Extremes of Space-Time Processes With Heavy-Tailed Distributions

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Outline

- A Class of Space-Time Processes: $X_t(\mathbf{s}) = \sum_{i=0}^{\infty} \psi_i(\mathbf{s})\, Z_{t-i}(\mathbf{s})\,, \quad \mathbf{s} \in [0,1]^d$
  - Dependence properties
- Preliminaries on Regular Variation on $\mathbb{D}([0,1]^d)$
  - Examples
- Point Process Convergence
  - Basic properties
- Application

EVT for Space-Time Processes

Basic set-up: 2 components, spatial and temporal.

Spatial part. Let $Z(\mathbf{s})$ be a random field on $[0,1]^d$.

- Usually $d = 1$ (transect) or $d = 2$ (two-dimensional space).
- $Z(\mathbf{s})$ is value of the random field at location $\mathbf{s} \in [0,1]^d$.
- View $Z(\mathbf{s})$ as a random element of $\mathbb{D} = \mathbb{D}([0,1]^d)$ of càdlàg functions J_1-topology; see Bickel and Wichura (1971).
- Will assume that Z has *regularly varying tail probabilities* –to be described later.

EVT for Space-Time Processes

Temporal part. Build in serial dependence by filtering the random field at each location $\mathbf{s} \in [0,1]^d$. That is, set

$$X_t(\mathbf{s}) = \sum_{i=0}^{\infty} \psi_i(\mathbf{s})\, Z_{t-i}(\mathbf{s})\,, \quad \mathbf{s} \in [0,1]^d\,,$$

where

- $(Z_t)_{t\in\mathbb{Z}}$ are iid copies of the random field Z on $[0,1]^d$
- ψ_i's are deterministic *càdlàg* real-valued fields on $[0,1]^d$.

Note: For $\mathbf{s}_1, \ldots, \mathbf{s}_k$ fixed,

$$\mathbf{X}_t := \begin{bmatrix} X_t(\mathbf{s}_1) \\ \vdots \\ X_t(\mathbf{s}_k) \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{\infty} \psi_i(\mathbf{s}_1) Z_{t-i}(\mathbf{s}_1) \\ \vdots \\ \sum_{i=0}^{\infty} \psi_i(\mathbf{s}_k) Z_{t-i}(\mathbf{s}_k) \end{bmatrix} = \sum_{i=0}^{\infty} A_i \mathbf{Z}_{t-i}$$

is a multivariate linear time series.

Dependence structure of (X_t)

Suppose the random field $Z(\mathbf{s})$ is stationary with covariance function $\gamma_Z(\mathbf{u})$,

$$\mathrm{Cov}(Z(\mathbf{s}+\mathbf{u}), Z(\mathbf{s})) = \gamma_Z(\mathbf{u}).$$

Spatial covariance of X_t.

$$\mathrm{Cov}(X_t(\mathbf{s}+\mathbf{u}), X_t(\mathbf{s})) = \left(\sum_{j=0}^{\infty} \psi_j(\mathbf{s}+\mathbf{u})\psi_j(\mathbf{s})\right) \gamma_Z(\mathbf{u}),$$

which is stationary in space (independent of $\mathbf{s}$) if the ψ_j's are constant functions. In this case,

$$\mathrm{Cov}(X_t(\mathbf{s}+\mathbf{u}), X_t(\mathbf{s})) = \left(\sum_{j=0}^{\infty} \psi_j^2\right) \gamma_Z(\mathbf{u}).$$

Dependence structure of (X_t)

Time covariance function of $X_t(\mathbf{s})$. For each $\mathbf{s} \in [0,1]^d$, the time series $X_t(\mathbf{s})$ is a *linear process* with covariance function

$$\mathrm{Cov}(X_{t+h}(\mathbf{s}), X_t(\mathbf{s})) = \left(\sum_{j=0}^{\infty} \psi_{j+h}(\mathbf{s}) \psi_j(\mathbf{s}) \right) \gamma_Z(\mathbf{0})$$

If the ψ_j's are constant functions, then the serial correlation does not depend on $\mathbf{s}$.

Note: In fact, the time series X_t defined on $\mathbb{D}([0,1]^d)$ is strictly stationary.

Space-time covariance function of $X_t(\mathbf{s})$.

$$\mathrm{Cov}(X_{t+h}(\mathbf{s}+\mathbf{u}), X_t(\mathbf{s})) = \left(\sum_{j=0}^{\infty} \psi_{j+h}(\mathbf{s}+\mathbf{u}) \psi_j(\mathbf{s}) \right) \gamma_Z(\mathbf{u})$$

which, if the ψ_j's are constant functions, is equal to

$$\begin{aligned}\gamma_X(h,\mathbf{u}) &= \mathrm{Cov}(X_{t+h}(\mathbf{s}+\mathbf{u}), X_t(\mathbf{s})) \\ &= \left(\sum_{j=0}^{\infty} \psi_{j+h}\psi_j\right)\gamma_Z(\mathbf{u}) \\ &= \gamma_T(h)\gamma_Z(\mathbf{u})\end{aligned}$$

Remarks:

(1) The filter functions ψ_j influence both the spatial and temporal covariances.

(2) If the ψ_j's are constant functions, then X_t has a multiplicative covariance function, i.e.,

$$\begin{aligned}\gamma_X(h,\mathbf{u}) &= \mathrm{Cov}(X_{t+h}(\mathbf{s}+\mathbf{u}), X_t(\mathbf{s})) \\ &= \gamma_T(h)\gamma_Z(\mathbf{u})\end{aligned}$$

Examples and Applications

1. Maximum ozone levels. Suppose there exists a standard L for annual maxima of ozone levels over the rectangular region $[0,1]^2$. Set

$$X_t(\mathbf{s}) \;=\; \text{maximum ozone level at site } \mathbf{s} \text{ during year } t.$$

Then the probability the standard L is not exceeded in n consecutive years is

$$P(\max_{t=1,\ldots,n} X_t(\mathbf{s}) \leq L, \text{ for all } \mathbf{s} \in [0,1]^2)\,.$$

2. Sea level (de Haan and Lin (2001)). Let $f(s)$ represent the height of a dyke off the Dutch coast at location s and set

$$X_t(s) \;=\; \text{maximum sea level at site } s \text{ during day } t$$

The probability that the dyke is not breached along the coast for n consecutive days is

$$P(\max_{t=1,\ldots,n} X_t(s) \leq f(s), \text{ for all } s \in [0,1])\,.$$

Regular Variation on $\mathbb{D}([0,1]^d)$ — Preliminaries:

Regular variation of $\mathbf{Z} = (Z_1, \ldots, Z_m)'$. There exists a random vector $\boldsymbol{\theta}$ defined on $\mathbb{S}^{m-1}$ such that for all $z > 0$

$$P(\|\mathbf{Z}\| > tz, \mathbf{Z}/\|\mathbf{Z}\| \in \cdot)/P(\|\mathbf{Z}\| > t) \xrightarrow{w} z^{-\alpha} P(\boldsymbol{\theta} \in \cdot)\,,$$

as $t \to \infty$ where $\xrightarrow{w}$ is weak convergence on $\mathbb{S}^{m-1}$, the unit sphere in $\mathbb{R}^m$.

- $P(\boldsymbol{\theta} \in \cdot)$ is called the spectral measure.
- α is the index of regular variation.

Equivalence: There exists $a_n > 0$ such that for all $z > 0$

$$nP(\|\mathbf{Z}\| > a_n z, \mathbf{Z}/\|\mathbf{Z}\| \in \cdot) \xrightarrow{w} z^{-\alpha} P(\boldsymbol{\theta} \in \cdot)$$

or, equivalently,

$$nP(a_n^{-1}\mathbf{Z} \in \cdot) \xrightarrow{v} m(\cdot)$$

for some Radon measure m on $\mathcal{B}(\overline{\mathbb{R}}^m \setminus \{\mathbf{0}\})$.

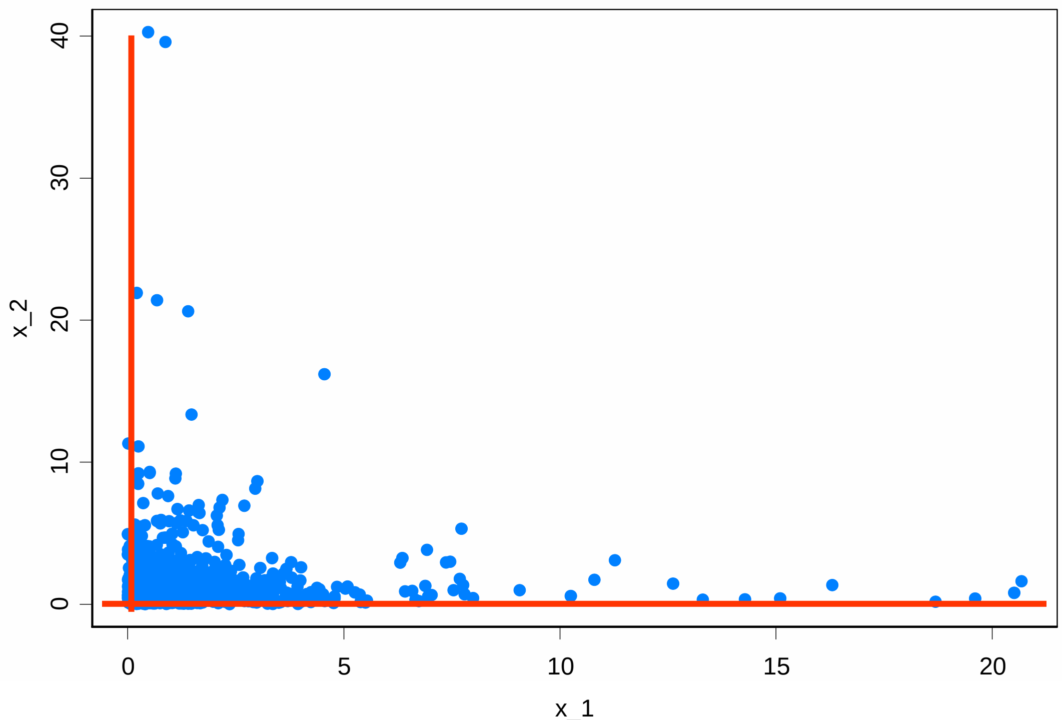
Examples of Regular Variation on $\mathbb{R}^2$:

1. If $Z_1 > 0$ and $Z_2 > 0$ are iid $RV(\alpha)$, then $\mathbf{Z} = (Z_1, Z_2)$ is regularly varying with index α and spectral distribution

$$P(\boldsymbol{\theta} = (0,1)) = P(\boldsymbol{\theta} = (1,0)) = .5 \text{ (mass on axes)}.$$

Interpretation: Unlikely that Z_1 and Z_2 are both large at the same time.

Figure: plot of (Z_{t1}, Z_{t2}) for realization of 10,000.



Examples of Regular Variation on $\mathbb{R}^2$:

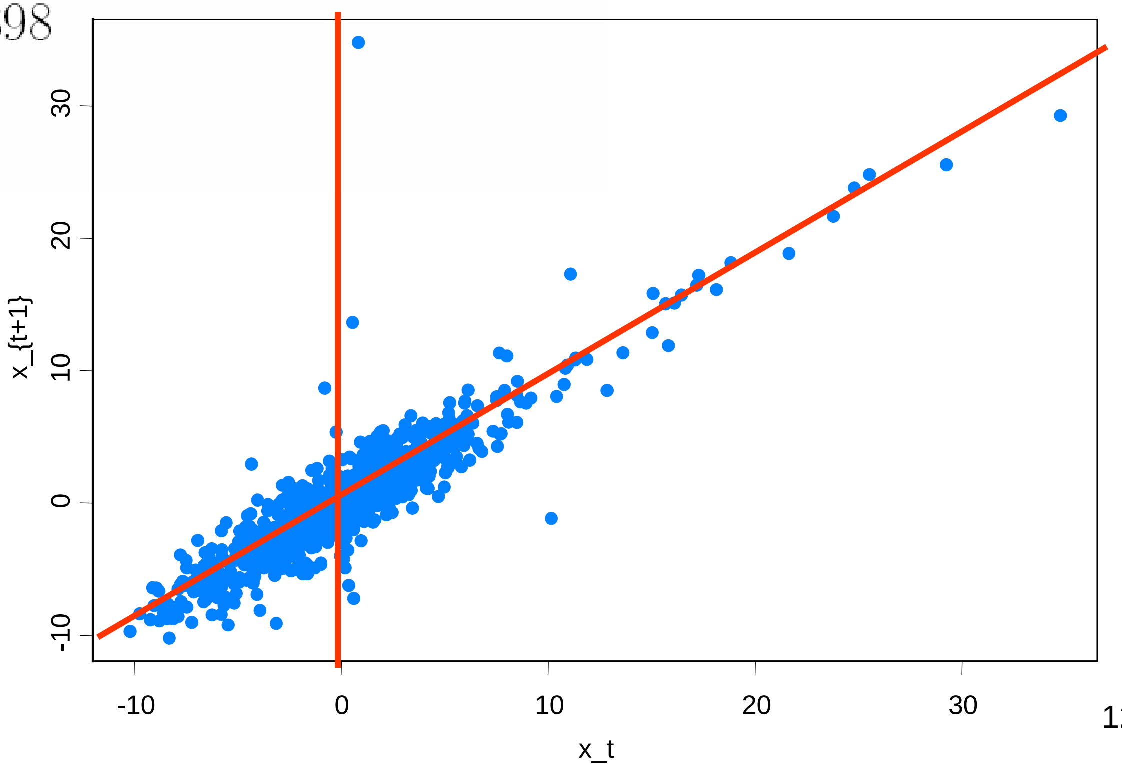
2. If $Z_1 = Z_2 > 0$ and RV(α), then $\mathbf{Z} = (Z_1, Z_2)$ is regularly varying with index α and spectral distribution

$$P(\boldsymbol{\theta} = (1/\sqrt{2}, 1/\sqrt{2})) = 1.$$

3. AR(1): $Z_t = .9Z_{t-1} + \epsilon_t$, ϵ_t ~IID symmetric stable (1.8). Then $\mathbf{Z} = (Z_1, Z_2)$ is RV(1.8) with spectral measure

$$\begin{cases} P(\boldsymbol{\theta} = (1, .9)/\sqrt{1.81}) = .9898 \\ P(\boldsymbol{\theta} = (0, 1)) = .0102 \end{cases}$$

Figure: plot of (Z_t, Z_{t+1}) for realization of 10,000.



Regular Variation on $\overline{\mathbb{D}}([0,1]^d)$

Polar coordinate transformation: For the càdlàg field $x \in \mathbb{D}\backslash\{0\}$

$$x \Leftrightarrow (\|x\|_\infty, \widetilde{x}) \ , \quad \widetilde{x} = x/\|x\|_\infty \, ,$$

where $\|x\|_\infty$ is the sup-norm of x, and 0 represents the zero function: We write

$$\overline{\mathbb{D}} = (0,\infty] \times \mathbb{S} \, , \text{where } \mathbb{S} = \{\widetilde{x} : x \in \mathbb{D}\backslash\{0\}\} \, .$$

Reg variation on $\overline{\mathbb{D}} = \overline{\mathbb{D}}([0,1]^d)$ (de Haan and Lin '01; Hult and Lindskog '05).

X is *regularly varying* with *spectral measure* σ on $\mathbb{S}$ and index $\alpha > 0$, if there exists $a_n > 0$ such that for all $t > 0$,

$$n\, P(\|X\|_\infty > t\, a_n \, , \widetilde{X} \in \cdot) \stackrel{w}{\rightarrow} t^{-\alpha}\, \sigma(\cdot) \, ,$$

where $\stackrel{w}{\rightarrow}$ denotes weak convergence on $\mathcal{B}(\mathbb{S})$. This convergence is equivalent to (Hult and Lindskog (2005))

$$n\, P(a_n^{-1}\, X \in \cdot) \stackrel{\widehat{w}}{\rightarrow} m(\cdot) \, .$$

Here $\xrightarrow{\hat{w}}$ denotes weak convergence of measures in the sense

$$m_n(f) = \int f dm_n \rightarrow \int f dm = m(f)$$

for all bounded continuous functions f on $\overline{\mathbb{D}}\backslash\{0\}$ which vanish outside a bounded set (see Appendix A2.6 in Daley and Vere-Jones (1988)), and m is a measure such that $\mu(\overline{\mathbb{D}}\backslash\mathbb{D}) = 0$;

Examples of Regular Variation on $\mathbb{D}([0,1]^d)$

1. sαs random field. Let $(\Gamma_i)_{i=1,2,\ldots}$ be the points of a unit rate Poisson process on $(0,\infty)$, (r_i) be an iid Rademacher sequence, (V_i) be iid $\mathbb{D}$-valued with $E(\|V_1\|^\alpha) < \infty$, all 3 sequences be independent. Then the infinite series

$$X = \sum_{i=1}^{\infty} r_i \, \Gamma_i^{-1/\alpha} \, V_i$$

for $\alpha \in (0,2)$ represents a sαs random field with spectral measure

$$\frac{E\left(\|V_1\|_\infty^\alpha I_S(\widetilde{V}_1)\right)}{E(\|V_1\|_\infty^\alpha)}, \quad S \in \mathcal{B}(\mathbb{S}) \, .$$

Regular variation is only determined by the first term in the series representation.

Examples of regular variation on $\mathbb{D}([0,1]^d)$

2. Max-stable random field. Let $(\Gamma_i)_{i=1,2,\dots}$ be the points of a unit rate Poisson process on $(0,\infty)$, independent of the iid $\mathbb{D}$-valued random fields Y_i with $E\|Y\|_\infty < \infty$. Then

$$X = \sup_{j\geq 1} \Gamma_j^{-1} Y_j$$

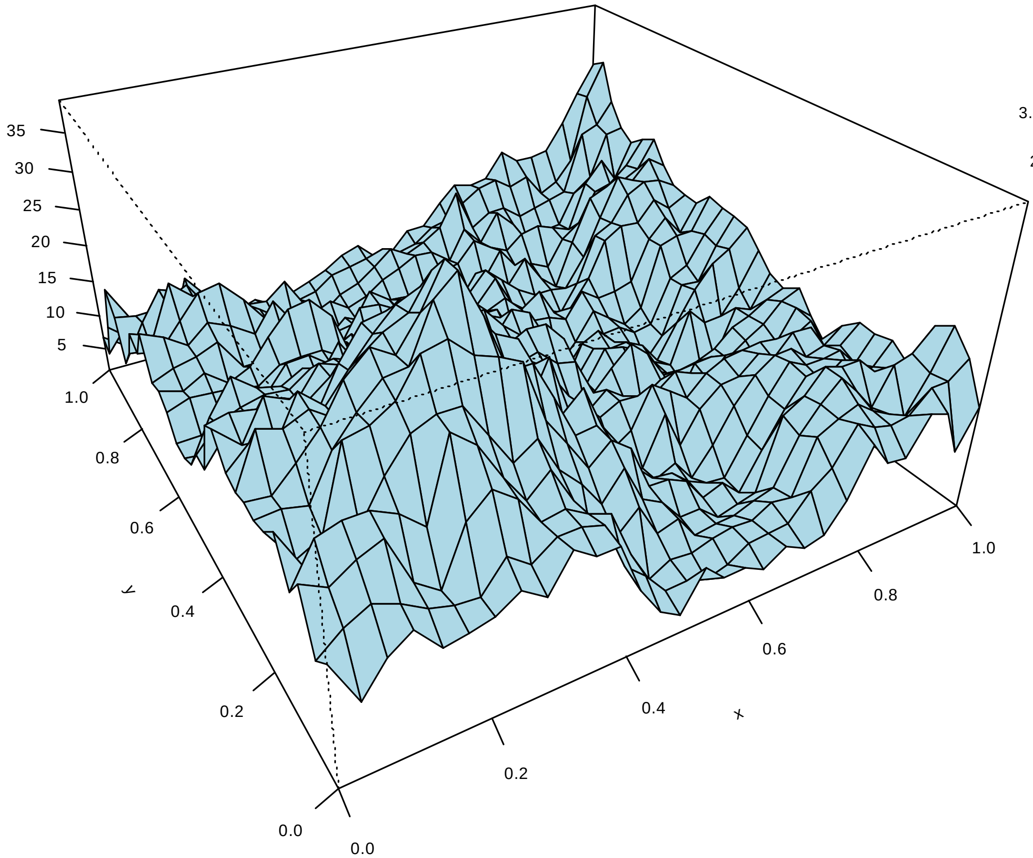
is a max-stable field (with unit Fréchet marginals) in the sense that for iid copies X_i of X, any $k \geq 1$,

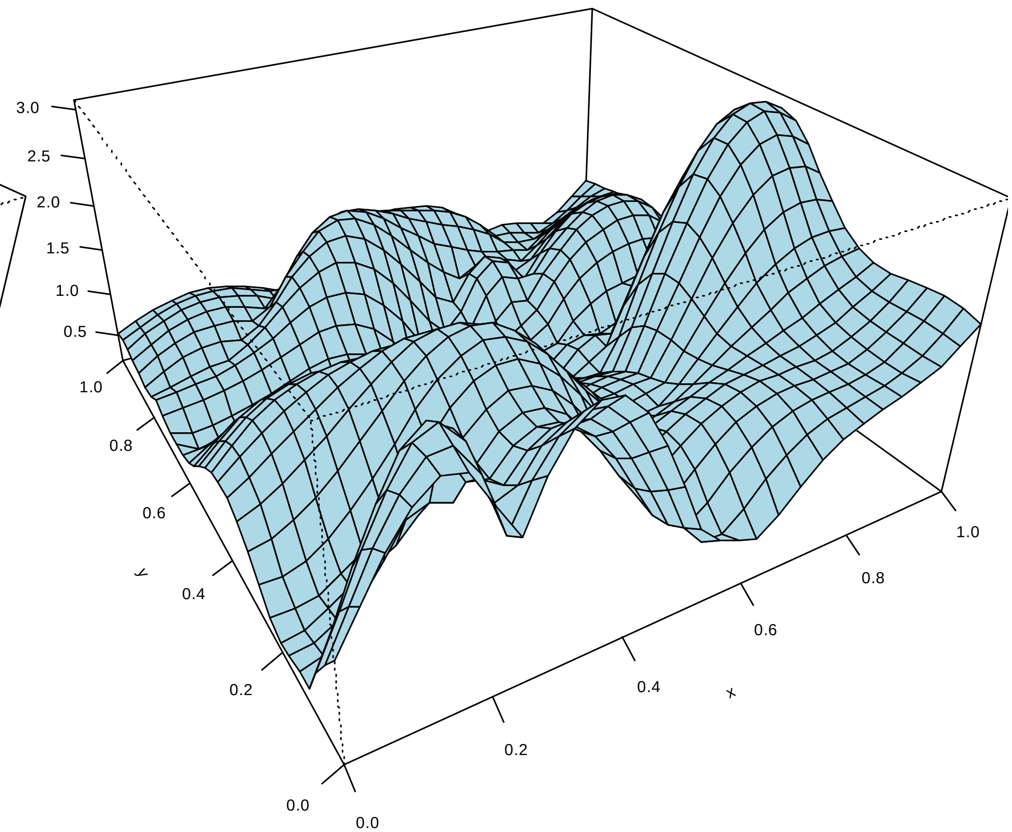
$$k\, X \stackrel{d}{=} \max_{i=1,\dots,k} X_i$$

in the sense of equality of the finite-dimensional distributions. See also Schlather (2002). Every max-stable $\mathbb{D}$-valued random field X has the representation above. The spectral measure of X is given by

$$E(\|Y_1\|_\infty I_S(\widetilde{Y}_1))/E\|Y_1\|_\infty\,, \quad S \in \mathcal{B}(\mathbb{S})\,.$$

Regular variation of X is determined by $\Gamma_1^{-1} Y_1$.





Max-stable random fields with Gaussian Y

Characterization of Regular Variation on $\mathbb{D}$

Proposition 1. (Hult and Lindskog (2005)) Z is regularly varying if and only if there exist $a_n > 0$ and a collection of Radon measures $m_{\mathbf{s}_1,\ldots,\mathbf{s}_k}$, $\mathbf{s}_i \in [0,1]^d$, not all of them being the null measure, with $m_{\mathbf{s}_1,\ldots,\mathbf{s}_k}(\overline{\mathbb{R}}^k \backslash \mathbb{R}^k) = 0$, such that the following conditions hold:

1) Finite-dimensional convergence:

$$n\, P(a_n^{-1}(Z(\mathbf{s}_1), \ldots, Z(\mathbf{s}_k)) \in \cdot) \xrightarrow{v} m_{\mathbf{s}_1,\ldots,\mathbf{s}_k}(\cdot)\,.$$

2) Tightness. For any $\epsilon, \eta > 0$ there exist $\delta \in (0, 0.5)$ and n_0 such that for $n \geq n_0$,

$$n\, P(w''(Z,\delta) > a_n\epsilon) \;\leq\; \eta\,,$$

$$n\, P(w(Z, [0,1]^d \backslash [\delta, 1-\delta]^d) > a_n\epsilon) \;\leq\; \eta\,.$$

Note. The measures $m_{\mathbf{s}_1,\ldots,\mathbf{s}_k}$, $\mathbf{s}_i \in [0,1]^d$, determine the limiting measure m in the definition of regular variation of Z.

Characterization of Regular Variation on $\mathbb{D}$

- In general, tightness in the regular variation sense (property 2) is not equivalent to tightness in $\mathbb{D}$ of the sequence

$$a_n^{-1} \max_{t=1,\dots,n} X_t \,. \tag{0.1}$$

  for an iid $\mathbb{D}$-valued sequence with regularly varying finite-dimensional distributions.
- There exists a regularly varying field X which is regularly varying in $\mathbb{D}$, for which (0.1) holds in the sense of finite-dimensional distributions but not in $\mathbb{D}$.

Application to Space-Time Processes

Proposition 2. Assume that (Z_t) is an iid sequence of random fields on $\mathbb{D}$ such that Z is regularly varying with index α and limiting measure m_Z. Suppose (ψ_i) is a sequence of càdlàg fields with

$$\sum_{i=0}^{\infty} \|\psi_i\|_{\infty}^{\min(1,\alpha-\epsilon)} < \infty$$

for some $\epsilon \in (0, \alpha)$. Then the infinite series

$$X = \sum_{i=0}^{\infty} \psi_i \, Z_i$$

converges a.s. in $\mathbb{D}$ and is regularly varying with index α and limiting measure

$$m = \sum_{i=0}^{\infty} m_Z \circ \psi_i^{-1} \, .$$

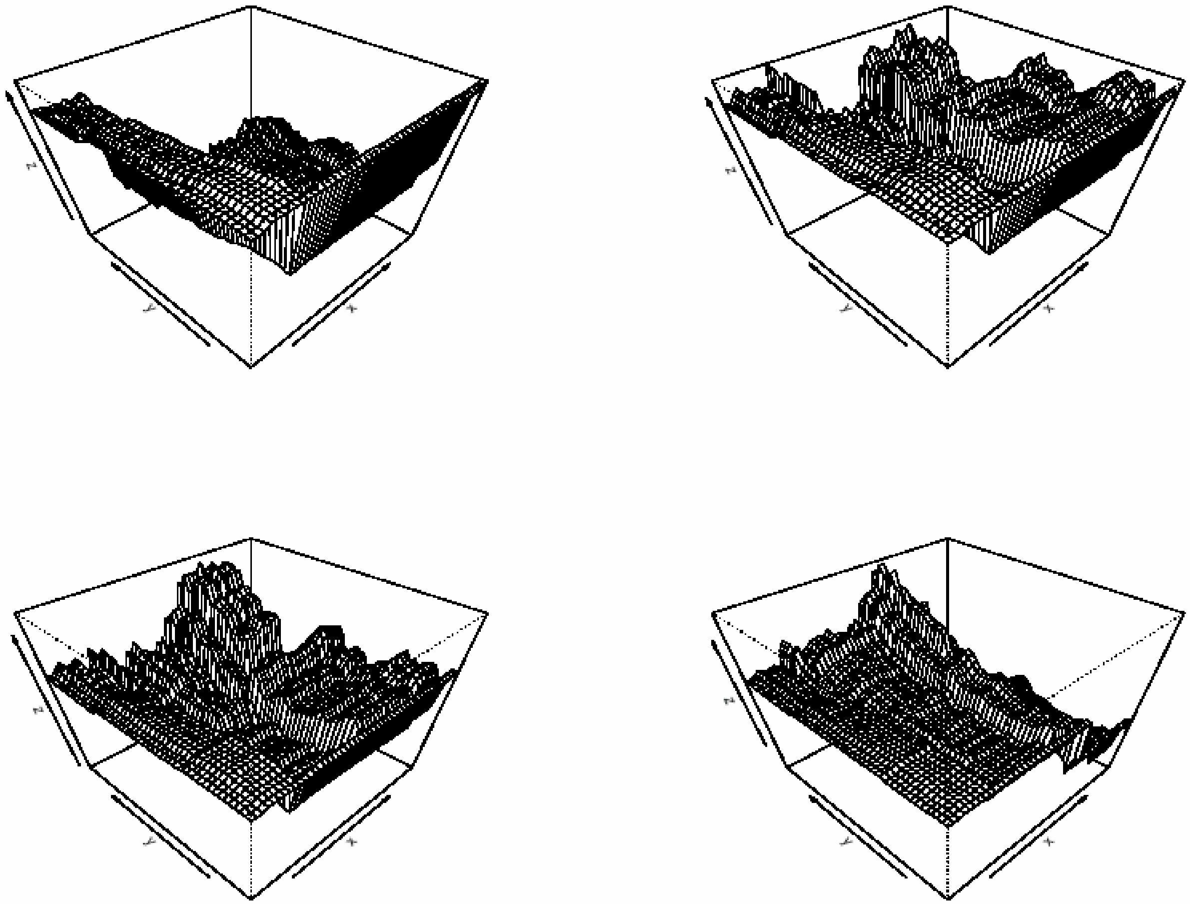


FIGURE 3. The autoregressive field $X_t = 0.9X_{t-1} + Z_t$ for $t = 0, 1, 2, 3$. The process Z is symmetric 1-stable Lévy motion.

Application to Space-Time Processes

Main ideas behind proof:

- Show convergence by bounding the sup norm and using the fact that $\|Z_i\|_\infty$ is regularly varying.
- First establish regular variation for finite sums by checking conditions (fidi convergence and tightness) of Proposition 1.
- Extend to infinite sums by approximating the tail sums.

Point Process Convergence

Point process convergence for the Z_t's. From Proposition 1, it follows that

$$I_n = \sum_{t=1}^{n} \varepsilon_{a_n^{-1} Z_t} \xrightarrow{d} I = \sum_{j=1}^{\infty} \varepsilon_{P_j} \,.$$

where $\xrightarrow{d}$ denotes convergence in distribution of point processes on the space $\widehat{M}(\overline{\mathbb{D}} \backslash \{\mathbf{0}\})$ and $\sum_{j=1}^{\infty} \varepsilon_{P_j}$ is a Poisson random measure on $\overline{\mathbb{D}} \backslash \{0\}$ with intensity measure m_Z.

Note: The space $\widehat{M}(\overline{\mathbb{D}}^m \backslash \{\mathbf{0}\})$ is the space of point measures on $\overline{\mathbb{D}}^m \backslash \{\mathbf{0}\}$ endowed with the topology generated by $\widehat{w}$-convergence.

Theorem.

$$N_n = \sum_{t=1}^{n} \varepsilon_{a_n^{-1} X_t} \xrightarrow{d} N = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \varepsilon_{\psi_i P_j} \,.$$

Point Process Convergence

Remark: This theorem generalizes the Davis and Resnick (1985) point process convergence result for linear processes.

Application

From the Theorem, we have

$$P(a_n^{-1} \max_{t=1,\ldots,n} \|X_t\|_\infty \le x) \to P(\sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \varepsilon_{\|\psi_i P_j\|_\infty}(x,\infty) = 0)$$
$$= \exp\{-m_Z(B)\},$$

where

$$B = \{y : \|\psi_i y\|_\infty > x, \text{ for some } i = 0, 1, \ldots\}.$$

If the ψ_i's are constant functions, then

$$B = \{y : \|y\|_\infty > x/\psi_+\}$$

and

$$\exp\{-m_Z(B)\} = \exp\{-x^{-\alpha}\psi_+^\alpha\},$$

where $\psi_+ = \max_j |\psi_j|$.

Extremal index $= \psi_+^\alpha / \sum_{i=0}^\infty |\psi_i|^\alpha$.