Model Selection for Geostatistical Models

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Model Selection for Geostatistical Models

$$Z(s) = \beta_0 + X_1(s)\beta_1 + \cdots + X_p(s)\beta_p + \delta(s)$$

- Which explanatory variables should be included?
- What is the form of $\delta(s)$?

Model Selection for Geostatistical Models

$$Z(s) = \underbrace{\beta_0 + X_1(s)\beta_1 + \cdots + X_p(s)\beta_p}_{\text{deterministic}} + \underbrace{\delta(s)}_{\text{stochastic}}$$

- Which explanatory variables should be included?
- What is the form of $\delta(s)$?

Model Selection for Geostatistical Models

Problem: How does one choose the "best" set of covariates and family of covariance functions?

Potential Objectives of Model Selection

1. Choose the correct model (consistency)
   - There exists a "true" finite-dimensional model.
   - If not a finite-dimensional model, at least include the key explanatory variables.
2. Choose the model that is best for prediction (efficiency)
   - Find a model that predicts well at un-observed locations.
3. Choose the model that maximizes data compression.
   - Find a model that summarizes the data in the most compact fashion.

The Geostatistical Model

Let $\mathbf{Z} = (Z(s_1), \ldots, Z(s_n))'$ be a partial realization of a random field $\mathbf{Z}(s)$, where $s \in D$, a fixed finite area under study.

A model for the random field at any location s is given by

$$Z(s) = \mathbf{X}'(s)\boldsymbol{\beta} + \delta(s),$$

where

- $\mathbf{X}(s) = (1, X_1(s), \ldots, X_p(s))'$ is a vector of explanatory variables observed at location s,
- $\boldsymbol{\beta}$ is a $p+1$ vector of unknown coefficients
- We assume that the error process $\delta(s)$ is a stationary, isotropic Gaussian process with mean zero and covariance function $\mathrm{Cov}(\delta(s), \delta(t)) = \sigma^2 \rho(||s-t||, \boldsymbol{\theta})$, where σ^2 is the variance of the process, $\rho(\cdot, \boldsymbol{\theta})$ is an isotropic correlation function, and $||\cdot||$ denotes Euclidean distance.

Autocorrelation Functions

Some of the standard autocorrelation functions:

1. **Exponential**

$$\rho(d) = \exp\left(\frac{-d}{\theta_1}\right)$$

2. **Gaussian**

$$\rho(d) = \exp\left(\frac{-d^2}{\theta_1^2}\right)$$

3. **Matern**

$$\rho(d) = \frac{1}{2^{\theta_2 - 1}\Gamma\left(\theta_2\right)}\left(\frac{2d\sqrt{\theta_2}}{\theta_1}\right)^{\theta_2} \mathcal{K}_{\theta_2}\left(\frac{2d\sqrt{\theta_2}}{\theta_1}\right), \qquad \theta_1 > 0,\ \theta_2 > 0,$$

where $\mathcal{K}_{\theta_2}(\cdot)$ is the modified Bessel function.

- Range parameter, θ_1, controls the rate of decay of the correlation between observations as distance increases.
- Smoothness parameter, θ_2, controls the smoothness of the random field.

AIC for Spatial Models

Background on AIC

Burnham and Anderson (1998), and McQuarrie and Tsai (1998)

Suppose

- $\boldsymbol{Z} \sim f_T$
- $\{f(\cdot;\psi),\ \psi \in \Psi\}$ is a family of candidate probability density functions

The Kullback-Leibler information between $f(\cdot;\psi)$ and f_T

$$I(\psi) = \int -2\log\left(\frac{f(\boldsymbol{z}\,|\psi)}{f_T(\boldsymbol{z})}\right) f_T(\boldsymbol{z})d\boldsymbol{z}\,.$$

- distance between $f(\cdot;\psi)$ and f_T
- similar to the notion of relative entropy
- loss of information when $f(\cdot;\psi)$ is used instead of f_T.

AIC for Spatial Models

By Jensen's inequality,

$$I(\psi) \;\geq\; 0 \quad \text{if and only if} \quad f(\boldsymbol{z};\psi) = f_T(\boldsymbol{z}) \quad \text{a.e. } [f_T]$$

Basic idea: minimize the Kullback-Leibler index

$$\begin{aligned}\Delta(\psi) \;&=\; \int -2\log\left(f\left(\boldsymbol{z}\,|\psi\right)\right) f_T(\boldsymbol{z}) d\boldsymbol{z} \\ &=\; E_T(-2\log L_Z(\psi))\,,\end{aligned}$$

where $L_Z(\psi)$ is the likelihood based on the data $\boldsymbol{Z}$.

Model Selection and Spatial Correlation

Traditional approach to model selection:

1. Select explanatory variables to model the large scale variation.
2. Estimate parameters using residuals from model in step 1.
3. Iterate.

Limitations:

- Ignores potential confounding between explanatory variables and correlation in spatial process
- Ignoring autocorrelation function can mask importance of explanatory variables

Simulations: Compare model selection performance of AIC for independent error regression model and geostatistical model

Model Selection: Simulation Set-up

1. **Sampling Design:** 100 locations simulated in a random pattern.

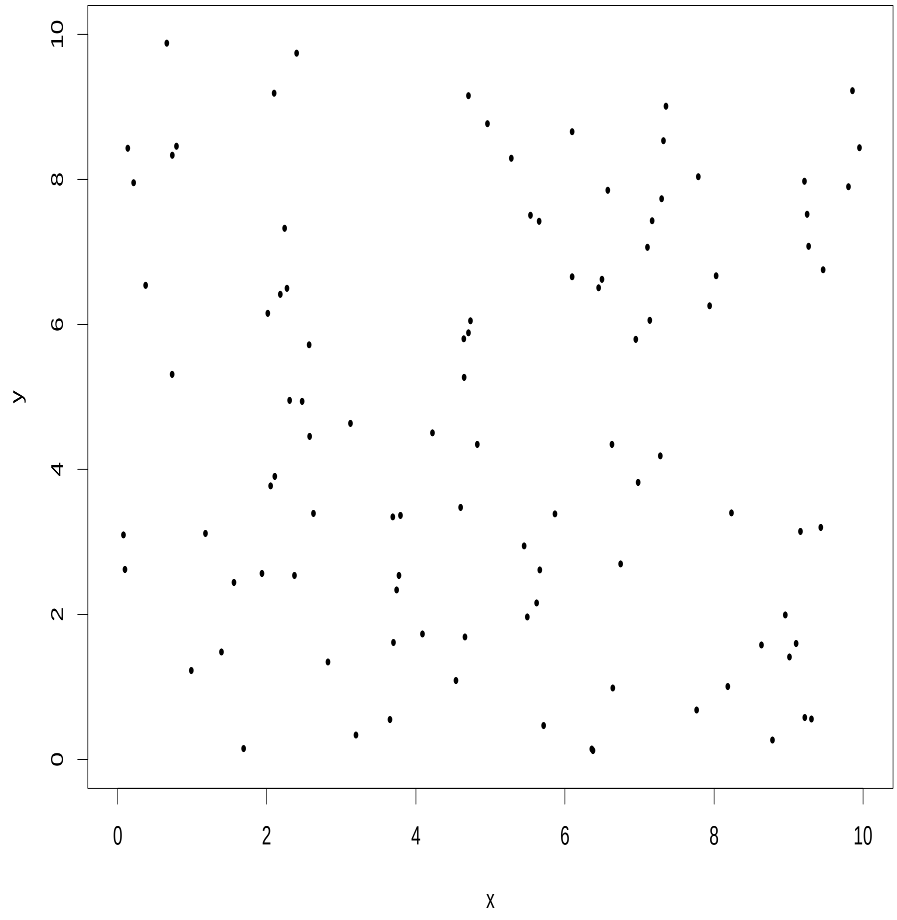
2. **Explanatory Variables:** Five possible explanatory variables:
   $X_1, X_2, X_3, X_4, X_5 \sim \sqrt{\frac{12}{10}}\, t_{12}$

3. **Response:**
   $$\boldsymbol{Z} = 2 + 0.75\boldsymbol{X}_1 + 0.50\boldsymbol{X}_2 + 0.25\boldsymbol{X}_3 + \boldsymbol{\delta},$$
   where $\boldsymbol{\delta}$ is a Gaussian random field with mean zero, $\sigma^2 = 50$, and autocorrelation Matern with parameters $\theta_1 = 4$ and $\theta_2 = 1$.

4. **Replicates:** 500 replicates were simulated with a new Gaussian random field generated for each replication.

5. **AIC:** Computed for $2^5 = 32$ possible models per replicate

Model Selection: Random Pattern Sampling Design

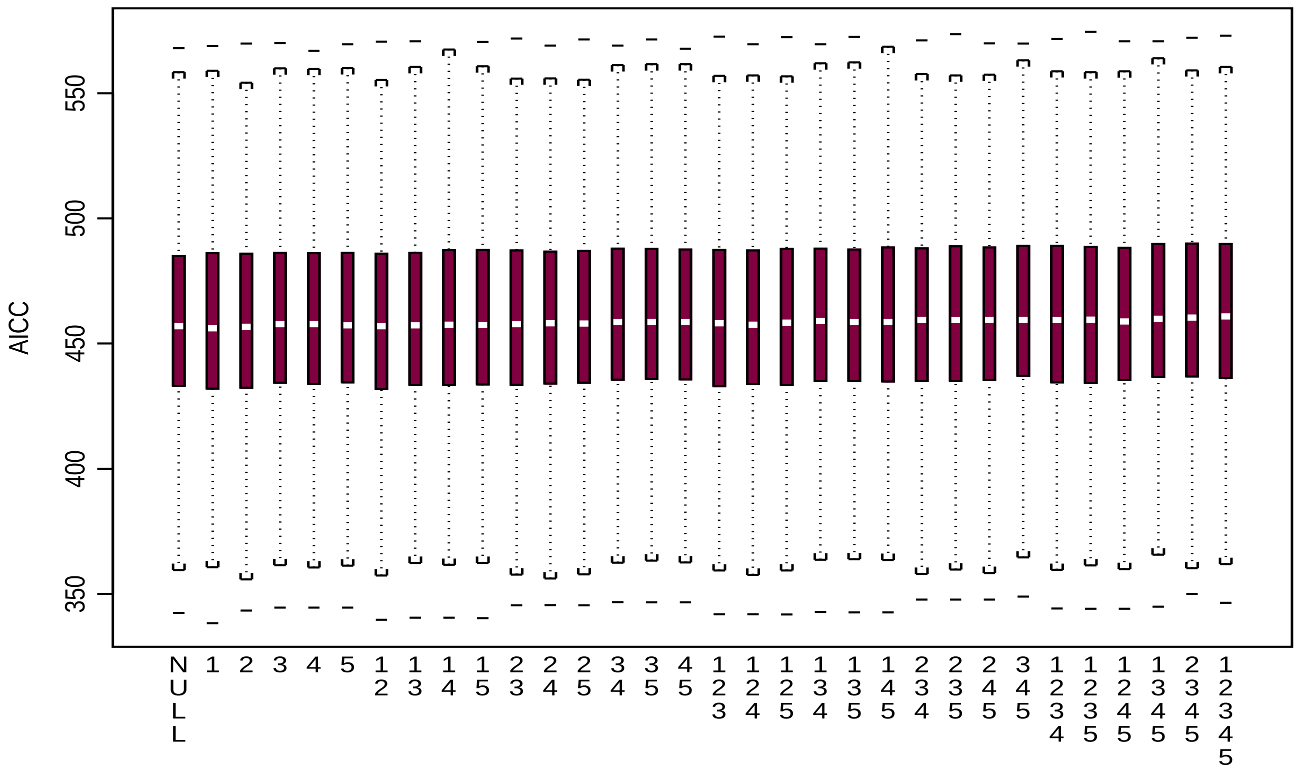


Model Selection: Simulation Results for the Random Pattern

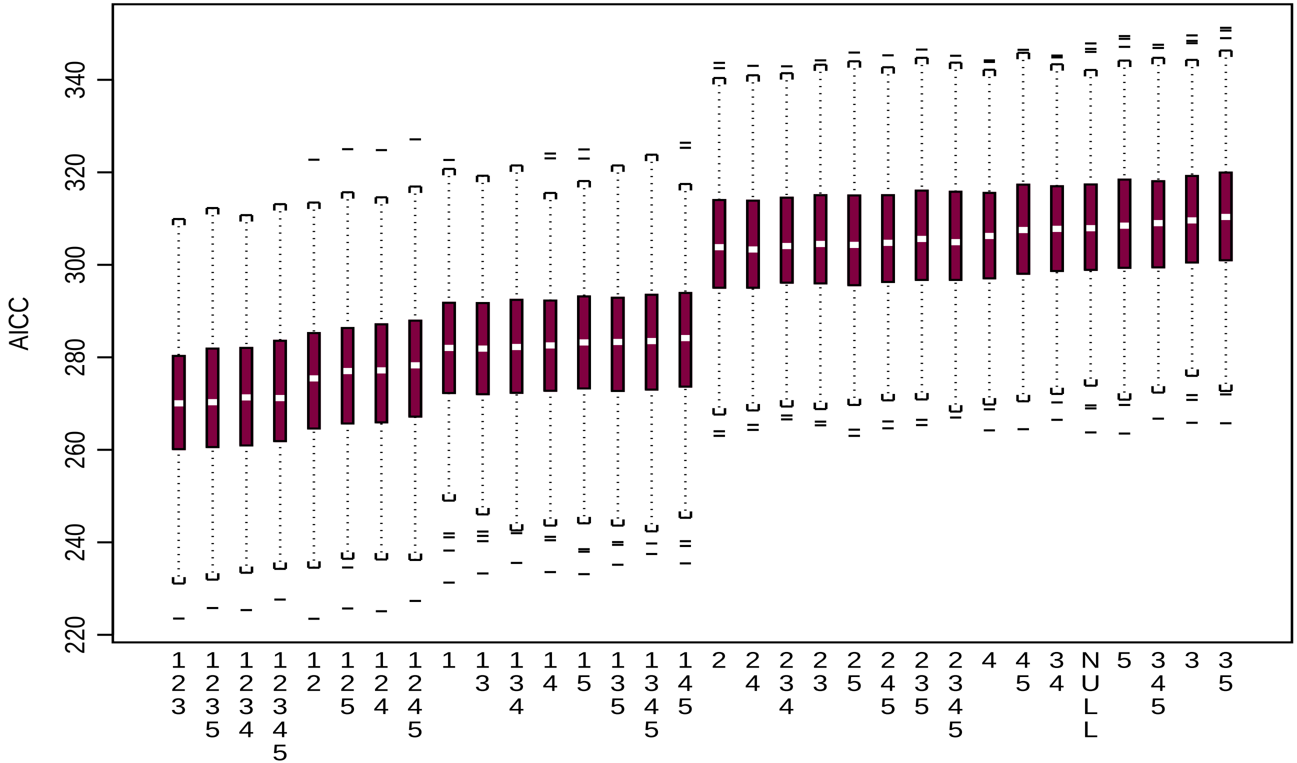
- Independent AIC and Spatial AIC report the percentage of simulations that each model was selected.
- Of the 32 possible models, the results given here include only those with 10% or more support for one of the models.

Variables in Model	Spatial AIC	Independent AIC
$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3$	46	0
$\boldsymbol{X}_1, \boldsymbol{X}_2$	18	6
$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_5$	11	0
Intercept only	0	37
$\boldsymbol{X}_1$	1	18
$\boldsymbol{X}_2$	0	12

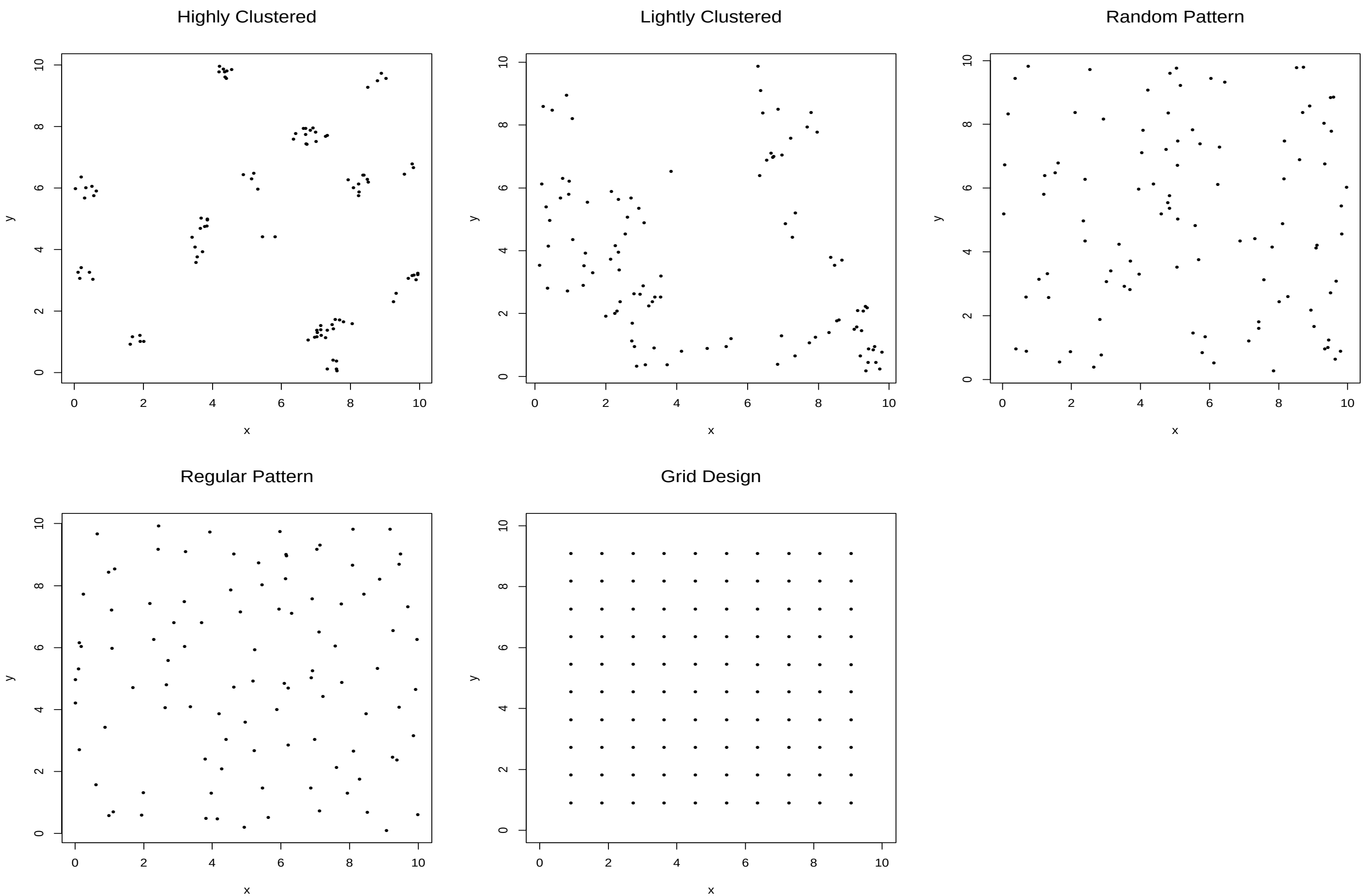
Model Selection: Independent model AIC Values



Model Selection: Spatial model AIC Values



Sampling Patterns



Model Selection: Effect of Sampling Design

Summary of model selection results for 5 different sampling patterns

Variables in Model	Highly Clustered	Lightly Clustered	Random	Regular Pattern	Grid Design
$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3$	73	65	46	43	16
$\boldsymbol{X}_1, \boldsymbol{X}_2$	0	2	18	21	35
$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4$	12	13	8	8	3
$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_5$	10	13	11	7	7

- Each column reports the percentage of simulations that each model was selected.
- Of the 32 possible models, the results given here include only those with 10% or more support for at least one of the sampling patterns.

Prediction

Efficient prediction

- Time series (Shibata (1980), Brockwell and Davis (1991)). AIC is an efficient order selection procedure for autoregressive models.
- Regression (see McQuarrie and Tsai (1998)).
- Other notions of efficiency, e.g., Kullback-Leibler efficiency and L_2 efficiency (see McQuarrie and Tsai (1998)).

Prediction: Prediction Error

Simulations:

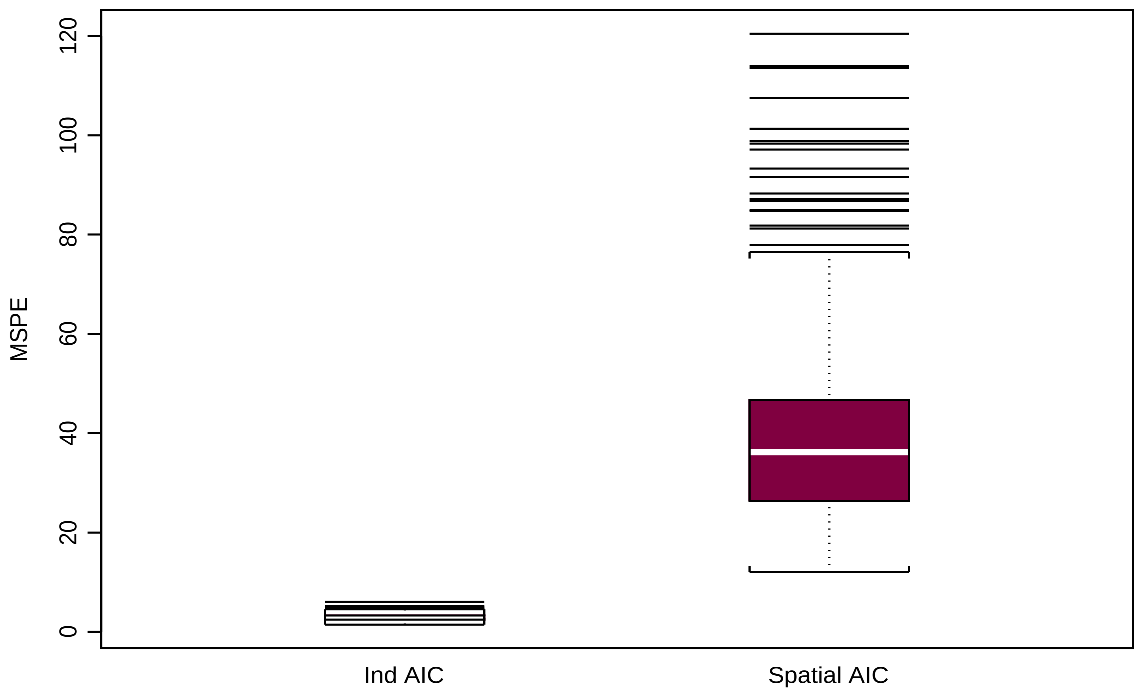
- Performed model selection and estimation using 100 observations and evaluated prediction performance using 100 additional observations simulated as above.
- Evaluated predictive performance

Mean Square Prediction Error:

$$\mathrm{MSPE} = \frac{1}{100} \sum_{j=1}^{100} \left(Z_j - \hat{Z}_j \right)^2$$

where $\tilde{Z}_j$ is the universal kriging predictor for the j^{th} prediction location using the true parameter values.

Prediction: MSPE



Prediction: Predictive Coverage

Predictive Coverage: for a 95% prediction interval, do 95% of the observed data fall in their corresponding prediction intervals?

Simulations:
For each of the 500 simulations, we compute predictive coverage. Then, over all 500 simulations, we examine:

- Mean predictive coverage
- Standard deviation of predictive coverage

Model	Mean	Std Dev
Independent error AIC	0.95	0.18
Spatial error AIC	0.92	0.25

Example: Lizard abundance

Abundance for the orange-throated whiptail lizard in southern California
Ver Hoef et al. (2001)

Data:

- 147 locations
- $Z = \log(\text{ave \# of lizards caught per day})$
- Explanatory variables: ant abundance (three levels), log(% sandy soils), elevation, barerock indicator, % cover, log(% chapparal plants)

Example: Lizard abundance

- Explanatory variables:
  ant abundance (three levels), log(% sandy soils), % cover, elevation, barerock indicator, log(% chapparal plants)
- 160 possible models

Predictors	AIC	Spatial Rank	Ind Rank
Ant_1, % sand	54.8	1	66
Ant_1,Ants_2, % sand	54.8	2	56
Ant_1, % sand, % cover	55.7	3	59
Ant_1,Ant_2, % sand, % cover, elevation, barerock, % chaparral	92.2	41	1
Ant_1,Ant_2, % sand, % cover, elevation, barerock, % chaparral	95.5	33	2
Ant_1, % sand, % cover, elevation, barerock,	95.7	38	3

Some Other Approaches to Model Selection and Prediction

- Bayesian Model Averaging
  - Model uncertainty is typically ignored in inference
  - Protect from over-confident inferences by averaging over models
- Minimum Description Length (MDL)
  - Goal: Find model that achieves maximum data compression.

    The code length (CL) of the data (Lee 2001) is the amount of memory required to store the data. Decomposition of CL:

    $$CL(\text{“}data\text{”}) = CL(\text{“}fitted\ model\text{”}) + CL(\text{“}data\ given\ fitted\ model\text{”}).$$

    Here $CL(\text{“}fitted\ model\text{”})$ might be interpreted as the code length of the model parameters and $CL(\text{“}data\ given\ fitted\ model\text{”})$ as the code length of the residuals from the fitted model.

Conclusions

- Ignoring spatial correlation can influence model selection results for both covariate selection and prediction
- Sampling patterns that offer observation pairs at small and larger distances may be advantageous for model selection
- Preliminary results suggest that accounting for spatial correlation can have large effects on prediction errors, but perhaps smaller impacts on predictive coverage.