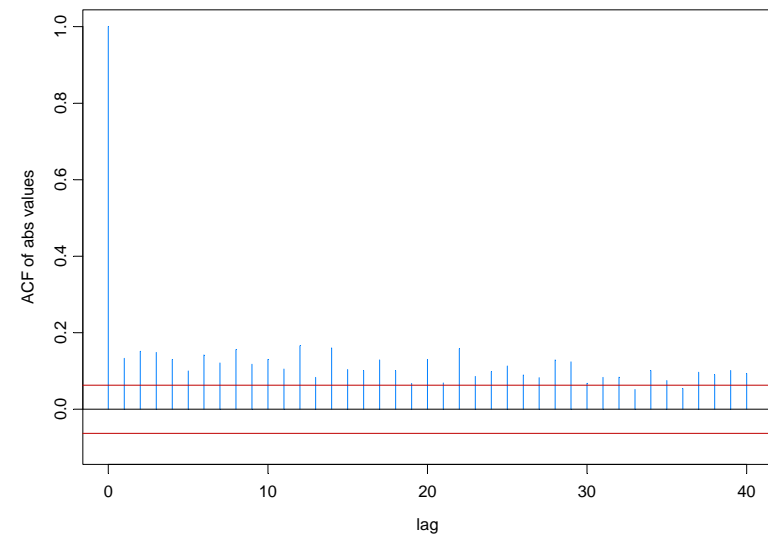
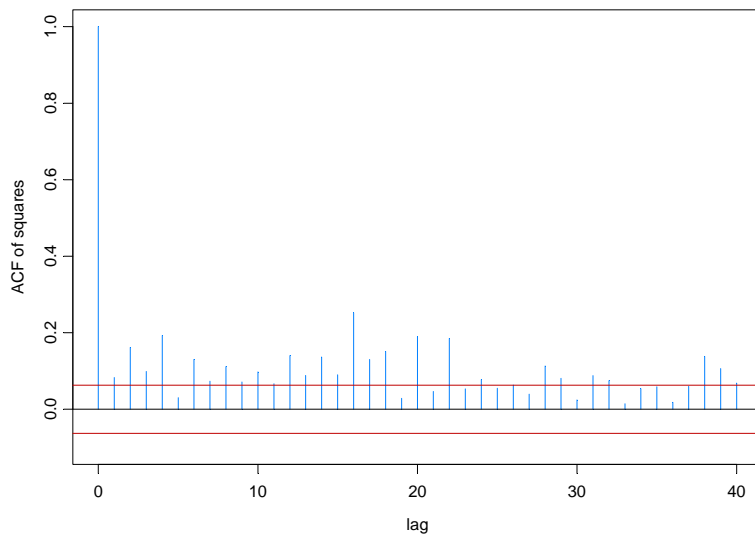
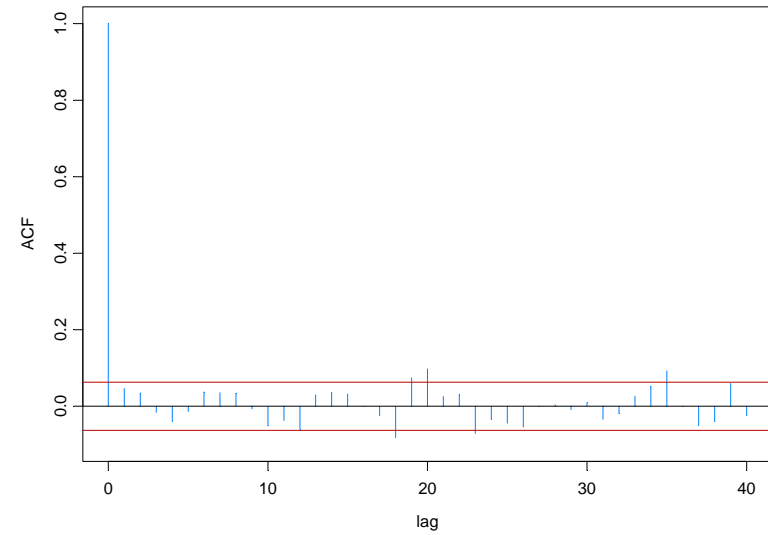
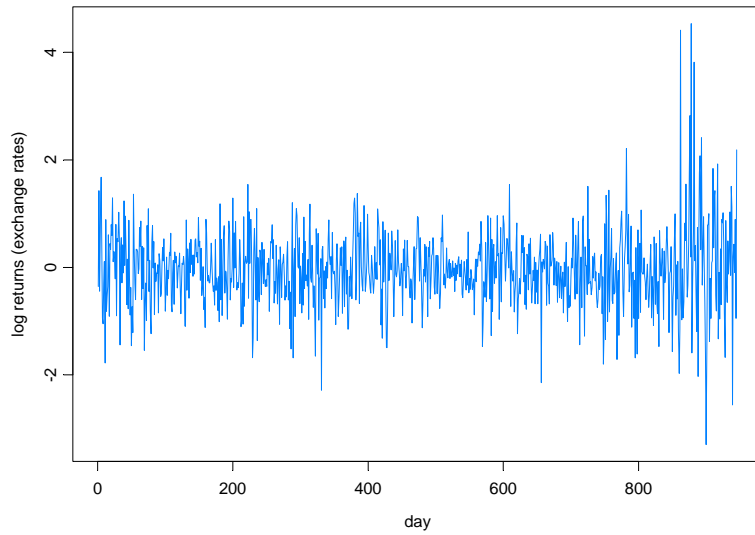
William Dunsmuir, University of New South Wales

Ying Wang, Dept of Public Health, W. Virginia

## Example: Daily Asthma Presentations (1990:1993)



# Example: Pound-Dollar Exchange Rates (Oct 1, 1981 - June 28, 1985; Koopman website)



## ➤ Motivating Examples

- Time series of counts
- Stochastic volatility

## ➤ Model setup and estimation

- Exponential family
  - ☞ 2 examples
- Estimation
  - ☞ GLM
    - Davis, Dunsmuir, Wang
  - ☞ Importance sampling
    - Durbin and Koopman
  - ☞ Approximation to the likelihood
    - Davis, Dunsmuir, and Wang

## ➤ Simulation and Application

- Time series of counts
- Stochastic volatility

## ➤ How good is the posterior approximation?

- posterior mode vs posterior mean

## Exponential Family Setup for Parameter-Driven Model

Time series data:  $Y_1, \dots, Y_n$

Regression (explanatory) variable:  $\mathbf{x}_t$

Observation equation:

$$p(y_t | \alpha_t) = \exp\{(\alpha_t + \beta^T \mathbf{x}_t) y_t - b(\alpha_t + \beta^T \mathbf{x}_t) + c(y_t)\}.$$

State equation:  $\{\alpha_t\}$  follows an autoregressive process satisfying the recursions

$$\alpha_t = \gamma + \phi_1 \alpha_{t-1} + \phi_2 \alpha_{t-2} + \dots + \phi_p \alpha_{t-p} + \varepsilon_t,$$

where  $\{\varepsilon_t\} \sim \text{IID } N(0, \sigma^2)$ .

Note:  $\alpha_t = 0$  corresponds to standard generalized linear model.

Original primary objective: Inference about  $\beta$ .

## Examples of parameter driven models

### Poisson model for time series of counts

Observation equation:

$$p(y_t | \alpha_t) = \frac{e^{(\beta^T x_t + \alpha_t) y_t} e^{-e^{(\beta^T x_t + \alpha_t)}}}{y_t!}, \quad y_t = 0, 1, \dots,$$

State equation: State variables follow a Gaussian AR(1) process

$$\alpha_t = \phi \alpha_{t-1} + \varepsilon_t, \quad \{\varepsilon_t\} \sim \text{IID } N(0, \sigma^2)$$

The resulting transition density of the state variables is

$$p(\alpha_{t+1} | \alpha_t) = n(\alpha_{t+1}; \phi \alpha_t, \sigma^2)$$

Remark: The case  $\sigma^2 = 0$  corresponds to a log-linear model with Poisson noise.

## Examples of parameter driven models-cont

A stochastic volatility model for financial data (Taylor '86):

Model:

$$Y_t = \sigma_t Z_t, \{Z_t\} \sim \text{IID } N(0,1)$$

$$\alpha_t = \phi \alpha_{t-1} + \varepsilon_t, \{\varepsilon_t\} \sim \text{IID } N(0, \sigma^2),$$

where  $\alpha_t = 2 \log \sigma_t$ .

The resulting observation and state transition densities are

$$p(y_t | \alpha_t) = n(y_t; 0, \exp(2\alpha_t))$$

$$p(\alpha_{t+1} | \alpha_t) = n(\alpha_{t+1}; \phi \alpha_t, \sigma^2)$$

Properties:

- Martingale difference sequence.
- Stationary.
- Strongly mixing at a geometric rate.

## Estimation Methods for Parameter Driven Models

- Estimating equations (Zeger '88): Let  $\hat{\beta}$  be the solution to the equation

$$\frac{\partial \mu}{\partial \beta} \Gamma_n (y_n - \mu) = 0,$$

where  $\mu = \exp(X \beta)$  and  $\Gamma_n = \text{var}(Y_n)$ .

- Monte Carlo EM (Chan and Ledolter '95)
- GLM (ignores the presence of the latent process, i.e.,  $\alpha_t = 0$ .)
- Importance sampling (Durbin & Koopman '01, Kuk '99, Kuk & Chen '97):
- Approximate likelihood (Davis, Dunsmuir & Wang '98)



## Estimation Methods — Importance Sampling (Durbin and Koopman)

Model:

$$Y_t | \alpha_t, \mathbf{x}_t \sim \text{Pois}(\exp(\mathbf{x}_t^\top \beta + \alpha_t))$$

$$\alpha_t = \phi \alpha_{t-1} + \varepsilon_t, \quad \{\varepsilon_t\} \sim \text{IID } N(0, \sigma^2)$$

**Relative Likelihood:** Let  $\psi = (\beta, \phi, \sigma^2)$  and suppose  $g(y_n, \alpha_n; \psi_0)$  is an approximating joint density for  $Y_n = (Y_1, \dots, Y_n)'$  and  $\alpha_n = (\alpha_1, \dots, \alpha_n)'$ .

$$\begin{aligned} L(\psi) &= \int p(y_n | \alpha_n) p(\alpha_n) d\alpha_n \\ &= \int \frac{p(y_n | \alpha_n) p(\alpha_n)}{g(y_n, \alpha_n; \psi_0)} g(y_n, \alpha_n; \psi_0) d\alpha_n \\ &= \int \frac{p(y_n | \alpha_n) p(\alpha_n)}{g(y_n, \alpha_n; \psi_0)} g(\alpha_n | y_n; \psi_0) g(y_n; \psi_0) d\alpha_n \\ \frac{L(\psi)}{L_g(\psi_0)} &= \int \frac{p(y_n | \alpha_n) p(\alpha_n)}{g(y_n, \alpha_n; \psi_0)} g(\alpha_n | y_n; \psi_0) d\alpha_n \end{aligned}$$

## Importance Sampling (cont)

$$\begin{aligned}\frac{L(\psi)}{L_g(\psi_0)} &= \int \frac{p(y_n | \alpha_n) p(\alpha_n)}{g(y_n, \alpha_n; \psi_0)} g(\alpha_n | y_n; \psi_0) d\alpha_n \\ &= E_g \left[ \frac{p(y_n | \alpha_n) p(\alpha_n)}{g(y_n, \alpha_n; \psi_0)} \mid y_n; \psi_0 \right] \\ &\sim \frac{1}{N} \sum_{j=1}^N \frac{p(y_n | \alpha_n^{(j)}) p(\alpha_n^{(j)})}{g(y_n, \alpha_n^{(j)}; \psi_0)},\end{aligned}$$

where  $\{\alpha_n^{(j)}; j = 1, \dots, N\} \sim \text{iid } g(\alpha_n | y_n; \psi_0)$ .

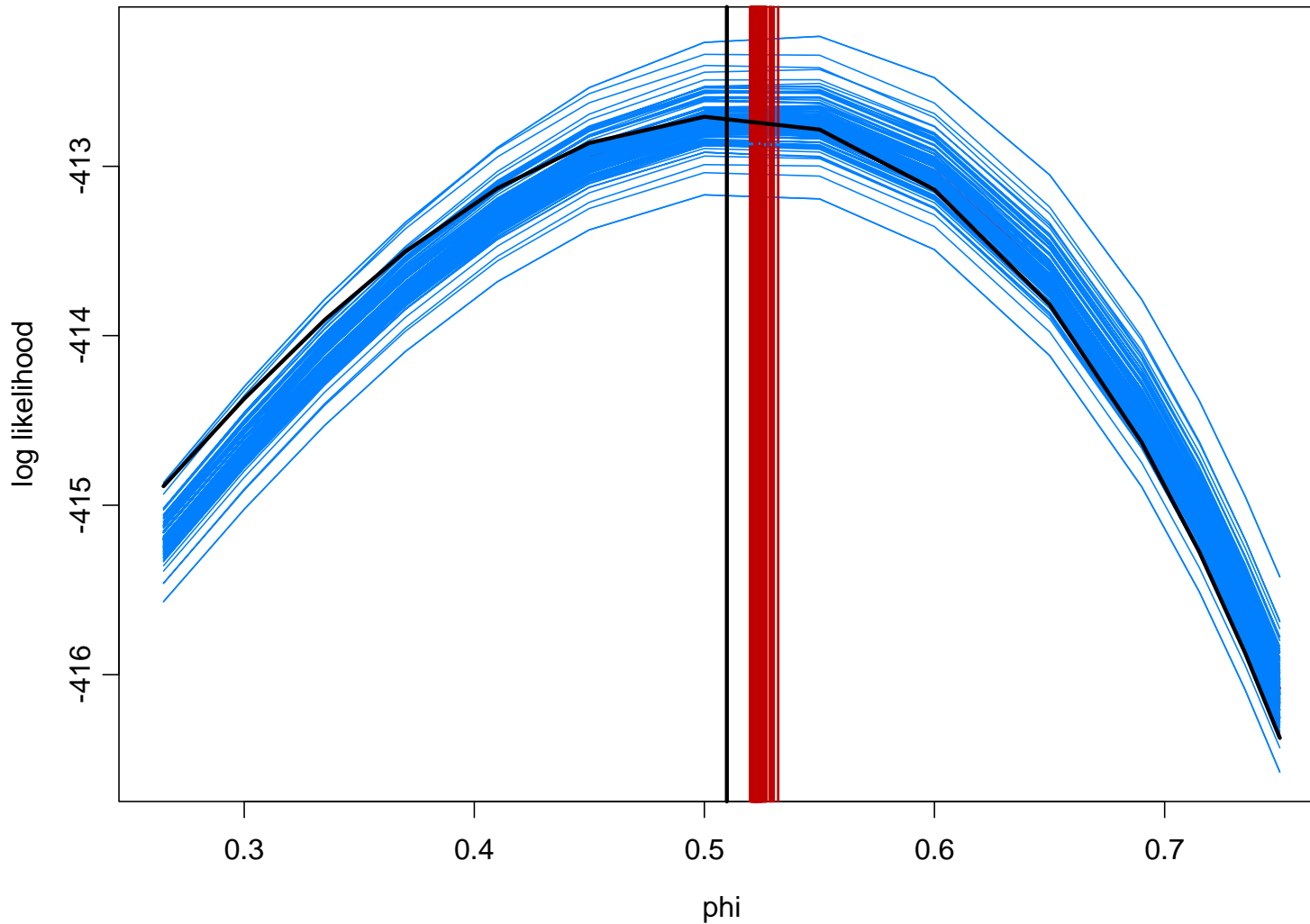
### Notes:

- This is a “one-sample” approximation to the relative likelihood. That is, for one realization of the  $\alpha$ 's, we have, in principle, an approximation to the whole likelihood function.
- Approximation is only good in a neighborhood of  $\psi_0$ . Geyer suggests maximizing ratio wrt  $\psi$  and iterate replacing  $\psi_0$  with  $\hat{\psi}$ .

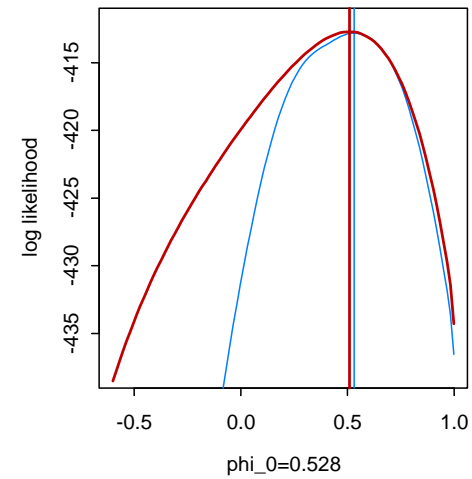
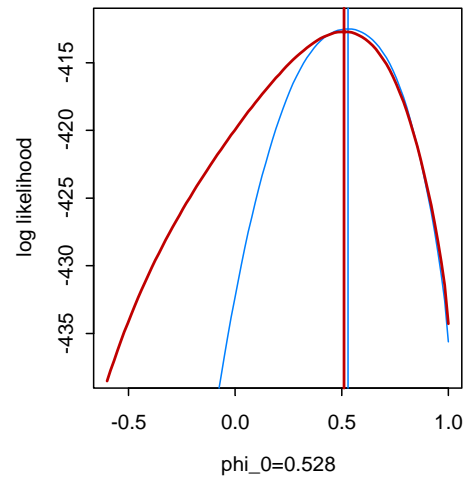
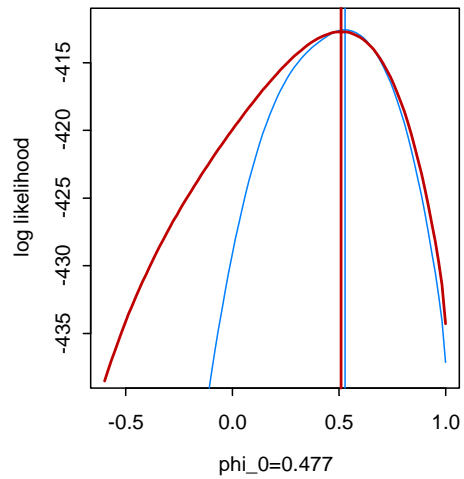
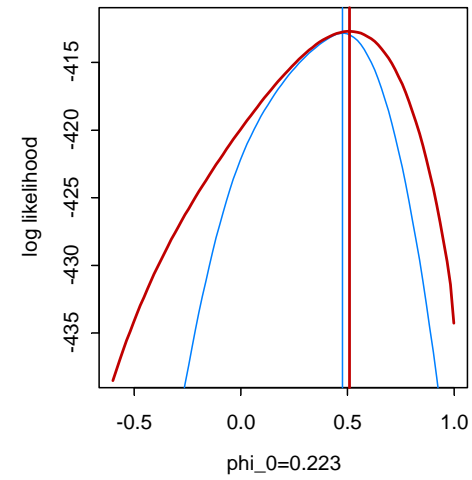
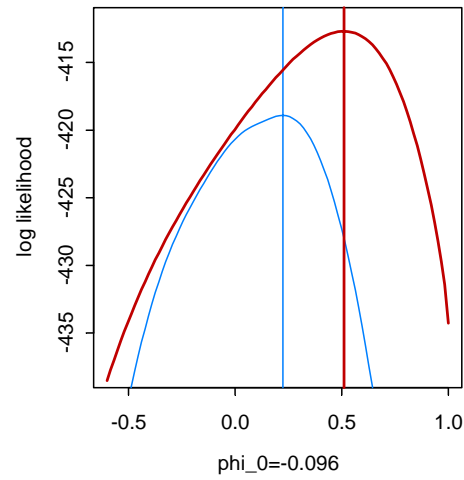
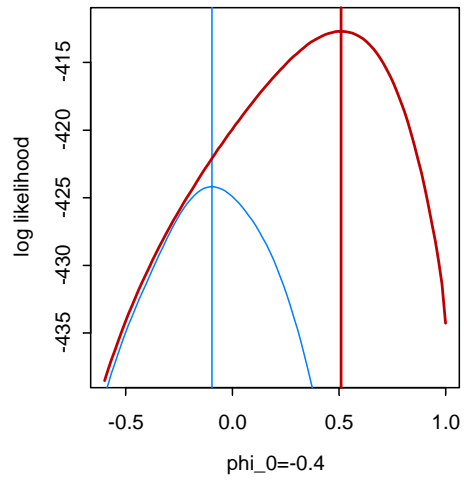
# Importance Sampling — example

Simulation example:  $Y_t | \alpha_t \sim \text{Pois}(\exp(.7 + \alpha_t))$ ,

$$\alpha_t = .5 \alpha_{t-1} + \varepsilon_t, \quad \{\varepsilon_t\} \sim \text{IID } N(0, .3), \quad n = 200, \quad N = 1000$$

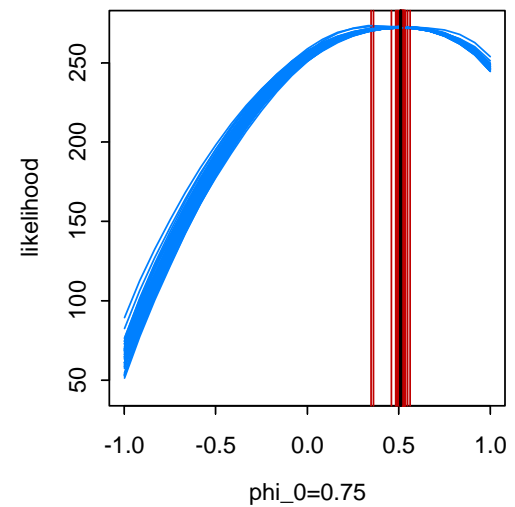
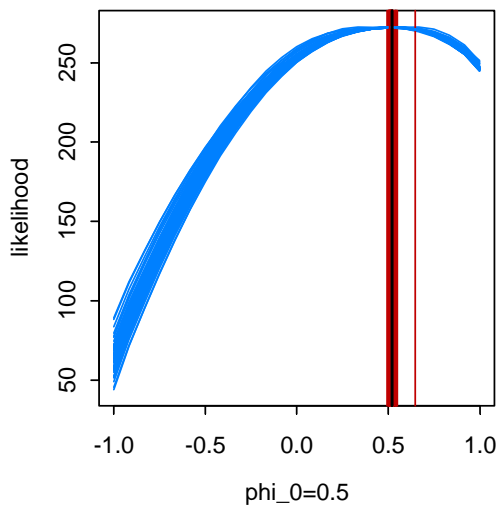
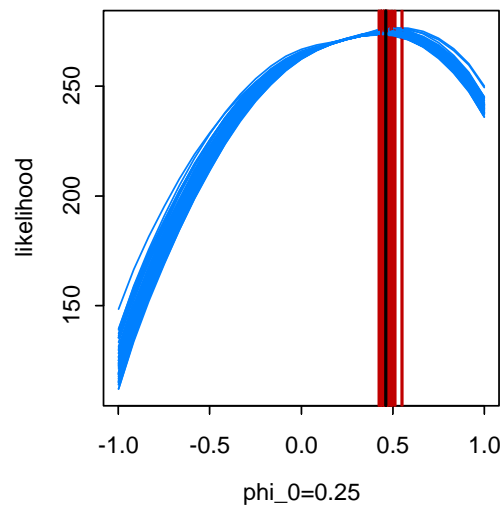
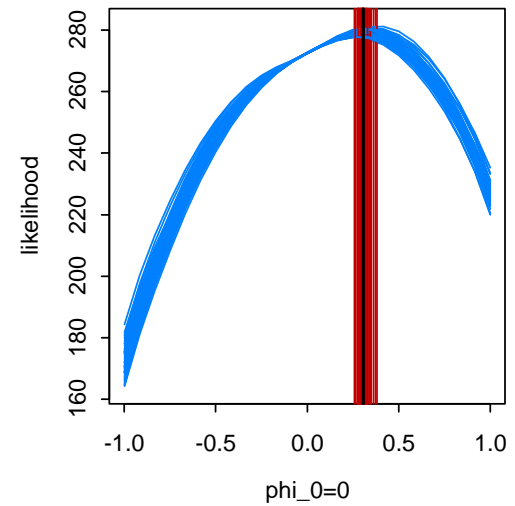
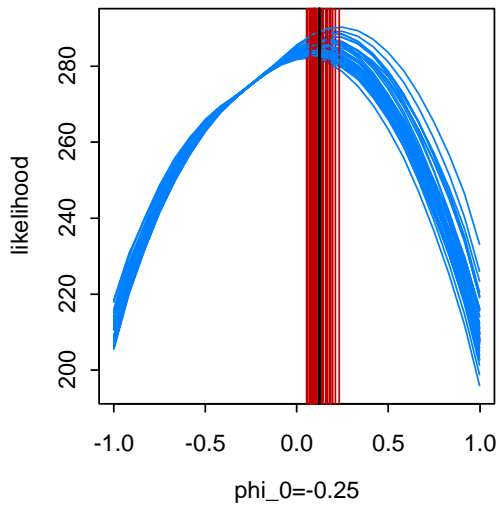
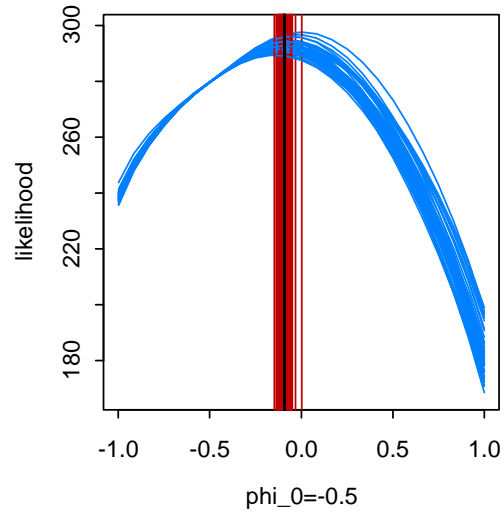


Simulation example:  $Y_t | \alpha_t \sim \text{Pois}(\exp(.7 + \alpha_t))$ ,  $\phi = .5$ ,  $\sigma^2 = .3$ ,  $n = 200$ ,  $N = 1000$



## Importance Sampling — example

Simulation example:  $\beta = .7$ ,  $\phi = .5$ ,  $\sigma^2 = .3$ ,  $n = 200$ ,  $N = 1000$ , 50 realizations plotted



## Importance Sampling (cont)

Choice of *importance density*  $g$ :

Durbin and Koopman suggest a linear state-space approximating model

$$Y_t = \mu_t + \mathbf{x}_t^T \beta + \alpha_t + Z_t, \quad Z_t \sim N(0, H_t),$$

with

$$\mu_t = y_t - \hat{\alpha}_t - \mathbf{x}'_t y_t e^{-(\hat{\alpha}_t + \mathbf{x}'_t \beta)} + 1,$$

$$H_t = e^{-(\hat{\alpha}_t + \mathbf{x}'_t \beta)},$$

where the  $\hat{\alpha}_t = E_g(\alpha_t | y_n)$  are calculated recursively under the approximating model until convergence.

With this choice of approximating model, it turns out that

$$g(\alpha_n | y_n; \Psi_0) \sim N(\Gamma_n^{-1} \tilde{y}_n, \Gamma_n^{-1}),$$

where

$$\tilde{y}_n = y_n - e^{X\beta + \hat{\alpha}_n} + e^{X\beta + \hat{\alpha}_n} \hat{\alpha}_n,$$

$$\Gamma_n = \text{diag}(e^{X\beta + \hat{\alpha}_n}) + (E(\alpha_n \alpha'_n))^{-1}.$$

## Importance Sampling (cont)

Components required in the calculation.

- $g(\mathbf{y}_n, \boldsymbol{\alpha}_n)$ 
  - ◆  $\tilde{\mathbf{y}}_n' \Gamma_n^{-1} \tilde{\mathbf{y}}_n$
  - ◆  $\det(\Gamma_n)$
- simulate from  $N(\Gamma_n^{-1} \tilde{\mathbf{y}}_n, \Gamma_n^{-1})$ 
  - ◆ compute  $\Gamma_n^{-1} \tilde{\mathbf{y}}_n$
  - ◆ simulate from  $N(\mathbf{0}, \Gamma_n^{-1})$

**Remark:** These quantities can be computed quickly using a version of the innovations algorithm or the Kalman smoothing recursions.

## Estimation Methods — Approximation to the likelihood

General setup:

$$p(y_n, \alpha_n) \propto p(y_n | \alpha_n) \det(G_n)^{1/2} \exp\{-(\alpha_n - \mu)^T G_n (\alpha_n - \mu) / 2\}$$

where

$$G_n^{-1} = E(\alpha_n - \mu)^T (\alpha_n - \mu)$$

Likelihood:

$$L(\psi) = \int p(y_n | \alpha_n) p(\alpha_n) d\alpha_n$$

Consider a Gaussian approximation  $p_a(\alpha_n | y_n) = \phi(\alpha_n; \mu_0, \Sigma_0)$  to the posterior

$$p(\alpha_n | y_n) \propto p(\alpha_n | y_n) p(\alpha_n)$$

Setting equal the respective posterior modes  $\alpha_a^*$  and  $\alpha^*$  of  $p_a(\alpha_n | y_n)$  and  $p(\alpha_n | y_n)$ , we have  $\mu_0 = \alpha^*$ , where  $\alpha^*$  is the solution of the equation

$$\frac{\partial}{\partial \alpha_n} \log p(y_n | \alpha_n, \psi) - G_n (\alpha_n - \mu) = 0$$



## Estimation Methods — Approximation to the likelihood (cont)

Matching Fisher information matrices:

$$\Sigma_0 = \left( - \frac{\partial^2}{\partial \alpha \partial \alpha^T} \log p(y_n | \alpha_n, \Psi) \Big|_{\alpha^*} + G_n \right)^{-1}$$

Approximating posterior:

$$p_a(\alpha_n | y_n, \Psi) = \phi(\alpha_n; \alpha^*, \left( - \frac{\partial^2}{\partial \alpha \partial \alpha^T} \log p(y_n | \alpha_n, \Psi) \Big|_{\alpha^*} + G_n \right)^{-1})$$

Notes:

1. This approximating posterior is identical to the importance sampling density used by Durbin and Koopman.
2. In traditional Bayesian setting, posterior is approximately  $p_a$  for large  $n$  (see Bernardo and Smith, 1994).
3. Obtain same result if one applies a Taylor series expansion to the joint likelihood and ignore terms of order  $> 2$ .

## Estimation Methods — Approximation to the likelihood (cont)

Approximate likelihood: Note that

$$p(\alpha_n | y_n) = \frac{p(y_n | \alpha_n) p(\alpha_n)}{L(\psi; y_n)},$$

which by solving for L in the expression,

$$p_a(\alpha_n^* | y_n, \psi) = p(\alpha_n^* | y_n, \psi),$$

we obtain

$$\begin{aligned} L_a(\psi; y_n) &= p(y_n | \alpha^*, \psi) p(\alpha^*, \psi) / p_a(\alpha^* | y_n, \psi) \\ &= \frac{|G_n|^{1/2} p(y_n | \alpha^*, \psi) \exp\{-(\alpha^* - \mu)^T G_n (\alpha^* - \mu) / 2\}}{\det\left(-\frac{\partial^2}{\partial \alpha \partial \alpha^T} \log p(y_n | \alpha_n, \psi) \Big|_{\alpha^*} + G_n\right)^{1/2}} \end{aligned}$$

## Estimation Methods — Approximation to the likelihood (cont)

Case of exponential family:

$$L_a(\psi; \mathbf{y}_n) = \frac{|\mathbf{G}_n|^{1/2}}{(\mathbf{K} + \mathbf{G}_n)^{1/2}} \exp \left\{ \mathbf{y}_n^T \boldsymbol{\alpha}^* - \mathbf{1}^T \{ \mathbf{b}(\boldsymbol{\alpha}^*) - c(\mathbf{y}_n) \} - (\boldsymbol{\alpha}^* - \boldsymbol{\mu})^T \mathbf{G}_n (\boldsymbol{\alpha}^* - \boldsymbol{\mu}) / 2 \right\},$$

where

$$\mathbf{K} = \text{diag} \left\{ \left. \frac{\partial^2}{\partial \alpha_t^2} b_t(\alpha_t) \right|_{\alpha_t^*} \right\},$$

and  $\boldsymbol{\alpha}^*$  is the solution to the equation

$$\mathbf{y}_n - \frac{\partial}{\partial \boldsymbol{\alpha}_n} \mathbf{b}(\boldsymbol{\alpha}_n) - \mathbf{G}_n (\boldsymbol{\alpha}_n - \boldsymbol{\mu}) = 0.$$

Using a Taylor expansion, the latter equation can be solved iteratively.

## Estimation Methods — Approximation to the likelihood

Implementation:

1. Let  $\alpha^* = \alpha^*(\psi)$  be the converged value of  $\alpha^{(j)}(\psi)$ , where

$$\alpha^{(j+1)}(\psi) = (\ddot{\mathbf{b}}^j + G_n)^{-1} \tilde{\mathbf{y}}_n^j(\psi),$$

and

$$\tilde{\mathbf{y}}_n^j = \mathbf{y}_n - \dot{\mathbf{b}}^j + \ddot{\mathbf{b}}^j \alpha^{(j)} + G_n \boldsymbol{\mu}.$$

2. Maximize  $L_a(\psi; \mathbf{y}_n)$  with respect to  $\psi$ .

## Simulation Results

**Model:**  $Y_t | \alpha_t \sim \text{Pois}(\exp(.7 + \alpha_t))$ ,  $\alpha_t = .5 \alpha_{t-1} + \varepsilon_t$ ,  $\{\varepsilon_t\} \sim \text{IID } N(0, .3)$ ,  $n = 200$

**Estimation methods:**

- Importance sampling (N=1000,  $\psi_0$  updated a maximum of 10 times )

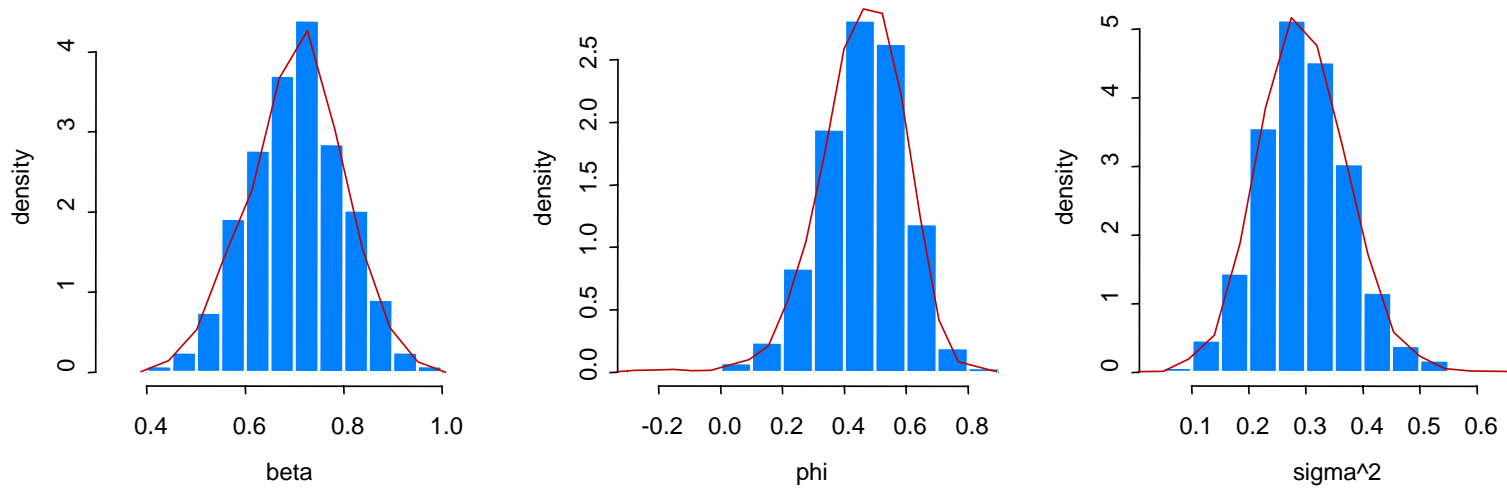
	beta	phi	sigma2
mean	0.6982	0.4718	0.3008
std	<b>0.1059</b>	<b>0.1476</b>	<b>0.0899</b>

- Approximation to likelihood

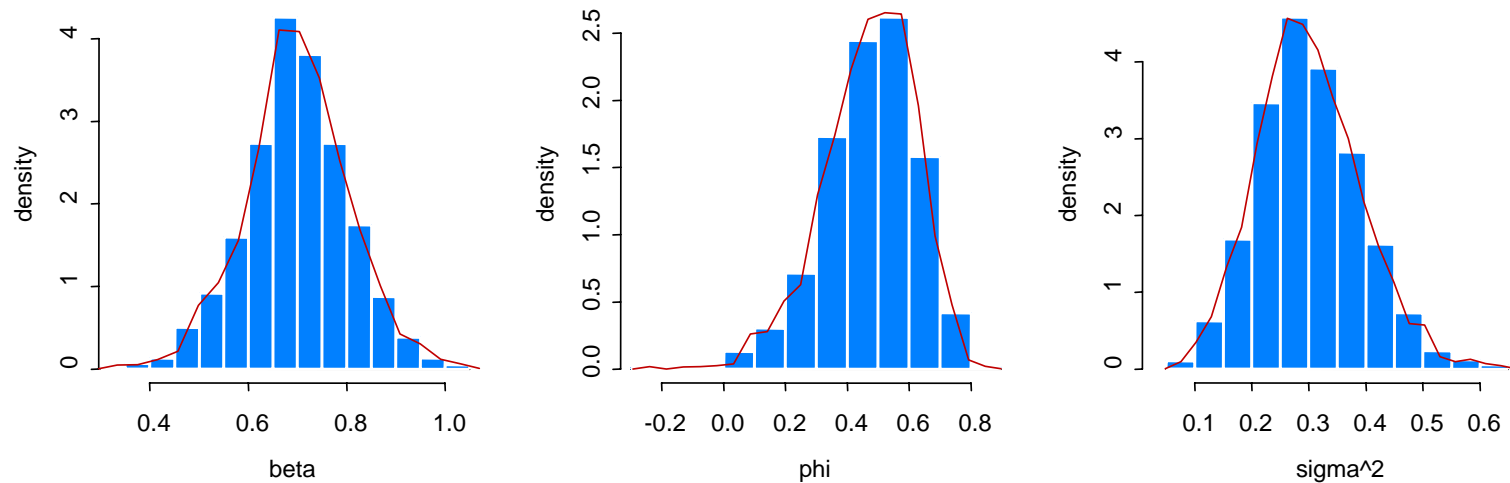
	beta	phi	sigma2
mean	0.7036	0.4579	0.2962
std	<b>0.0951</b>	<b>0.1365</b>	<b>0.0784</b>

**Model:**  $Y_t | \alpha_t \sim \text{Pois}(\exp(.7 + \alpha_t))$ ,  $\alpha_t = .5 \alpha_{t-1} + \varepsilon_t$ ,  $\{\varepsilon_t\} \sim \text{IID } N(0, .3)$ ,  $n = 200$

### Approx likelihood



### Importance Sampling



## Application to Model Fitting for the Polio Data

Model for  $\{\alpha_t\}$ :

$$\alpha_t = \phi\alpha_{t-1} + \varepsilon_t, \quad \{\varepsilon_t\} \sim \text{IID } N(0, \sigma^2).$$

- Importance sampling (  $\psi_0$  updated 5 times for each  $N=100, 500, 1000,$  )
- Simulation based on 1000 replications and the fitted AL model.

	Import Sampling			Approx Like			GLM	
	$\hat{\beta}_{IS}$	Simulation		$\hat{\beta}_{AL}$	Simulation		$\hat{\beta}_{GLM}$	SD
		Mean	SD		Mean	SD		
Intercept	<b>0.203</b>	0.223	0.381	<b>0.202</b>	0.210	0.343	<b>.207</b>	0.078
Trend( $\times 10^{-3}$ )	<b>-2.675</b>	-2.778	3.979	<b>-2.690</b>	-2.720	3.415	<b>-4.18</b>	1.400
$\cos(2\pi t/12)$	<b>0.110</b>	0.103	0.124	<b>0.113</b>	0.111	0.123	<b>-.152</b>	0.097
$\sin(2\pi t/12)$	<b>-0.456</b>	-0.456	0.151	<b>-0.454</b>	-0.454	0.143	<b>-.532</b>	0.109
$\cos(2\pi t/6)$	<b>0.399</b>	0.401	0.123	<b>0.396</b>	0.400	0.114	<b>.169</b>	0.098
$\sin(2\pi t/6)$	<b>0.015</b>	0.024	0.118	<b>0.016</b>	0.012	0.110	<b>-.432</b>	0.101
$\phi$	<b>0.865</b>	0.777	0.198	<b>0.845</b>	0.764	0.165		
$\sigma^2$	<b>0.088</b>	0.100	0.068	<b>0.104</b>	0.114	0.075		

## Simulation Results

Stochastic volatility model:

$$Y_t = \sigma_t Z_t, \{Z_t\} \sim \text{IID } N(0,1)$$

$$\alpha_t = \gamma + \phi \alpha_{t-1} + \varepsilon_t, \{\varepsilon_t\} \sim \text{IID } N(0, \sigma^2), \text{ where } \alpha_t = 2 \log \sigma_t; n=1000, \text{NR}=500$$

CV=10

	True	AL	RMSE	IS	RMSE
$\gamma$	-.411	<b>-.491</b>	.210	<b>-.490</b>	.216
$\phi$	0.950	<b>0.940</b>	.025	<b>0.940</b>	.026
$\sigma$	0.484	<b>0.478</b>	.065	<b>0.481</b>	.073

CV=1

	True	AL	RMSE	IS	RMSE
$\gamma$	-.368	<b>-.499</b>	.341	<b>-.485</b>	.324
$\phi$	0.950	<b>0.932</b>	.046	<b>0.934</b>	.043
$\sigma$	0.260	<b>0.270</b>	.068	<b>0.268</b>	.068



## Application to Sydney Asthma Count Data

**Data:**  $Y_1, \dots, Y_{1461}$  daily asthma presentations in a Campbelltown hospital.

Preliminary analysis identified.

- no upward or downward trend
- **annual cycle** modeled by  $\cos(2\pi t/365), \sin(2\pi t/365)$
- **seasonal effect** modeled by

$$P_{ij}(t) = \frac{1}{B(2.5,5)} \left( \frac{t - T_{ij}}{100} \right)^{2.5} \left( 1 - \frac{t - T_{ij}}{100} \right)^5$$

where  $B(2.5,5)$  is the beta function and  $T_{ij}$  is the start of the  $j^{\text{th}}$  school term in year  $i$ .

- day of the week effect modeled by separate indicator variables for **Sunday** and **Monday** (increase in admittance on these days compared to Tues-Sat).
- Of the meteorological variables (max/min temp, humidity) and pollution variables (ozone, NO, NO<sub>2</sub>), only **humidity** at lags of 12-20 days and **NO<sub>2</sub>(max)** appear to have an association.

## Results for Asthma Data—(IS & AL)

Term	IS	AL	Mean	SD
Intercept	0.590	0.591	0.593	.0658
Sunday effect	0.138	0.138	0.139	.0531
Monday effect	0.229	0.231	0.230	.0495
$\cos(2\pi t/365)$	-0.218	-0.218	-0.217	.0415
$\sin(2\pi t/365)$	0.200	0.179	0.181	.0437
Term 1, 1990	0.188	0.198	0.194	.0638
Term 2, 1990	0.183	0.130	0.129	.0664
Term 1, 1991	0.080	0.075	0.070	.0733
Term 2, 1991	0.177	0.164	0.157	.0665
Term 1, 1992	0.223	0.221	0.214	.0667
Term 2, 1992	0.243	0.239	0.237	.0620
Term 1, 1993	0.379	0.397	0.394	.0625
Term 2, 1993	0.127	0.111	0.108	.0682
Humidity $H_t/20$	0.009	0.010	0.007	.0032
$\text{NO}_2$ max	-0.125	-0.107	-0.108	.0347
AR(1), $\phi$	0.385	0.788	0.468	.3790
$\sigma^2$	0.053	0.010	0.018	.0153

## Is the posterior distribution close to normal?

**Compare posterior mean with posterior mode:** Can compute the posterior mean using *SIR* (sampling importance-resampling)

**Posterior mode:** The mode of  $p(\alpha_n | y_n)$  is  $\alpha^*$  found at the last iteration.

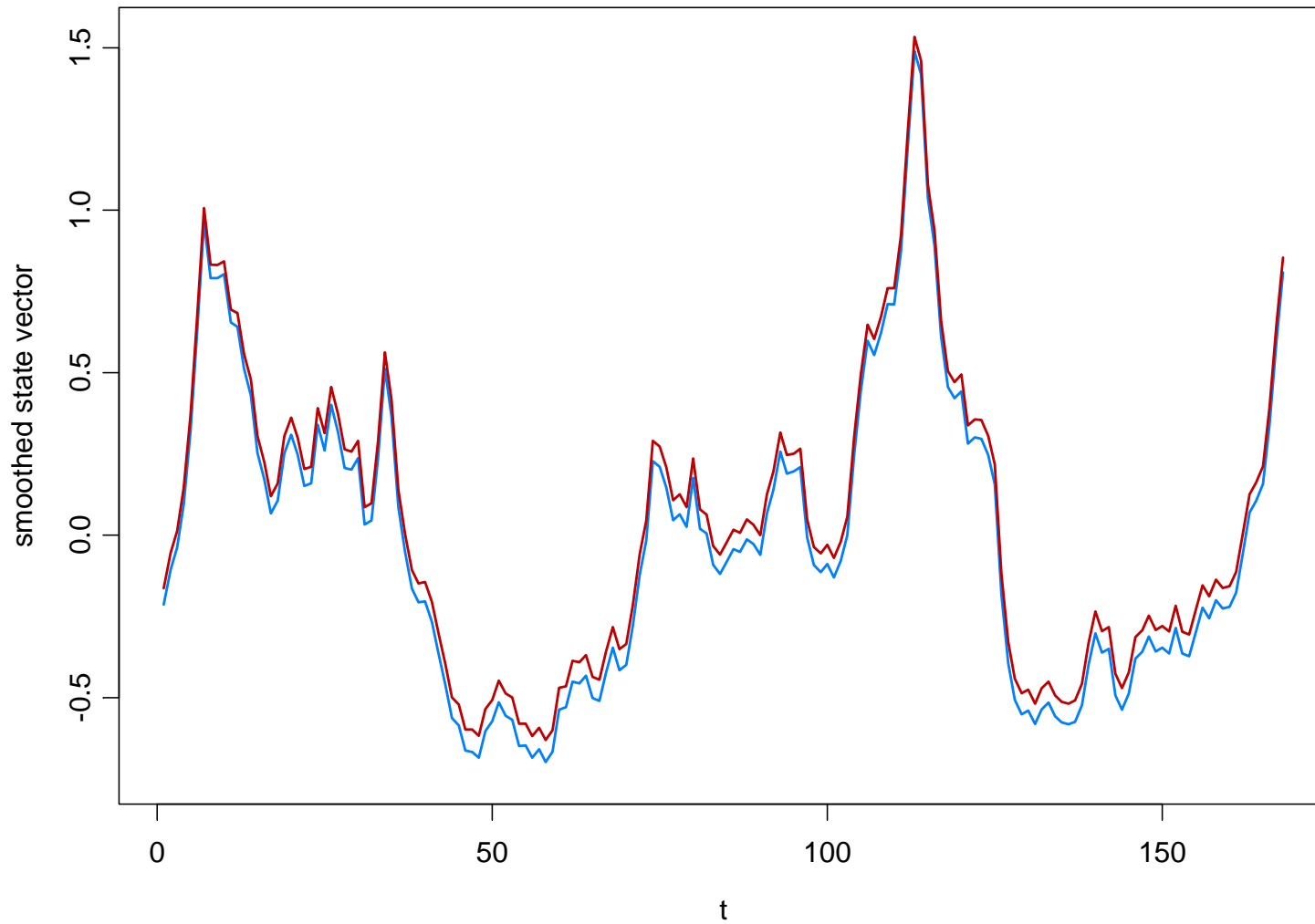
**Posterior mean:** The mean of  $p(\alpha_n | y_n)$  can be found using *SIR*.

Let  $\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(N)}$  be independent draws from the multivariate distr  $p_a(\alpha_n | y_n)$ . For  $N$  large, an approximate iid sample from  $p(\alpha_n | y_n)$  can be obtained by drawing a random sample from  $\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(N)}$  with probabilities

$$p_i = \frac{w_i}{\sum_{i=1}^N w_i}, \quad w_i = \frac{p(\alpha^{(i)} | y_n)}{p_a(\alpha^{(i)} | y_n)} \propto \frac{L(\psi; y_n, \alpha^{(i)})}{p_a(\alpha^{(i)} | y_n)}, \quad i = 1, \dots, N.$$

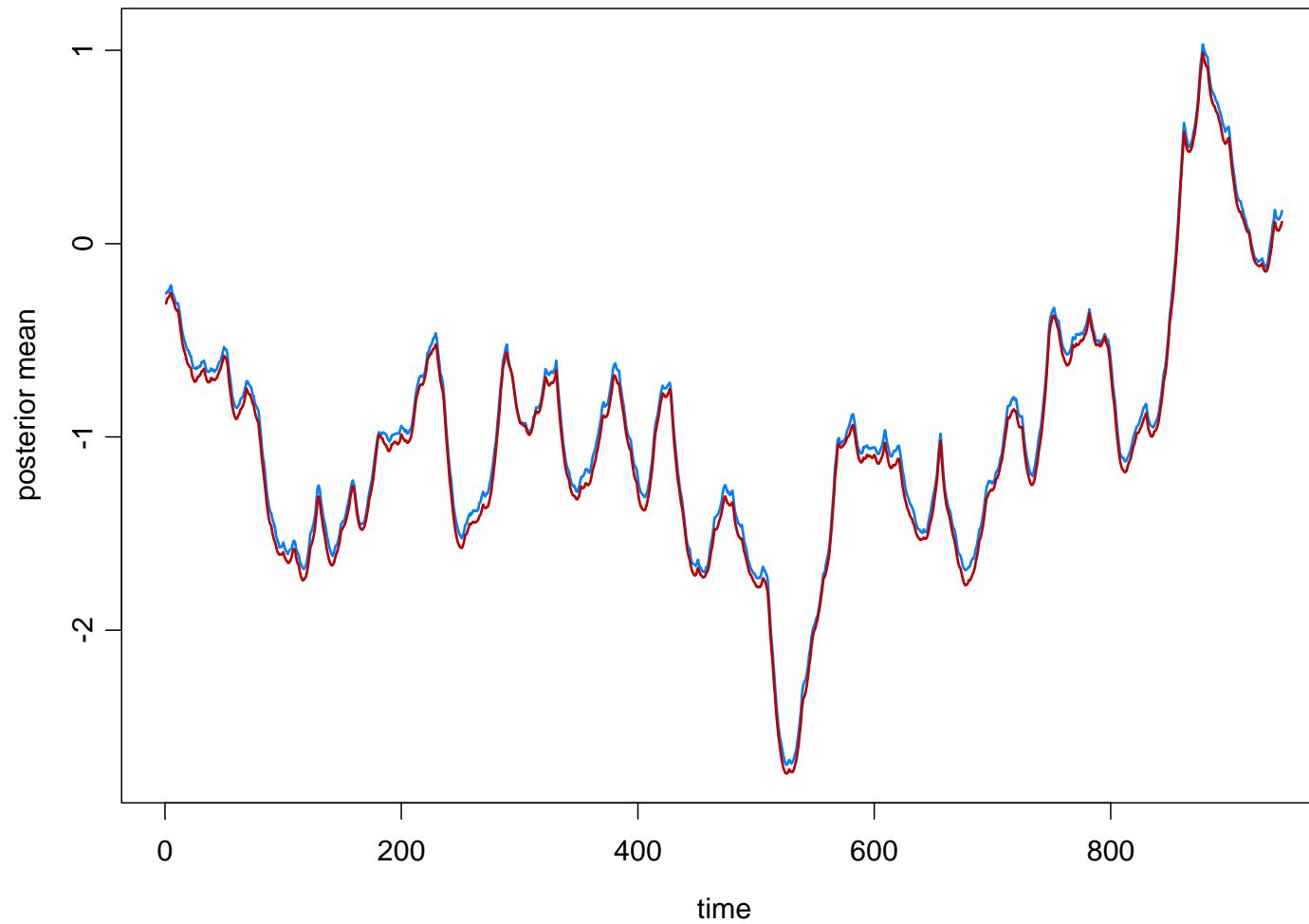
## Posterior mean vs posterior mode?

Polio data: blue = mean, red = mode



## Posterior mean vs posterior mode?

Pound/US exchange rate data: blue = mean, red = mode



## Summary Remarks

1. Importance sampling offers a nice clean method for estimation in parameter driven models.
2. Relative likelihood approach is a one-sample based procedure, but may have convergence problems.
3. Approximation to the likelihood is a non-simulation based procedure which may have great potential especially with large sample sizes and/or large number of explanatory variables.
5. Approximation likelihood approach is amenable to bootstrapping procedures for bias correction.
6. Posterior mode matches posterior mean reasonably well.
7. Extension to more general latent process models (e.g., long memory) is in progress.