Regular Variation and Financial Time Series Models

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Outline

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  • Multiplicative models for log-returns (GARCH, SV)

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  • Extremes and extremal index
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Characteristics of some financial time series

Define $X_t = \ln (P_t) - \ln (P_{t-1})$ (log returns)

- heavy tailed

  $$P(|X_1| > x) \sim C x^{-\alpha}, \quad 0 < \alpha < 4.$$  

- uncorrelated

  $$\hat{\rho}_{X^2}(h) \text{ near 0 for all lags } h > 0 \text{ (MGD sequence)}$$

- $|X_t|$ and $X_t^2$ have slowly decaying autocorrelations

  $$\hat{\rho}_{|X|}(h) \text{ and } \hat{\rho}_{X^2}(h) \text{ converge to 0 slowly as } h \text{ increases.}$$

- process exhibits ‘volatility clustering’.
Basic model

$$X_t = \ln(P_t) - \ln(P_{t-1}) \quad \text{(log returns)}$$

$$= \sigma_t Z_t,$$

where

- $\{Z_t\}$ is IID with mean 0, variance 1 (if exists). (e.g. N(0,1) or a $t$-distribution with $\nu$ df.)
- $\{\sigma_t\}$ is the volatility process
- $\sigma_t$ and $Z_t$ are independent.

Properties:

- $EX_t = 0$, $Cov(X_t, X_{t+h}) = 0$, $h > 0$ (uncorrelated if $\text{Var}(X_t) < \infty$)
- conditional heteroscedastic (condition on $\sigma_t$).
Multiplicative models for log(returns)-cont

\[ X_t = \sigma_t Z_t \] (observation eqn in state-space formulation)

Two classes of models for volatility:

(i) GARCH(p,q) process (General AutoRegressive Conditional Heteroscedastic-observation-driven specification)

\[ \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_p X_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_q \sigma_{t-q}^2. \]

Special case: ARCH(1):

\[ X_t^2 = (\alpha_0 + \alpha_1 X_{t-1}^2) Z_t^2 \]
\[ = \alpha_1 Z_t^2 X_{t-1}^2 + \alpha_0 Z_t^2 \]
\[ = A_t X_{t-1}^2 + B_t \] (stochastic recurrence eqn)

\[ \rho_{X_t^2} (h) = \alpha_1^h, \text{ if } \alpha_1^2 < 1/3. \]
Multiplicative models for log(returns)-cont

\[ X_t = \sigma_t Z_t \] (observation eqn in state-space formulation)

(ii) stochastic volatility process (parameter-driven specification)

\[ \log \sigma_t^2 = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}, \quad \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty, \{\varepsilon_t\} \sim \text{IIDN}(0, \sigma^2) \]

\[ \rho_{X^2}(h) = \text{Cor}(\sigma_t^2, \sigma_{t+h}^2) / EZ_1^4 \]

Question:

• Joint distributions of process regularly varying if distr of \( Z_1 \) is regularly varying?
Multivariate regular variation of $X=(X_1, \ldots, X_m)$: There exists a random vector $\theta \in S^{m-1}$ such that

$$P(|X| > t, X/|X| \in \bullet)/P(|X| > t) \xrightarrow{v} x^{-\alpha} P(\theta \in \bullet)$$

($\xrightarrow{v}$ vague convergence on $S^{m-1}$, unit sphere in $\mathbb{R}^m$).

- $P(\theta \in \bullet)$ is called the spectral measure
- $\alpha$ is the index of $X$.

**Equivalence:**

$$\frac{P(X \in t\bullet)}{P(|X| > t)} \xrightarrow{v} \mu(\bullet)$$

$\mu$ is a measure on $\mathbb{R}^m$ which satisfies for $x > 0$ and $A$ bounded away from 0,

$$\mu(xB) = x^{-\alpha} \mu(xA).$$
Examples:

1. If $X_1 > 0$ and $X_2 > 0$ are iid RV($\alpha$), then $\mathbf{X} = (X_1, X_2)$ is multivariate regularly varying with index $\alpha$ and spectral distribution

$$P(\theta = (0,1) ) = P(\theta = (1,0) ) = .5 \text{ (mass on axes).}$$

Interpretation: Unlikely that $X_1$ and $X_2$ are very large at the same time.

Figure: plot of $(X_{t1}, X_{t2})$ for realization of 10,000.
2. If \( X_1 = X_2 > 0 \), then \( X = (X_1, X_2) \) is multivariate regularly varying with index \( \alpha \) and spectral distribution

\[
P( \theta = (1/\sqrt{2}, 1/\sqrt{2}) ) = 1.
\]

3. AR(1): \( X_t = 0.9 X_{t-1} + Z_t \), \( \{Z_t\} \sim \text{IID symmetric stable (1.8)} \)

\[
\text{Distr of } \theta: \begin{cases} 
\pm(1,.9)/\sqrt{1.81}, \text{ W.P. .9898} \\
\pm(0,1), \text{ W.P. .0102} 
\end{cases}
\]

**Figure:** plot of \((X_t, X_{t+1})\) for realization of 10,000.
Applications of multivariate regular variation

• Domain of attraction for *sums of iid random vectors* (Rvaceva, 1962). That is, when does the partial sum

\[ a_n^{-1} \sum_{t=1}^{n} X_t \]

converge for some constants \( a_n \)?

• *Spectral measure* of multivariate stable vectors.

• *Domain of attraction* for componentwise maxima of iid random vectors (Resnick, 1987). Limit behavior of

\[ a_n^{-1} \bigvee_{t=1}^{n} X_t \]

• Weak convergence of *point processes* with iid points.

• Solution to *stochastic recurrence equations*, \( Y_t = A_t Y_{t-1} + B_t \)

• Weak convergence of *sample autocovariances*. 
Linear combinations:

$X \sim RV(\alpha) \Rightarrow$ all linear combinations of $X$ are regularly varying

i.e., there exist $\alpha$ and slowly varying fcn $L(.)$, s.t.

$$P(c^T X > t)/(t^\alpha L(t)) \rightarrow w(c), \text{ exists for all real-valued } c,$$

where

$$w(tc) = t^{-\alpha}w(c).$$

Use vague convergence with $A_c = \{y: c^T y > 1\}$, i.e.,

$$\frac{P(X \in tA_c)}{t^{-\alpha}L(t)} = \frac{P(c^T X > t)}{P(|X| > t)} \rightarrow \mu(A_c) = w(c),$$

where $t^\alpha L(t) = P(|X| > t)$. 
Applications of multivariate regular variation (cont)

Converse?

\( \mathbf{X} \sim \text{RV}(\alpha) \iff \text{all linear combinations of } \mathbf{X} \text{ are regularly varying?} \)

There exist \( \alpha \) and slowly varying fcn \( L(.), \) s.t.

\[
(LC) \quad P(c^T \mathbf{X} > t)/(t^\alpha L(t)) \rightarrow w(c), \text{ exists for all real-valued } c.
\]

Theorem (Basrak, Davis, Mikosch, `02). Let \( \mathbf{X} \) be a random vector.

1. If \( \mathbf{X} \) satisfies (LC) with \( \alpha \) non-integer, then \( \mathbf{X} \) is \( \text{RV}(\alpha) \).

2. If \( \mathbf{X} > 0 \) satisfies (LC) for non-negative \( c \) and \( \alpha \) is non-integer, then \( \mathbf{X} \) is \( \text{RV}(\alpha) \).

3. If \( \mathbf{X} > 0 \) satisfies (LC) with \( \alpha \) an odd integer, then \( \mathbf{X} \) is \( \text{RV}(\alpha) \).
Applications of theorem

1. Kesten (1973). Under general conditions, (LC) holds with \( L(t)=1 \) for stochastic recurrence equations of the form

\[
Y_t = A_t Y_{t-1} + B_t, \quad (A_t, B_t) \sim \text{IID},
\]

\( A_t \) \( d \times d \) random matrices, \( B_t \) random \( d \)-vectors.

It follows that the distributions of \( Y_t \), and in fact all of the finite dim’l distrs of \( Y_t \) are regularly varying (if \( \alpha \) is non-even).

2. GARCH processes. Since squares of a GARCH process can be embedded in a SRE, the \textit{finite dimensional distributions} of a \textit{GARCH} are regularly varying.
Example of ARCH(1): \( \alpha_0=1, \alpha_1=1, \alpha=2, X_t=(\alpha_0+\alpha_1 X_{t-1}^2)^{1/2}Z_t, \) \( \{Z_t\}\sim\text{IID} \)

**Figures:** plots of \( (X_t, X_{t+1}) \) and estimated distribution of \( \theta \) for realization of 10,000.
Example: SV model $X_t = \sigma_t Z_t$

Suppose $Z_t \sim \text{RV}(\alpha)$ and

$$\log \sigma_t^2 = \sum_{j=-\infty}^{\infty} \psi_j \epsilon_{t-j}, \quad \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty, \{\epsilon_t\} \sim \text{IIDN}(0,\sigma^2).$$

Then $Z_n = (Z_1, \ldots, Z_n)'$ is regularly varying with index $\alpha$ and so is $X_n = (X_1, \ldots, X_n)' = \text{diag}(\sigma_1, \ldots, \sigma_n) Z_n$

with spectral distribution concentrated on $(\pm 1, 0), (0, \pm 1)$.

Figure: plot of $(X_t, X_{t+1})$ for realization of 10,000.
Extremes for GARCH and SV processes

Setup

- \( X_t = \sigma_t Z_t \), \( \{Z_t\} \sim \text{IID } (0,1) \)
- \( X_t \) is RV \((\alpha)\)
- Choose \( \{b_n\} \) s.t. \( nP(X_t > b_n) \rightarrow 1 \)

Then

\[
P^n(b_n^{-1} X_1 \leq x) \rightarrow \exp\{-x^{-\alpha}\}.
\]

Then, with \( M_n = \max\{X_1, \ldots, X_n\} \),

(i) GARCH:

\[
P(b_n^{-1} M_n \leq x) \rightarrow \exp\{-\gamma x^{-\alpha}\},
\]

\( \gamma \) is extremal index \((0 < \gamma < 1)\).

(ii) SV model:

\[
P(b_n^{-1} M_n \leq x) \rightarrow \exp\{-x^{-\alpha}\},
\]

extremal index \( \gamma = 1 \) no clustering.
Extremes for GARCH and SV processes (cont)

(i) **GARCH:** \( P( b_n^{-1} M_n \leq x ) \rightarrow \exp \{-\gamma x^{-\alpha} \} \)

(ii) **SV model:** \( P( b_n^{-1} M_n \leq x ) \rightarrow \exp \{-x^{-\alpha} \} \)

Remarks about extremal index.

(i) \( \gamma < 1 \) implies clustering of exceedances

(ii) Numerical example. Suppose \( c \) is a threshold such that

\[
P^n ( b_n^{-1} X_1 \leq c ) \sim .95
\]

Then, if \( \gamma = .5 \), \( P( b_n^{-1} M_n \leq c ) \sim ( .95 )^5 = .975 \)

(iii) \( 1/\gamma \) is the *mean cluster size* of exceedances.

(iv) Use \( \gamma \) to *discriminate* between GARCH and SV models.

(v) Even for the light-tailed SV model (i.e., \( \{ Z_t \} \sim \text{iid} \ N(0,1) \), the *extremal index* is 1 (see Breidt and Davis `98)
Extremes for GARCH and SV processes (cont)
Summary of results for ACF of GARCH(p,q) and SV models

**GARCH(p,q)**

- **$\alpha \in (0, 2)$:**
  \[
  (\hat{\rho}_x(h))_{h=1,\ldots,m} \xrightarrow{d} (V_h / V_0)_{h=1,\ldots,m},
  \]

- **$\alpha \in (2, 4)$:**
  \[
  \left( n^{1-2/\alpha} \hat{\rho}_x(h) \right)_{h=1,\ldots,m} \xrightarrow{d} \gamma_{X}^{-1}(0)(V_h)_{h=1,\ldots,m}.
  \]

- **$\alpha \in (4, \infty)$:**
  \[
  \left( n^{1/2} \hat{\rho}_x(h) \right)_{h=1,\ldots,m} \xrightarrow{d} \gamma_{X}^{-1}(0)(G_h)_{h=1,\ldots,m}.
  \]

**Remark:** Similar results hold for the sample ACF based on $|X_t|$ and $X_t^2$. 
Summary of results for ACF of GARCH(p,q) and SV models (cont)

**SV Model**

\( \alpha \in (0, 2) \):

\[
\left( \frac{n}{\ln n} \right)^{1/\alpha} \hat{\rho}_{\sigma}(h) \xrightarrow{d} \frac{\|\sigma_1 \sigma_{h+1}\|_\alpha}{\|\sigma_1\|_\alpha^2} \frac{S_h}{S_0}.
\]

\( \alpha \in (2, \infty) \):

\[
\left( n^{1/2} \hat{\rho}_{\sigma}(h) \right)_{h=1,\ldots,m} \xrightarrow{d} \gamma^{-1}_\sigma(0)(G_h)_{h=1,\ldots,m}.
\]
Wrap-up

• *Regular variation* is a flexible tool for modeling both *dependence* and *tail heaviness*.

• Useful for establishing *point process convergence* of heavy-tailed time series.

• *Extremal index* $\gamma < 1$ for GARCH and $\gamma = 1$ for SV.

Unresolved issues related to $\text{RV} \Leftrightarrow (\text{LC})$

• $\alpha = 2n$?

• there is an example for which $X_1, X_2 > 0$, and $(c, X_1)$ and $(c, X_2)$ have the same limits for all $c > 0$.

• $\alpha = 2n-1$ and $X \not> 0$ (not true in general).