Extremes of Space-Time Processes
With Heavy-Tailed Distributions

Richard A. Davis
Colorado State University
www.stat.colostate.edu/~rdavis

Thomas Mikosch
University of Copenhagen
Outline

• A Class of Space-Time Processes: \( X_t(s) = \sum_{i=0}^{\infty} \psi_i(s) Z_{t-i}(s) \), \( s \in [0, 1]^d \)
  – Dependence properties

• Preliminaries on Regular Variation on \( \mathbb{D}([0, 1]^d) \)
  – Examples

• Point Process Convergence
  – Basic properties

• Application
EVT for Space-Time Processes

Basic set-up: 2 components, spatial and temporal.

Spatial part. Let $Z(s)$ be a random field on $[0, 1]^d$.

- Usually $d = 1$ (transect) or $d = 2$ (two-dimensional space).
- $Z(s)$ is value of the random field at location $s \in [0, 1]^d$.
- View $Z(s)$ as a random element of $\mathbb{D} = \mathbb{D}([0, 1]^d)$ of càdlàg functions $J_1$-topology; see Bickel and Wichura (1971).
- Will assume that $Z$ has regularly varying tail probabilities – to be described later.
EVT for Space-Time Processes

Temporal part. Build in serial dependence by filtering the random field at each location \( s \in [0, 1]^d \). That is, set

\[
X_t(s) = \sum_{i=0}^{\infty} \psi_i(s) Z_{t-i}(s), \quad s \in [0, 1]^d,
\]

where

- \((Z_t)_{t \in \mathbb{Z}}\) are iid copies of the random field \( Z \) on \([0, 1]^d\)
- \( \psi_i \)'s are deterministic \( càdlàg \) real-valued fields on \([0, 1]^d\).

**Note:** For \( s_1, \ldots, s_k \) fixed,

\[
X_t := \begin{bmatrix}
X_t(s_1) \\
\vdots \\
X_t(s_k)
\end{bmatrix} = \begin{bmatrix}
\sum_{i=0}^{\infty} \psi_i(s_1) Z_{t-i}(s_1) \\
\vdots \\
\sum_{i=0}^{\infty} \psi_i(s_k) Z_{t-i}(s_k)
\end{bmatrix} = \sum_{i=0}^{\infty} A_i Z_{t-i}
\]

is a multivariate linear time series.
Dependence structure of \((X_t)\)

Suppose the random field \(Z(s)\) is stationary with covariance function \(\gamma_Z(u)\),

\[
\text{Cov}(Z(s + u), Z(s)) = \gamma_Z(u).
\]

Spatial covariance of \(X_t\).

\[
\text{Cov}(X_t(s + u), X_t(s)) = \left( \sum_{j=0}^{\infty} \psi_j(s + u)\psi_j(s) \right) \gamma_Z(u),
\]

which is stationary in space (independent of \(s\)) if the \(\psi_j\)'s are constant functions. In this case,

\[
\text{Cov}(X_t(s + u), X_t(s)) = \left( \sum_{j=0}^{\infty} \psi_j^2 \right) \gamma_Z(u).
\]
Dependence structure of \((X_t)\)

**Time covariance function of** \(X_t(s)\). For each \(s \in [0, 1]^d\), the time series \(X_t(s)\) is a **linear process** with covariance function

\[
\text{Cov}(X_{t+h}(s), X_t(s)) = \left( \sum_{j=0}^{\infty} \psi_{j+h}(s) \psi_j(s) \right) \gamma_Z(0)
\]

If the \(\psi_j\)'s are constant functions, then the serial correlation does not depend on \(s\).

**Note:** In fact, the time series \(X_t\) defined on \(\mathbb{D}([0, 1]^d)\) is strictly stationary.

**Space-time covariance function of** \(X_t(s)\).

\[
\text{Cov}(X_{t+h}(s + u), X_t(s)) = \left( \sum_{j=0}^{\infty} \psi_{j+h}(s + u) \psi_j(s) \right) \gamma_Z(u)
\]
which, if the $\psi_j$’s are constant functions, is equal to

$$
\gamma_X(h, u) = \text{Cov}(X_{t+h}(s + u), X_t(s))
$$

$$
= \left( \sum_{j=0}^{\infty} \psi_j \psi_j \right) \gamma_Z(u)
$$

$$
= \gamma_T(h) \gamma_Z(u)
$$

Remarks:

(1) The filter functions $\psi_j$ influence both the spatial and temporal covariances.

(2) If the $\psi_j$’s are constant functions, then $X_t$ has a multiplicative covariance function, i.e.,

$$
\gamma_X(h, u) = \text{Cov}(X_{t+h}(s + u), X_t(s))
$$

$$
= \gamma_T(h) \gamma_Z(u)
$$
Examples and Applications

1. Maximum ozone levels. Suppose there exists a standard $L$ for annual maxima of ozone levels over the rectangular region $[0, 1]^2$. Set
\[ X_t(s) = \text{maximum ozone level at site } s \text{ during year } t. \]
Then the probability the standard $L$ is not exceeded in $n$ consecutive years is
\[ P\left( \max_{t=1,\ldots,n} X_t(s) \leq L, \text{ for all } s \in [0, 1]^2 \right). \]

2. Sea level (de Haan and Lin (2001)). Let $f(s)$ represent the height of a dyke off the Dutch coast at location $s$ and set
\[ X_t(s) = \text{maximum sea level at site } s \text{ during day } t \]
The probability that the dyke is not breached along the coast for $n$ consecutive days is
\[ P\left( \max_{t=1,\ldots,n} X_t(s) \leq f(s), \text{ for all } s \in [0, 1] \right). \]
Regular Variation on $\mathbb{D}([0,1]^d) —$ Preliminaries:

Regular variation of $Z = (Z_1, \ldots, Z_m)'$. There exists a random vector $\theta$ defined on $\mathbb{S}^{m-1}$ such that for all $z > 0$

$$P(\|Z\| > t \cdot z, Z/\|Z\| \in \cdot)/P(\|Z\| > t) \xrightarrow{w} z^{-\alpha} P(\theta \in \cdot),$$

as $t \to \infty$ where $\xrightarrow{w}$ is weak convergence on $\mathbb{S}^{m-1}$, the unit sphere in $\mathbb{R}^m$.

- $P(\theta \in \cdot)$ is called the spectral measure.
- $\alpha$ is the index of regular variation.

Equivalence: There exists $a_n > 0$ such that for all $z > 0$

$$nP(\|Z\| > a_n \cdot z, Z/\|Z\| \in \cdot) \xrightarrow{w} z^{-\alpha} P(\theta \in \cdot)$$

or, equivalently,

$$nP(a_n^{-1}Z \in \cdot) \xrightarrow{w} m(\cdot)$$

for some Radon measure $m$ on $\mathcal{B}(\overline{\mathbb{R}}^m \setminus \{0\})$. 
Examples of Regular Variation on $\mathbb{R}^2$:

1. If $Z_1 > 0$ and $Z_2 > 0$ are iid $RV(\alpha)$, then $Z = (Z_1, Z_2)$ is regularly varying with index $\alpha$ and spectral distribution

$$P(\theta = (0, 1)) = P(\theta = (1, 0)) = .5 \text{ (mass on axes).}$$

Interpretation: Unlikely that $Z_1$ and $Z_2$ are both large at the same time.

Figure: plot of $(Z_{t_1}, Z_{t_2})$ for realization of 10,000.
Examples of Regular Variation on $\mathbb{R}^2$:

2. If $Z_1 = Z_2 > 0$ and $\text{RV}(\alpha)$, then $Z = (Z_1, Z_2)$ is regularly varying with index $\alpha$ and spectral distribution

$$P(\theta = (1/\sqrt{2}, 1/\sqrt{2})) = 1.$$ 

3. AR(1): $Z_t = 0.9Z_{t-1} + \epsilon_t$, $\epsilon_t \sim \text{IID symmetric stable (1.8)}$. Then $Z = (Z_1, Z_2)$ is $\text{RV}(1.8)$ with spectral measure

$$\begin{cases} 
P(\theta = (1, 0.9)/\sqrt{1.81}) = 0.9898 \\
P(\theta = (0, 1)) = 0.0102 
\end{cases}$$

Figure: plot of $(Z_t, Z_{t+1})$ for realization of 10,000.
Regular Variation on $\overline{\mathbb{D}}([0, 1]^d)$

**Polar coordinate transformation:** For the càdlàg field $x \in \mathbb{D}\setminus\{0\}$

$$x \leftrightarrow (\|x\|_\infty, \tilde{x}) , \quad \tilde{x} = x/\|x\|_\infty ,$$

where $\|x\|_\infty$ is the sup-norm of $x$, and 0 represents the zero function: We write

$$\overline{\mathbb{D}} = (0, \infty] \times \mathbb{S} , \text{ where } \mathbb{S} = \{\tilde{x} : x \in \mathbb{D}\setminus\{0\}\} .$$

**Reg variation on** $\overline{\mathbb{D}} = \overline{\mathbb{D}}([0, 1]^d)$ (de Haan and Lin ‘01; Hult and Lindskog ‘05).

$X$ is regularly varying with spectral measure $\sigma$ on $\mathbb{S}$ and index $\alpha > 0$, if there exists $a_n > 0$ such that for all $t > 0$,

$$n \, P(\|X\|_\infty > t \, a_n , \tilde{X} \in \cdot) \xrightarrow{w} t^{-\alpha} \sigma(\cdot) ,$$

where $\xrightarrow{w}$ denotes weak convergence on $\mathcal{B}(\mathbb{S})$. This convergence is equivalent to (Hult and Lindskog (2005))

$$n \, P(a_n^{-1} \, X \in \cdot) \xrightarrow{w} m(\cdot) .$$
Here $\hat{\nu}$ denotes weak convergence of measures in the sense

$$m_n(f) = \int f \, dm_n \rightarrow \int f \, dm = m(f)$$

for all bounded continuous functions $f$ on $\mathbb{D} \setminus \{0\}$ which vanish outside a bounded set (see Appendix A2.6 in Daley and Vere-Jones (1988)), and $m$ is a measure such that $\mu(\mathbb{D} \setminus \mathbb{D}) = 0$;
Examples of Regular Variation on $\mathbb{D}([0, 1]^d)$

1. sas random field. Let $(\Gamma_i)_{i=1,2,...}$ be the points of a unit rate Poisson process on $(0, \infty)$, $(r_i)$ be an iid Rademacher sequence, $(V_i)$ be iid $\mathbb{D}$-valued with $E(\|V_1\|^{\alpha}) < \infty$, all 3 sequences be independent. Then the infinite series

$$X = \sum_{i=1}^{\infty} r_i \Gamma_i^{-1/\alpha} V_i$$

for $\alpha \in (0, 2)$ represents a sas random field with spectral measure

$$\frac{E(\|V_1\|^{\alpha} I_S(\tilde{V}_1))}{E(\|V_1\|^{\alpha}_{\infty})}, \quad S \in \mathcal{B}(\mathbb{S}).$$

Regular variation is only determined by the first term in the series representation.
Examples of regular variation on $\mathbb{D}([0, 1]^d)$

2. Max-stable random field. Let $(\Gamma_i)_{i=1,2,...}$ be the points of a unit rate Poisson process on $(0, \infty)$, independent of the iid $\mathbb{D}$-valued random fields $Y_i$ with $E\|Y\|_\infty < \infty$. Then

$$X = \sup_{j \geq 1} \Gamma_j^{-1} Y_j$$

is a max-stable field (with unit Fréchet marginals) in the sense that for iid copies $X_i$ of $X$, any $k \geq 1$,

$$k \overset{d}{=} \max_{i=1,...,k} X_i$$

in the sense of equality of the finite-dimensional distributions. See also Schlather (2002). Every max-stable $\mathbb{D}$-valued random field $X$ has the representation above. The spectral measure of $X$ is given by

$$E(\|Y_1\|_\infty I_S(\tilde{Y}_1))/E\|Y_1\|_\infty, \quad S \in \mathcal{B}(\mathbb{S}).$$

Regular variation of $X$ is determined by $\Gamma_1^{-1} Y_1$. 
Max-stable random fields with Gaussian $Y$
Characterization of Regular Variation on $\mathbb{D}$

**Proposition 1.** (Hult and Lindskog (2005)) $Z$ is regularly varying if and only if there exist $a_n > 0$ and a collection of Radon measures $m_{s_1,\ldots,s_k}$, $s_i \in [0, 1]^d$, not all of them being the null measure, with $m_{s_1,\ldots,s_k}(\mathbb{R}^k \setminus \mathbb{R}^k) = 0$, such that the following conditions hold:

1) Finite-dimensional convergence:

$$n \ P(a_n^{-1}(Z(s_1), \ldots, Z(s_k)) \in \cdot) \xrightarrow{v} m_{s_1,\ldots,s_k}(\cdot).$$

2) Tightness. For any $\epsilon, \eta > 0$ there exist $\delta \in (0, 0.5)$ and $n_0$ such that for $n \geq n_0$,

$$n \ P(w''(Z, \delta) > a_n \epsilon) \leq \eta,$$

$$n \ P(w(Z, [0, 1]^d \setminus [\delta, 1 - \delta]^d) > a_n \epsilon) \leq \eta.$$ 

Note. The measures $m_{s_1,\ldots,s_k}$, $s_i \in [0, 1]^d$, determine the limiting measure $m$ in the definition of regular variation of $Z$. 

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Characterization of Regular Variation on $\mathbb{D}$

- In general, tightness in the regular variation sense (property 2) is not equivalent to tightness in $\mathbb{D}$ of the sequence

\[
\alpha_n^{-1} \max_{t=1,\ldots,n} X_t.
\]

for an iid $\mathbb{D}$-valued sequence with regularly varying finite-dimensional distributions.

- There exists a regularly varying field $X$ which is regularly varying in $\mathbb{D}$, for which (0.1) holds in the sense of finite-dimensional distributions but not in $\mathbb{D}$. 
Application to Space-Time Processes

Proposition 2. Assume that \((Z_t)\) is an iid sequence of random fields on \(\mathbb{D}\) such that \(Z\) is regularly varying with index \(\alpha\) and limiting measure \(m_Z\). Suppose \((\psi_i)\) is a sequence of càdlàg fields with

\[
\sum_{i=0}^{\infty} \|\psi_i\|_\infty^{\min(1,\alpha-\epsilon)} < \infty
\]

for some \(\epsilon \in (0, \alpha)\). Then the infinite series

\[
X = \sum_{i=0}^{\infty} \psi_i Z_i
\]

converges a.s. in \(\mathbb{D}\) and is regularly varying with index \(\alpha\) and limiting measure

\[
m = \sum_{i=0}^{\infty} m_Z \circ \psi_i^{-1}.
\]
Figure 3. The autoregressive field $X_t = 0.9X_{t-1} + Z_t$ for $t = 0, 1, 2, 3$. The process $Z$ is symmetric 1-stable Lévy motion.
Application to Space-Time Processes

Main ideas behind proof:

- Show convergence by bounding the sup norm and using the fact that $\|Z_i\|_\infty$ is regularly varying.
- First establish regular variation for finite sums by checking conditions (fidi convergence and tightness) of Proposition 1.
- Extend to infinite sums by approximating the tail sums.
Point Process Convergence

Point process convergence for the $Z_t$'s. From Proposition 1, it follows that

$$I_n = \sum_{t=1}^{n} \varepsilon_{a_n^{-1}Z_t} \xrightarrow{d} I = \sum_{j=1}^{\infty} \varepsilon_{P_j}.$$ 

where $\xrightarrow{d}$ denotes convergence in distribution of point processes on the space $\widetilde{M}(\overline{D}\setminus\{0\})$ and $\sum_{j=1}^{\infty} \varepsilon_{P_j}$ is a Poisson random measure on $\overline{D}\setminus\{0\}$ with intensity measure $m_Z$.

Note: The space $\widetilde{M}(\overline{D}^m\setminus\{0\})$ is the space of point measures on $\overline{D}^m\setminus\{0\}$ endowed with the topology generated by $\hat{w}$-convergence.

Theorem.

$$N_n = \sum_{t=1}^{n} \varepsilon_{a_n^{-1}X_t} \xrightarrow{d} N = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \varepsilon_{\psi_i P_j}.$$
Point Process Convergence

**Remark:** This theorem generalizes the Davis and Resnick (1985) point process convergence result for linear processes.
Application

From the Theorem, we have

\[ P(a_n^{-1} \max_{t=1, \ldots, n} \| X_t \|_\infty \leq x) \rightarrow P(\sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \varepsilon \| \psi_i P_j \|_\infty(x, \infty) = 0) \]

\[ = \exp\{-m_Z(B)\}, \]

where

\[ B = \{ y : \| \psi_i y \|_\infty > x, \text{ for some } i = 0, 1, \ldots \}. \]

If the \( \psi_i \)'s are constant functions, then

\[ B = \{ y : \| y \|_\infty > x / \psi_+ \} \]

and

\[ \exp\{-m_Z(B)\} = \exp\{-x^{-\alpha} \psi_+^\alpha\}, \]

where \( \psi_+ = \max_j |\psi_j| \).

Extremal index = \( \psi_+^{\alpha} / \sum_{i=0}^{\infty} |\psi_i|^{\alpha} \).