Regular Variation and Financial Time Series Models

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Outline

- Characteristics of some financial time series
  - IBM returns
  - Multiplicative models for log-returns (GARCH, SV)
- Regular variation
  - univariate case
  - multivariate case
  - new characterization: **X** is RV ⇔ c´**X** is RV **?**
- Applications of regular variation
  - Stochastic recurrence equations (GARCH)
  - Point process convergence
  - Extremes and extremal index
  - Limit behavior of sample correlations
- Wrap-up

Characteristics of some financial time series

Define $X_t = \ln(P_t) - \ln(P_{t-1})$ (log returns)

- heavy tailed

$$P(|X_1| > x) \sim C\, x^{-\alpha}, \quad 0 < \alpha < 4.$$

- uncorrelated

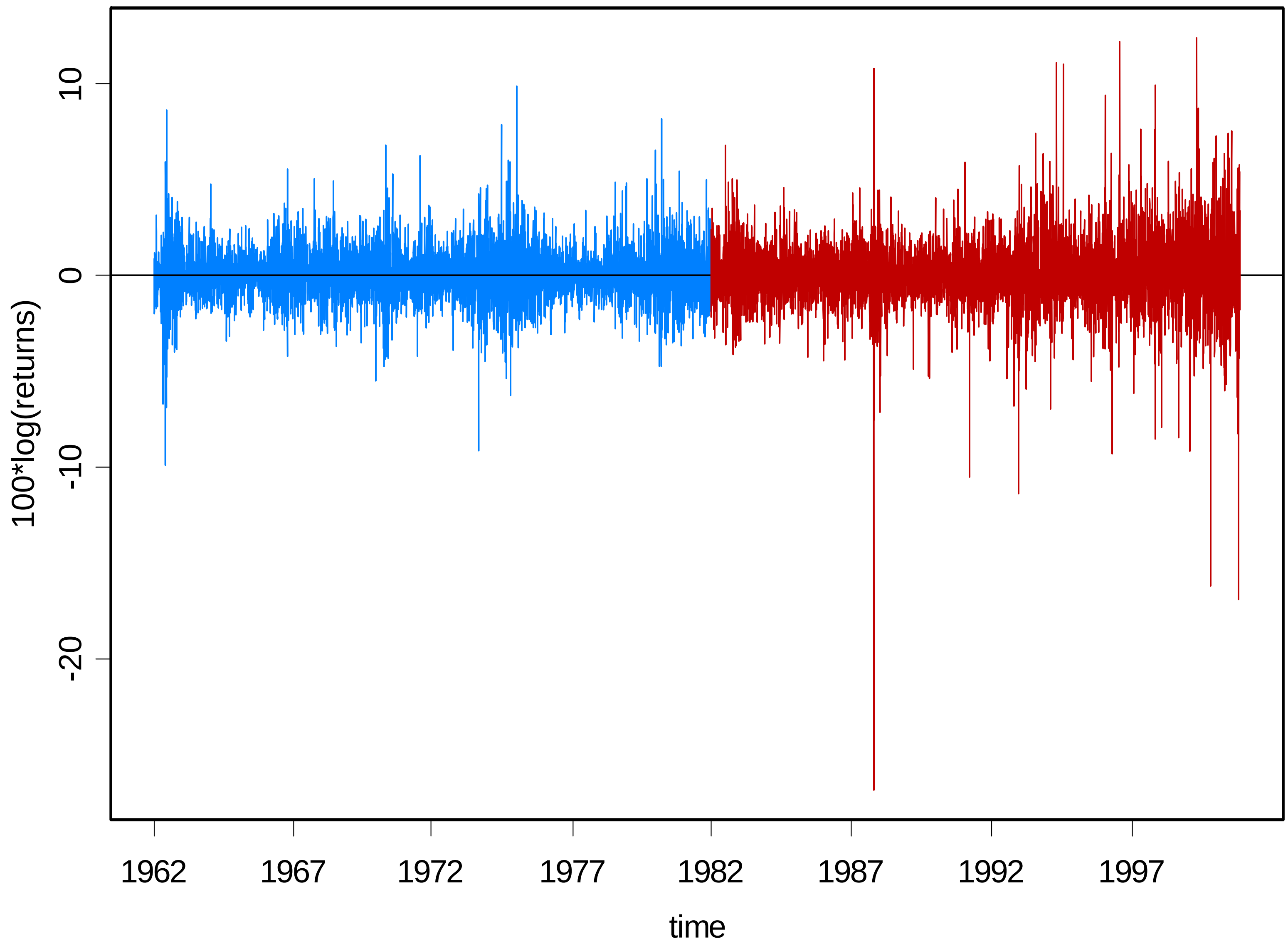
$\hat{\rho}_X(h)$ near 0 for all lags $h > 0$ (MGD sequence)

- $|X_t|$ and X_t^2 have slowly decaying autocorrelations

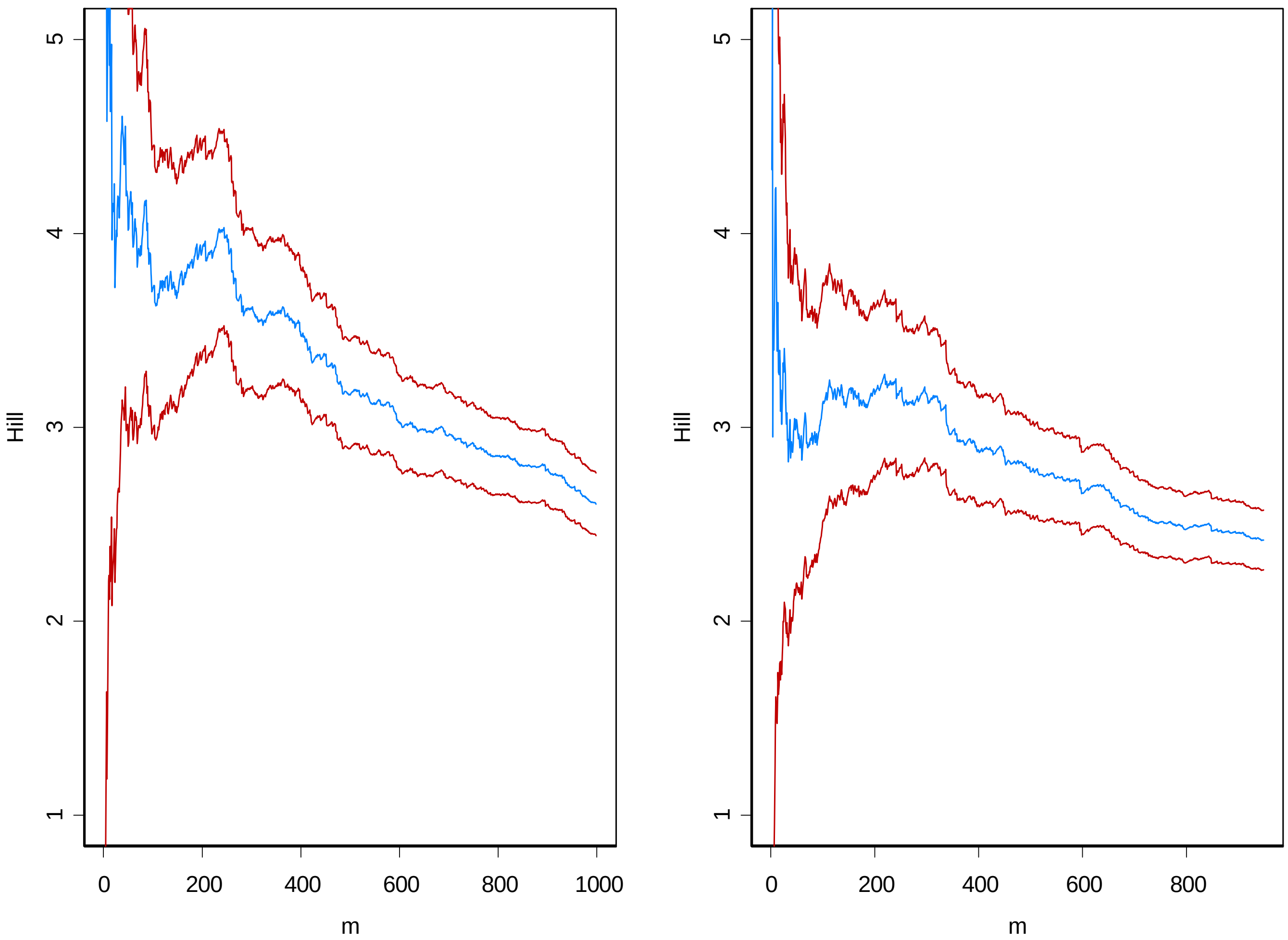
$\hat{\rho}_{|X|}(h)$ and $\hat{\rho}_{X^2}(h)$ converge to 0 *slowly* as h increases.

- process exhibits 'volatility clustering'.

Log returns for IBM 1/3/62-11/3/00 (blue=1961-1981)



Hill's plot of tail index for IBM (1962-1981, 1982-2000)



Multiplicative models for log(returns)

Basic model

$$X_t = \ln(P_t) - \ln(P_{t-1}) \quad \text{(log returns)}$$
$$= \sigma_t Z_t ,$$

where

- $\{Z_t\}$ is IID with mean 0, variance 1 (if exists). (e.g. N(0,1) or a *t*-distribution with ν df.)
- $\{\sigma_t\}$ is the volatility process
- σ_t and Z_t are independent.

Properties:

- $EX_t = 0$, $\mathrm{Cov}(X_t, X_{t+h}) = 0$, $h>0$ (uncorrelated if $\mathrm{Var}(X_t) < \infty$)
- conditional heteroscedastic (condition on σ_t).

Multiplicative models for log(returns)-cont

$X_t = \sigma_t Z_t$ (observation eqn in state-space formulation)

Two classes of models for volatility:

(i) GARCH(p,q) process (General AutoRegressive Conditional Heteroscedastic-observation-driven specification)

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_p X_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_q \sigma_{t-q}^2 \ .$$

Special case: ARCH(1):

$$\begin{aligned} X_t^2 &= (\alpha_0 + \alpha_1 X_{t-1}^2) Z_t^2 \\ &= \alpha_1 Z_t^2 X_{t-1}^2 + \alpha_0 Z_t^2 \\ &= A_t X_{t-1}^2 + B_t \end{aligned}$$

(stochastic recurrence eqn)

$$\rho_{X^2}(h) = \alpha_1^h, \text{ if } \alpha_1^2 < 1/3.$$

Multiplicative models for log(returns)-cont

$X_t = \sigma_t Z_t$ (observation eqn in state-space formulation)

(ii) stochastic volatility process (parameter-driven specification)

$$\log \sigma_t^2 = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}, \quad \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty, \{\varepsilon_t\} \sim \mathrm{IID\,N}(0,\sigma^2)$$

$$\rho_{X^2}(h) = Cor(\sigma_t^2, \sigma_{t+h}^2) / EZ_1^4$$

Question:

- Joint distributions of process regularly varying if distr of Z_1 is regularly varying?

Regular variation — multivariate case

Multivariate regular variation of $\mathbf{X}=(X_1, \ldots, X_m)$: There exists a random vector $\theta \in S^{m-1}$ such that

$$P(|\mathbf{X}|> t\,x,\ \mathbf{X}/|\mathbf{X}| \in \bullet\ )/P(|\mathbf{X}|>t) \to_v x^{-\alpha} P(\ \theta \in \bullet\ )$$

($\to_v$ vague convergence on S^{m-1}, unit sphere in R^m) .

- $P(\ \theta \in \bullet)$ is called the spectral measure
- α is the index of $\mathbf{X}$.

Equivalence:

$$\frac{P(\mathbf{X} \in t\bullet)}{P(|\mathbf{X}|> t)} \to_v \mu(\bullet)$$

μ is a measure on R^m which satisfies for $x > 0$ and A bounded away from 0,

$$\mu(xB) = x^{-\alpha} \mu(xA).$$

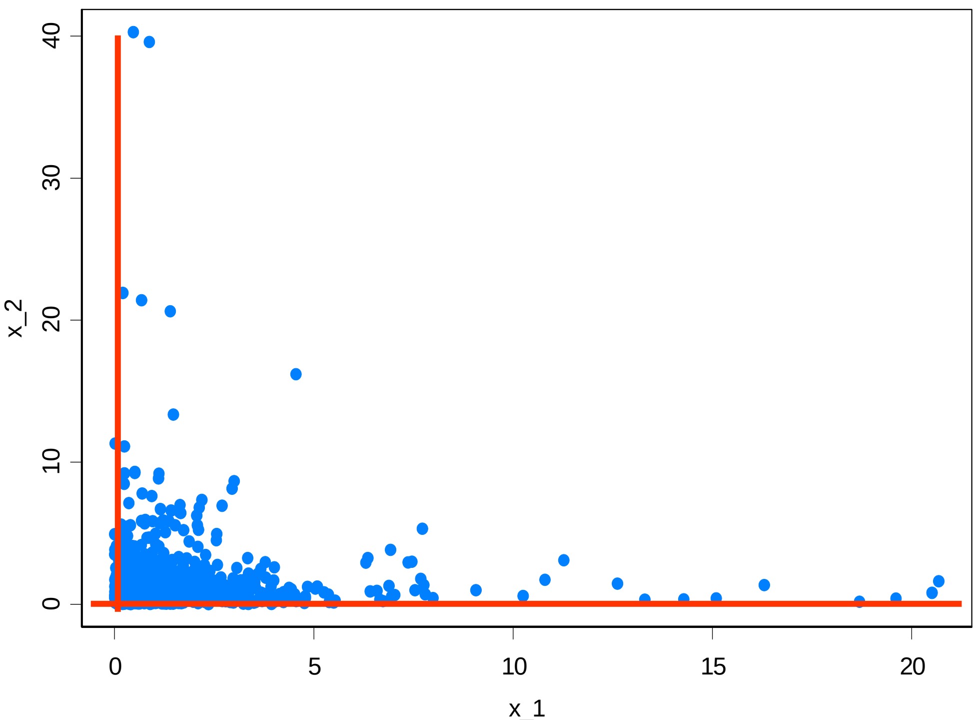
Examples:

1. If $X_1 > 0$ and $X_2 > 0$ are iid RV(α), then $\mathbf{X} = (X_1, X_2)$ is multivariate regularly varying with index α and *spectral distribution*

$$P(\theta = (0,1)) = P(\theta = (1,0)) = .5 \quad \text{(mass on axes)}.$$

Interpretation: Unlikely that X_1 and X_2 are very large at the same time.

Figure: plot of (X_{t1}, X_{t2}) for realization of 10,000.



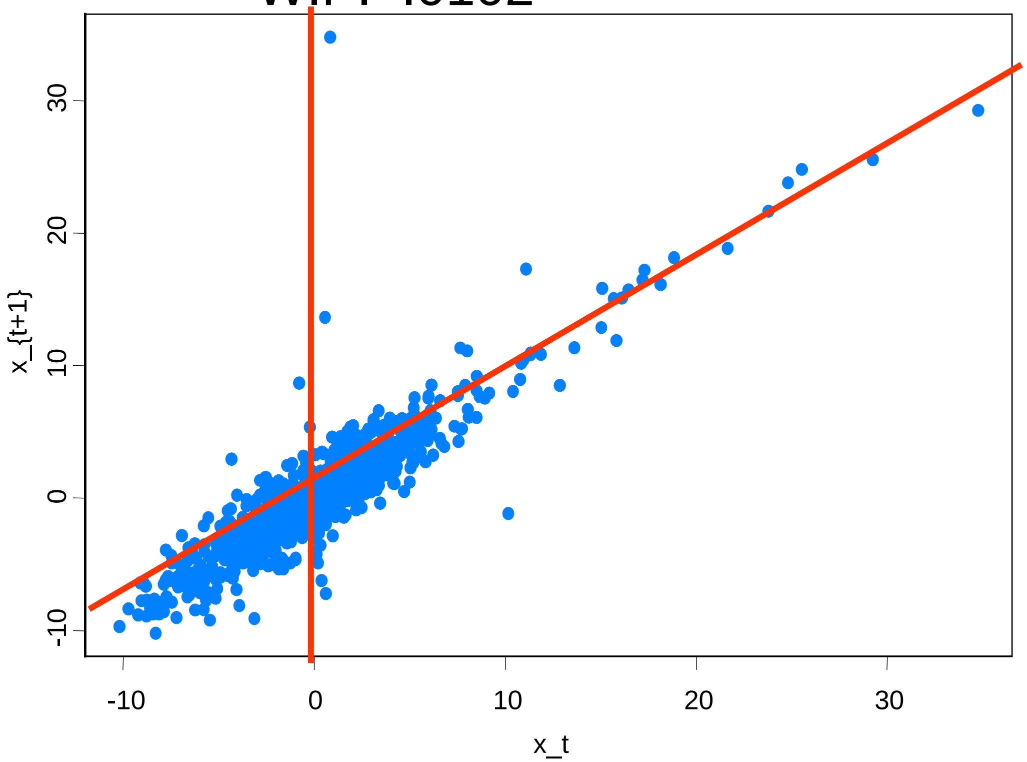
2. If $X_1 = X_2 > 0$, then $\mathbf{X} = (X_1, X_2)$ is multivariate regularly varying with index α and *spectral distribution*

$$P(\theta = (1/\sqrt{2}, 1/\sqrt{2})) = 1.$$

3. AR(1): $X_t = .9\, X_{t-1} + Z_t$, $\{Z_t\}$~IID symmetric stable (1.8)

$$\text{Distr of } \theta: \begin{cases} \pm(1,.9)/\text{sqrt}(1.81), & \text{W.P. } .9898 \\ \pm(0,1), & \text{W.P. } .0102 \end{cases}$$

Figure: plot of (X_t, X_{t+1}) for realization of 10,000.



Applications of multivariate regular variation

- Domain of attraction for *sums of iid random vectors* (Rvaceva, 1962). That is, when does the partial sum

$$a_n^{-1}\sum_{t=1}^{n}\mathbf{X}_t$$

converge for some constants a_n?

- *Spectral measure* of multivariate stable vectors.
- *Domain of attraction* for componentwise maxima of iid random vectors (Resnick, 1987). Limit behavior of

$$a_n^{-1}\bigvee_{t=1}^{n}\mathbf{X}_t$$

- Weak convergence of *point processes* with iid points.
- Solution to *stochastic recurrence equations*, $\mathbf{Y}_t = \mathbf{A}_t\,\mathbf{Y}_{t-1} + \mathbf{B}_t$
- Weak convergence of *sample autocovariances*.

Applications of multivariate regular variation (cont)

Linear combinations:

X ~RV(α) ⇒ all linear combinations of **X** are regularly varying

i.e., there exist α and slowly varying fcn L(.), s.t.

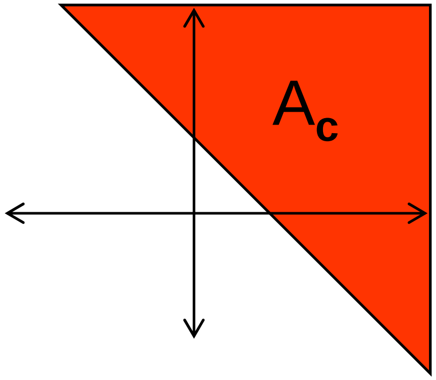
$$P(\mathbf{c}^T\mathbf{X} > t)/(t^{-\alpha}L(t)) \to w(\mathbf{c}), \text{ exists for all real-valued } \mathbf{c},$$

where

$$w(t\mathbf{c}) = t^{-\alpha}w(\mathbf{c}).$$

Use vague convergence with $A_\mathbf{c}=\{\mathbf{y}: \mathbf{c}^T\mathbf{y} > 1\}$, i.e.,

$$\frac{P(\mathbf{X} \in tA_\mathbf{c})}{t^{-\alpha}L(t)} = \frac{P(\mathbf{c}^T\mathbf{X} > t)}{P(|\mathbf{X}| > t)} \to \mu(A_\mathbf{c}) =: w(\mathbf{c}),$$



where $t^{-\alpha}L(t) = P(|\mathbf{X}| > t)$.

Applications of multivariate regular variation (cont)

Converse?

$\mathbf{X} \sim RV(\alpha) \Leftarrow$ all linear combinations of $\mathbf{X}$ are regularly varying?

There exist α and slowly varying fcn $L(.)$, s.t.

(LC) $P(\mathbf{c}^T\mathbf{X} > t)/(t^{-\alpha}L(t)) \to w(\mathbf{c})$, exists for all real-valued $\mathbf{c}$.

Theorem (Basrak, Davis, Mikosch, `02). Let $\mathbf{X}$ be a random vector.

1. If $\mathbf{X}$ satisfies (LC) with α non-integer, then $\mathbf{X}$ is $RV(\alpha)$.
2. If $\mathbf{X} > 0$ satisfies (LC) for non-negative $\mathbf{c}$ and α is non-integer, then $\mathbf{X}$ is $RV(\alpha)$.
3. If $\mathbf{X} > 0$ satisfies (LC) with α an odd integer, then $\mathbf{X}$ is $RV(\alpha)$.

Applications of theorem

1. Kesten (1973). Under general conditions, (LC) holds with L(t)=1 for stochastic recurrence equations of the form

$$\mathbf{Y}_t = \mathbf{A}_t \mathbf{Y}_{t-1} + \mathbf{B}_t, \quad (\mathbf{A}_t, \mathbf{B}_t) \sim \text{IID},$$

$\mathbf{A}_t$ $d \times d$ random matrices, $\mathbf{B}_t$ random d-vectors.

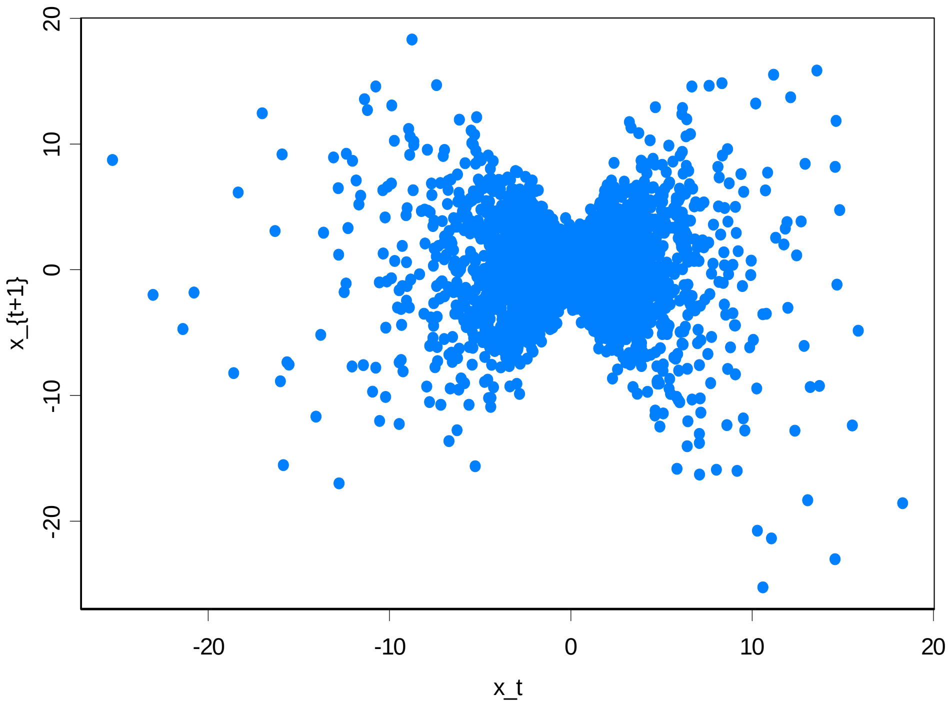
It follows that the distributions of $\mathbf{Y}_t$, and in fact all of the finite dim'l distrs of $\mathbf{Y}_t$ are regularly varying (if α is non-even).

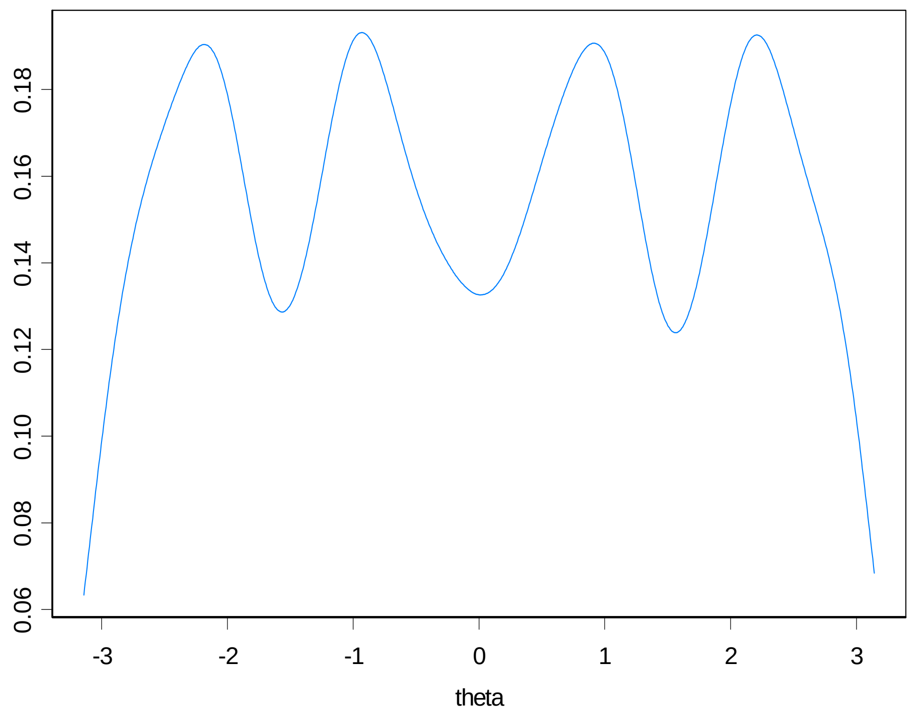
2. GARCH processes. Since squares of a GARCH process can be embedded in a SRE, the *finite dimensional distributions* of a *GARCH* are regularly varying.

Examples (cont)

Example of ARCH(1): $\alpha_0=1$, $\alpha_1=1$, $\alpha=2$, $X_t=(\alpha_0+\alpha_1 X^2_{t-1})^{1/2}Z_t$, $\{Z_t\}\sim$IID

Figures: plots of (X_t, X_{t+1}) and estimated distribution of θ for realization of 10,000.





Applications of theorem (cont)

Example: SV model $X_t = \sigma_t Z_t$

Suppose $Z_t \sim RV(\alpha)$ and

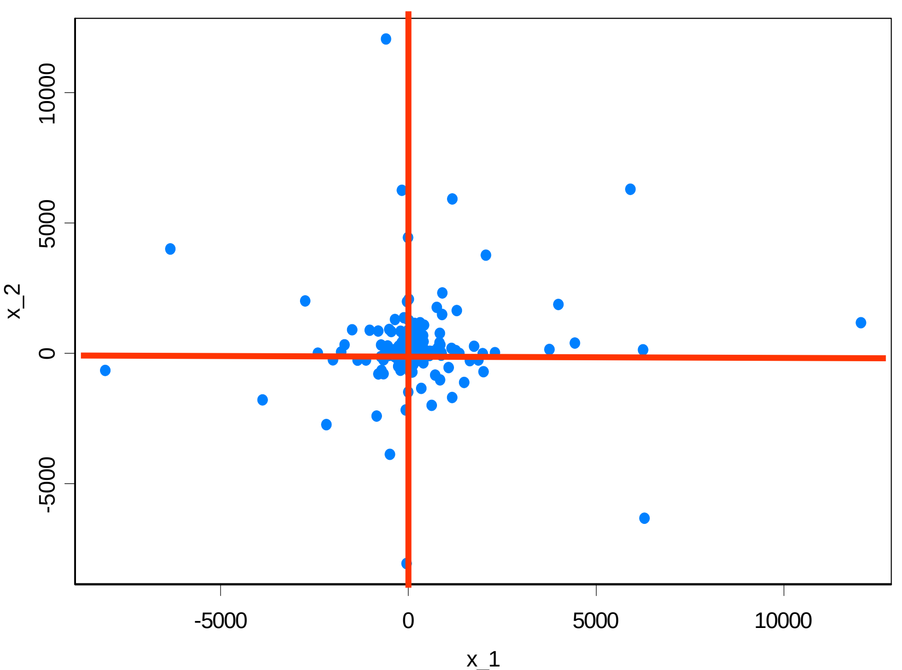
$$\log\sigma_t^2 = \sum_{j=-\infty}^{\infty}\psi_j\varepsilon_{t-j},\ \ \sum_{j=-\infty}^{\infty}\psi_j^2 < \infty, \{\varepsilon_t\} \sim \text{IID N}(0,\sigma^2).$$

Then $\mathbf{Z}_n=(Z_1,\ldots,Z_n)'$ is regulary varying with index α and so is

$$\mathbf{X}_n = (X_1,\ldots,X_n)' = \text{diag}(\sigma_1,\ldots,\sigma_n)\,\mathbf{Z}_n$$

with spectral distribution concentrated on $(\pm 1,0)$, $(0, \pm 1)$.

Figure: plot of (X_t, X_{t+1}) for realization of 10,000.



Extremes for GARCH and SV processes

Setup

- $X_t = \sigma_t Z_t$, $\{Z_t\} \sim$ IID (0,1)
- X_t is RV (α)
- Choose $\{b_n\}$ s.t. $nP(X_t > b_n) \to 1$

Then

$$P^n(b_n^{-1} X_1 \le x) \to \exp\{-x^{-\alpha}\}.$$

Then, with $M_n = \max\{X_1, \ldots, X_n\}$,

(i) GARCH:

$$P(b_n^{-1} M_n \le x) \to \exp\{-\gamma x^{-\alpha}\},$$

γ is extremal index ($0 < \gamma < 1$).

(ii) SV model:

$$P(b_n^{-1} M_n \le x) \to \exp\{-x^{-\alpha}\},$$

extremal index $\gamma = 1$ no clustering.

Extremes for GARCH and SV processes (cont)

(i) GARCH: $P(b_n^{-1}M_n \le x) \to \exp\{-\gamma x^{-\alpha}\}$

(ii) SV model: $P(b_n^{-1}M_n \le x) \to \exp\{-x^{-\alpha}\}$

Remarks about extremal index.

(i) $\gamma < 1$ implies clustering of exceedances

(ii) Numerical example. Suppose c is a threshold such that

$$P^n(b_n^{-1}X_1 \le c) \sim .95$$

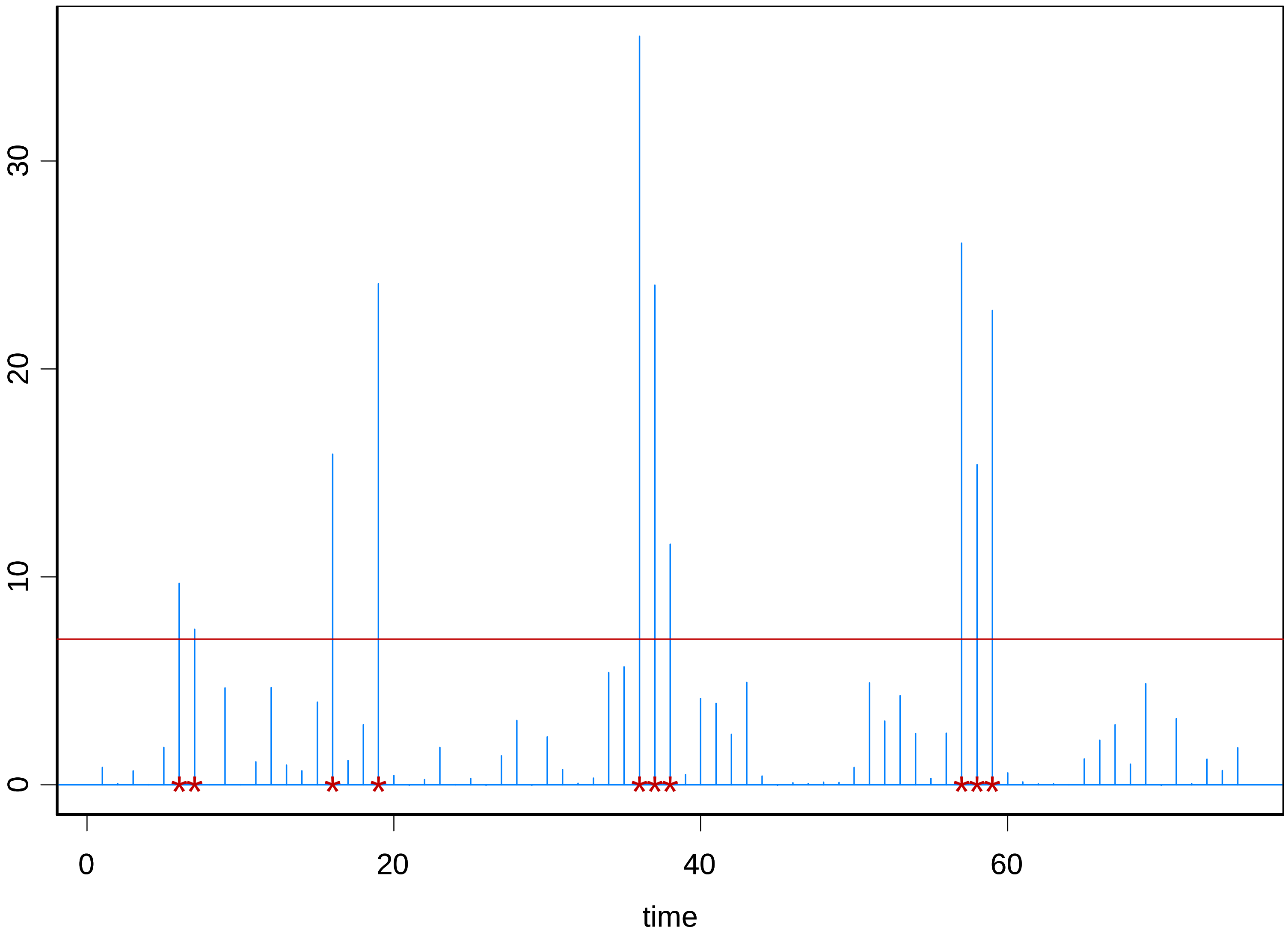
Then, if $\gamma = .5$, $P(b_n^{-1}M_n \le c) \sim (.95)^{.5} = .975$

(iii) $1/\gamma$ is the *mean cluster size* of exceedances.

(iv) Use γ to *discriminate* between GARCH and SV models.

(v) Even for the light-tailed SV model (i.e., $\{Z_t\}$ ~IID N(0,1), the *extremal index* is 1 (see Breidt and Davis `98)

Extremes for GARCH and SV processes (cont)



Summary of results for ACF of GARCH(p,q) and SV models

GARCH(p,q)

$\alpha \in (0,2)$:

$$(\hat{\rho}_X(h))_{h=1,\ldots,m} \xrightarrow{d} (V_h / V_0)_{h=1,\ldots,m},$$

$\alpha \in (2,4)$:

$$\left(n^{1-2/\alpha}\hat{\rho}_X(h)\right)_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1}(0)\left(V_h\right)_{h=1,\ldots,m}.$$

$\alpha \in (4,\infty)$:

$$\left(n^{1/2}\hat{\rho}_X(h)\right)_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1}(0)\left(G_h\right)_{h=1,\ldots,m}.$$

Remark: Similar results hold for the sample ACF based on $|X_t|$ and X_t^2.

Summary of results for ACF of GARCH(p,q) and SV models (cont)

SV Model

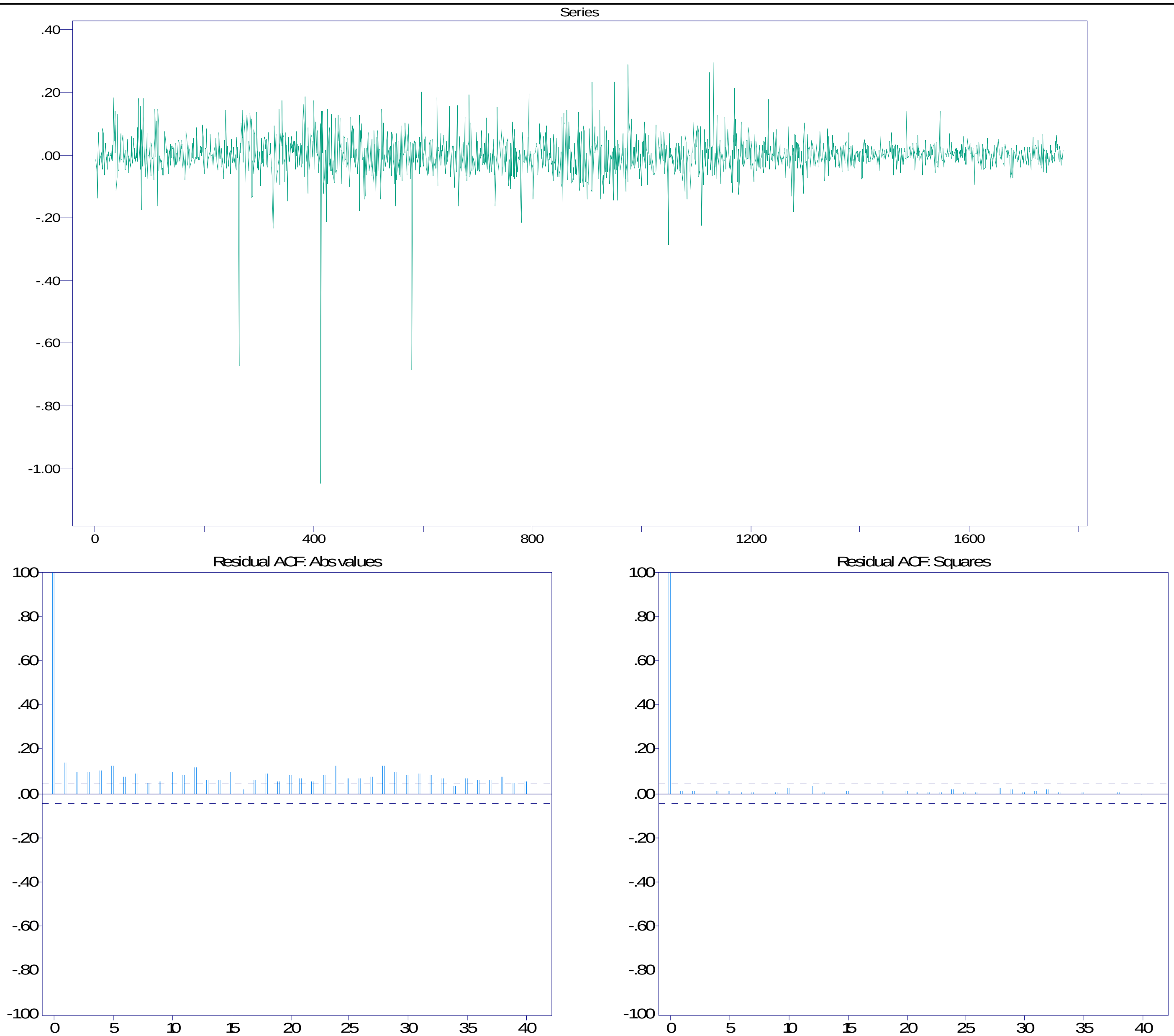
$\alpha \in (0,2)$:

$$(n/\ln n)^{1/\alpha}\hat{\rho}_X(h) \xrightarrow{d} \frac{\|\sigma_1\sigma_{h+1}\|_\alpha}{\|\sigma_1\|_\alpha^2}\frac{S_h}{S_0}.$$

$\alpha \in (2, \infty)$:

$$\left(n^{1/2}\hat{\rho}_X(h)\right)_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1}(0)\left(G_h\right)_{h=1,\ldots,m}.$$

Amazon returns May 16, 1997 to June 16, 2004.



Wrap-up

• *Regular variation* is a flexible tool for modeling both *dependence* and *tail heaviness.*

• Useful for establishing *point process convergence* of heavy-tailed time series.

• *Extremal index* $\gamma < 1$ for GARCH and $\gamma = 1$ for SV.

Unresolved issues related to RV $\Leftrightarrow$ (LC)

• $\alpha = 2n$?

• there is an example for which $\mathbf{X}_1, \mathbf{X}_2 > 0$, and $(\mathbf{c}, \mathbf{X}_1)$ and $(\mathbf{c}, \mathbf{X}_2)$ have the same limits for all $\mathbf{c} > \mathbf{0}$.

• $\alpha = 2n-1$ and $\mathbf{X} \not> 0$ (not true in general).