Inference for Lévy-Driven Continuous-Time ARMA Processes

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Outline

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- Canonical representation
- Estimation via the uniformly sampled process
- Estimation for non-negative CAR(1)
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Why study continuous-time models?

- For handling irregularly-spaced data.
- For financial applications—option pricing.
- For taking advantage of the now wide-spread availability of high-frequency data.
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In recent years, various attempts have been made to use continuous-time in order to capture the so-called stylized features of financial time series

- tail heaviness
- dependence without correlation
- volatility clustering
Background (cont)

log-returns for Nikkei (7/97 – 4/99)
Barndorff-Nielsen and Shephard (2001) introduced the following SV model for the log-asset price $X^*$:

$$dX^*(t) = (\mu + \beta V(t))dt + \sqrt{V(t)}dW(t),$$

where $W(t)$ is SBM. The volatility process $V$ is an independent stationary non-negative Lévy-driven Ornstein-Uhlenbeck process satisfying

$$dV(t) + aV(t)dt = \sigma dL(t), \quad a > 0,$$

i.e.,

$$V(t) = \sigma \int_{-\infty}^{t} e^{-a(t-u)}dL(u)$$

with $L(t)$ a Lévy process.
Daily Volatility Estimates for DM/$ (12/1/86 to 6/30/99) based on 5-minute returns (see Todorov).
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Lévy-driven CARMA processes

The covariance function of Barndorff-Nielsen Shephard model has limited behavior; namely covariance function must decrease exponentially. Instead, we consider the case that $V$ is a subordinator-driven non-negative continuous-time ARMA (CARMA).

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq \infty}, P)$ be a filtered probability space, where $\mathcal{F}_0$ contains all the $P$-null sets of $\mathcal{F}$ and $(\mathcal{F}_t)$ is right-continuous.

**Definition (Lévy Process).** $\{L(t), t \geq 0\}$ is an $(\mathcal{F}_t)$-adapted Lévy process if $L(t) \in \mathcal{F}_t$ for all $t \geq 0$ and

- $L(0) = 0$ a.s.,
- $L(t_0), L(t_1) - L(t_0), \ldots, L(t_n) - L(t_{n-1})$ are independent for $0 \leq t_0 < t_1 < \cdots < t_n$,
- the distribution of $\{L(s + t) - L(s) : t \geq 0\}$ does not depend on $s$,
- $L(t)$ is continuous in probability.
The characteristic function of $L(t)$, $\phi_t(\theta) := E(\exp(i\theta L(t)))$, has the Lévy-Khinchin representation,

$$\phi_t(\theta) = \exp(t\xi(\theta)), \quad \theta \in \mathbb{R},$$

where

$$\xi(\theta) = i\theta m - \frac{1}{2}\theta^2 \sigma^2 + \int_{\mathbb{R}_0} \left(e^{i\theta x} - 1 - \frac{ix\theta}{1 + x^2}\right) \nu(dx),$$

for some $m \in \mathbb{R}$, $\sigma > 0$, and the measure $\nu$ is on the Borel subsets of $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$, known as the Lévy measure of the process $L$, satisfying

$$\int_{\mathbb{R}_0} \frac{u^2}{1 + u^2} \nu(du) < \infty.$$
Some examples

\[ \xi(\theta) = i\theta m - \frac{1}{2} \theta^2 \sigma^2 + \int_{\mathbb{R}} \left( e^{i\theta x} - 1 - \frac{i x \theta}{1 + x^2} \right) \nu(dx), \]

- \( \nu = 0 \Rightarrow \text{Brownian motion.} \)
- \( m = \sigma^2 = 0, \int_{\mathbb{R}} \frac{|u|}{1+u^2} \nu(du) < \infty \Rightarrow \text{compound Poisson with drift.} \)
- \( \nu(du) = \alpha u^{-1} e^{-\beta u} du \Rightarrow \text{a gamma process with} \)
\[ \xi(\theta) = \int (e^{i\theta x^{-1}} - 1) \nu(dx) = (1 - i\theta / \beta)^{-\alpha t}, \]

- \( \nu(du) = \frac{1}{2} \alpha |u|^{-1} e^{-\beta |u|} du \Rightarrow \text{a symmetrized gamma process} \)
\( (L_1 - L_2). \)
- \( \xi(\theta) = \exp(-c|\theta|^\alpha), 0 < \alpha \leq 2, \Rightarrow \text{symmetric stable process.} \)
Formally, a CARMA process driven by a Lévy process is a stationary solution of the $p$th order linear differential equation

\begin{equation}
 a(D)Y(t) = \sigma b(D)DL(t),
\end{equation}

where $D$ denotes differentiation with respect to $t$,

\begin{align*}
 a(z) &= z^p + a_1z^{p-1} + \cdots + a_p, \\
 b(z) &= b_0 + b_1z^{1} + \cdots + b_{p-1}z^{p-1}, \\
 b_q &= 1, \quad b_j := 0 \text{ for } j > q, \text{ and } \{L(t)\} \text{ is a second-order Lévy process with } \text{Var}(L(1)) = 1.
\end{align*}
The defining SDE (1) is interpreted through the state-space formulation given by the observation and state equations,

\begin{align*}
Y(t) &= \sigma b' X(t), \quad t \geq 0, \quad (2) \\
\text{d}X(t) &= A X(t) \text{d}t + e \text{d}L(t), \quad (3)
\end{align*}

where

\[
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1
\end{bmatrix},
\]

\[
e' = \begin{bmatrix}
0 & 0 & \cdots & 0 & 1
\end{bmatrix}, \quad \text{and}
\]

\[
b' = \begin{bmatrix}
b_0 & b_1 & \cdots & b_q & 0 & \cdots & 0 & 1
\end{bmatrix}.
\]
The solution to (3) satisfies

\[ X(t) = e^{At}X(0) + \int_0^t e^{A(t-u)}e \, dL(u). \]

**Proposition.** If \( X(0) \) is independent of \( \{L(t)\} \), then \( \{X(t)\} \) given by (4) is strictly (and weakly) stationary if and only if the eigenvalues of the matrix \( A \) all have strictly negative real parts and \( X(0) \sim \int_0^\infty e^{Au}e \, dL(u). \)

**Remark.** It is easy to check that the eigenvalues of the matrix \( A \) are the zeroes of the autoregressive polynomial \( a(z) \).
Sometimes it is convenient to define the CARMA process for all real values of \( t \).

**Extension to all \( t \).** Let \( \{M(t), \ 0 \leq t < \infty \} \) be an independent copy of \( L \) and set

\[
L^*(t) = L(t)I_{[0,\infty)}(t) - M(-t-)I_{(-\infty,0]}(t).
\]

If the eigenvalues of \( A \) have negative real parts, then

\[
X(t) = \int_{-\infty}^{t} e^{A(t-u)} e \, dL^*(u).
\]

is a strictly stationary process satisfying

\[
X(t) = e^{A(t-s)}X(s) + \int_{s}^{t} e^{A(t-u)} e \, dL^*(u).
\]
Definition (Causal CARMA Process). If the eigenvalues of $A$ have negative real parts, then the CARMA process $Y$ is the strictly stationary process

$$Y(t) = \sigma b'X(t)$$

where

$$X(t) = \int_{-\infty}^{t} e^{A(t-u)}b' dL(u),$$

i.e.,

$$Y(t) = \sigma \int_{-\infty}^{t} b'e^{A(t-u)}b' dL(u).$$

That is, $\{Y(t)\}$ is a causal function of $\{L(t)\}$,

$$Y(t) = \sigma \int_{-\infty}^{\infty} g(t - u) dL(u), \text{ where } g(t) = \begin{cases} \sigma b'e^{At}b', & t > 0, \\ 0, & \text{otherwise.} \end{cases}$$
Using the causal representation of $Y(t)$, the ACVF of $Y$ is

$$\gamma(h) = \sigma^2 \int_{-\infty}^{\infty} \tilde{g}(h-u) g(u) \, du,$$

where $\tilde{g}(u) = g(-x)$. After some calculation, one can show that

$$\gamma(h) = \frac{\sigma^2}{2\pi} \int_{-\infty}^{\infty} e^{i\omega h} \left| \frac{b(i\omega)}{a(i\omega)} \right|^2 \, d\omega,$$

i.e., $Y$ has rational spectral density $f(\omega) = \frac{\sigma^2}{2\pi} \left| \frac{b(i\omega)}{a(i\omega)} \right|^2$.

**Remark.** Gaussian processes with rational spectral density have been of interest for many years. (See extensive study by Doob (1944) and a nice paper by Pham Din Duan (1977).) The SDE approach to such processes can be found in the engineering literature and was employed by Jones (1978) for modeling irregularly-spaced data.
When the zeroes $\lambda_1, \ldots, \lambda_p$ of the causal AR polynomial $a(z)$ are distinct, then the kernel function $g$ and ACVF $\gamma$ have the special form

\[
g(h) = \sigma \sum_{r=1}^{p} \frac{b(\lambda_r)}{a'(\lambda_r)} e^{\lambda_r h} I_{[0,\infty)}(h) \quad \text{and} \quad \gamma(h) = \sigma^2 \sum_{r=1}^{p} \frac{b(\lambda_r)b(-\lambda_r)}{a'(\lambda_r)a(-\lambda_r)} e^{\lambda_r |h|}.
\]

Now defining $\alpha_r = \sigma b(\lambda_r)/a'(\lambda_r)$, $r = 1, \ldots, p$, we can write

\[
Y(t) = \sum_{r=1}^{p} Y_r(t),
\]

where $Y_r$ is the CAR(1) process,

\[
Y_r(t) = \int_{-\infty}^{t} \alpha_r e^{\lambda_r(t-u)} dL(u).
\]
Equivalently,

\[ Y(t) = [1, \ldots, 1] Y(t), \quad t \geq 0, \]

where \( Y \) is the solution of

\[ dY(t) = \text{diag}[\lambda_i]_{i=1}^{p} Y dt + \sigma BR^{-1} e dL \]

with \( Y(0) = \sigma BR^{-1} X(0) \), \( B = \text{diag}[b(\lambda_i)] \), and \( R = [\lambda_i^{i-1}] \).

**Remark.** Simulation of a CARMA\((p, q)\) process with distinct AR roots can be achieved by the much simpler problem of simulating component CAR(1) processes and adding them together.
From the representation, \( Y(t) = \int_{-\infty}^{t} g(t - u) \, dL(u) \), the marginal distribution of \( Y \) has cumulant generating function

\[
\log E(\exp(i\theta Y(t))) = \int_{-\infty}^{\infty} \xi(\theta g(u)) \, du.
\]

Using independence of the increments, one can easily calculate the joint cgf of the fidis.

- \( \nu = 0 \Rightarrow \text{Gaussian CARMA}(p, q) \).
- \( \{L(t)\} \) compound Poisson with bilateral exponential jumps \( \Rightarrow \) (in the CAR(1) case) that \( \{Y(t)\} \) has marginal cgf, \( \kappa(\theta) = -\frac{\lambda}{2a_1} \ln \left(1 + \frac{\theta^2}{\beta^2}\right) \), i.e., \( Y(t) \) has a symmetrized gamma distribution (bilateral exponential if \( \lambda = 2a_1 \)).
- For CAR(1) with non-negative Lévy input, see Barndorff-Nielsen and Shephard (2001) and storage theory literature.
1. For linear Gaussian CARMA processes, MLE based on observations $Y(t_1), \ldots, Y(t_n)$ can be easily carried out using the state-space representation (see Jones (1981)).

2. For both the linear Lévy-driven CARMA and Gaussian CTAR processes, the likelihood can be computed using the state-space representation and then optimized.
If $Y$ is a Gaussian CARMA process, then it is well known (e.g., Doob (1944), Phillips (1959), Brockwell (1995)), that the sampled process $Y(nh), n = 0, \pm 1, \ldots$, for fixed spacing $h$ is a *strict* Gaussian ARMA($r, s$) process with $0 \leq s < r \leq p$.

If $L$ is non-Gaussian, the sampled process will have the same spectral density (and hence ACVF) as the analogous Gaussian CARMA process. So from a second order perspective, the two sampled processes are identical. However, (except in the CAR(1) case), the non-Gaussian CARMA will not generally be a *strict* ARMA process.
CAR(1) Example. If $Y$ is the CAR(1) process, then the sampled process is the strict AR(1) process

$$Y_n^{(h)} = \phi Y_{n-1}^{(h)} + Z_n, \quad n = 0, 1, \ldots,$$

where $\phi = \exp(-ah)$ and

$$Z_n = \sigma \int_{(n-1)h}^{nh} e^{-a(nh-u)} \ dL(u).$$

The noise sequence $\{Z_n\}$ is iid and $Z_n$ has the infinitely divisible distribution with cgf

$$\int_0^h \xi(\sigma\theta e^{-au}) \ du,$$

where $\xi(\theta)$ is the log-characteristic function of $L(1)$. 
For the CARMA\((p, q)\) process with \(p > 1\), the situation is more complicated. If the AR roots \(\lambda_1, \ldots, \lambda_p\) are all distinct then, from the canonical representation, the sampled process is

\[
Y(nh) = \sum_{r=1}^{p} Y_r(nh),
\]

where \(\{Y_r(nh)\}\) is the strict AR(1) process

\[
Y_r(nh) = e^{\lambda_r h} Y_r((n-1)h) + Z_r(n), \quad n = 0, \pm 1, \ldots,
\]

with

\[
Z_r(n) = \alpha_r \int_{(n-1)h}^{nh} e^{\lambda_r (nh-u)} dL(u).
\]

and

\[
\alpha_r = \sigma \frac{b(\lambda_r)}{a'(\lambda_r)}.
\]
Let $Y$ be the CAR(1) process driven by the Lévy process \{L(t), t \geq 0\} with non-negative increments, i.e., $Y$ is the stationary solution of the stochastic differential equation,

$$dY(t) + aY(t)dt = \sigma dL(t).$$

For any $h > 0$, the sampled process \{\(Y^{(h)}_n := Y(nh), n = 0, 1, \ldots\)\} is a discrete-time AR(1) process satisfying

$$Y^{(h)}_n = \phi Y^{(h)}_{n-1} + Z_n, \quad n = 0, 1, \ldots,$$

where $\phi = \exp(-ah)$ (obviously $0 < \phi < 1$), and

$$Z_n = \sigma \int_{(n-1)h}^{nh} e^{-a(nh-u)} dL(u).$$

the noise sequence \{\(Z_n\)\} is iid and positive since $L$ has stationary, independent, and positive increments.
If the process \( \{Y(t), 0 \leq t \leq T\} \) is observed at times \( 0, h, \ldots, Nh \), where \( N = \lfloor T/h \rfloor \), i.e., \( N \) is the integer part of \( T/h \), then, since the innovations \( Z_n \) of the process \( \{Y_n(h)\} \) are non-negative and \( 0 < \phi < 1 \), we can use the highly efficient Davis-McCormick estimator of \( \phi \),

\[
\hat{\phi}_N(h) = \min_{1 \leq n \leq N} \frac{Y_n(h)}{Y_{n-1}(h)}.
\]

To obtain the asymptotic distribution of \( \hat{\phi}_N(h) \) with \( h \) fixed, we need to suppose the distr \( F \) of \( Z_n \) satisfies \( F(0) = 0 \) and that \( F \) is regularly varying at zero with exponent \( \alpha \), i.e.,

\[
\lim_{t \downarrow 0} \frac{F(tx)}{F(t)} = x^\alpha, \quad \text{for all } x > 0.
\]

(These conditions are satisfied by the gamma-driven CAR(1) process as we shall show later.)
Under these conditions on $F$, the results of Davis and McCormick (1989) imply that $\hat{\phi}_N^{(h)} \to \phi$ a.s. as $N \to \infty$ with $h$ fixed and that

$$\lim_{N \to \infty} P \left[ k_N^{-1} \left( \hat{\phi}_N^{(h)} - \phi \right) c_\alpha \leq x \right] = G_\alpha(x)$$

where $k_N = F^{-1}(N^{-1})$, $c_\alpha = \left( EY_1^{(h)\alpha} \right)^{1/\alpha}$ and $G_\alpha$ is the Weibull distribution function,

$$G_\alpha(x) = \begin{cases} 1 - \exp \left\{ -x^\alpha \right\}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$
From the observations \( \{Y^{(h)}_n, \ n = 0, 1, \ldots, N\} \), we thus obtain the estimator \( \hat{\phi}^{(h)}_N \), and hence the estimator of \( a \) is

\[
\hat{a}^{(h)}_N = -h^{-1} \log \hat{\phi}^{(h)}_N.
\]

Using a Taylor series approximation, we find that

\[
\lim_{N \to \infty} P \left[ (-h)e^{-ah} k^{-1}_N (\hat{a}^{(h)}_N - a) c_\alpha \leq x \right] = G_\alpha(x),
\]

where \( G_\alpha \) is the Weibull distribution specified above.

Since \( \text{Var}(Y^{(h)}) = \sigma^2/(2a) \), we use the estimator

\[
\hat{\sigma}^2_N = \frac{2\hat{a}^{(h)}_N}{N} \sum_{i=0}^{N} (Y^{(h)}_i - \bar{Y}^{(h)}_N)^2
\]

to estimate \( \sigma^2 \),
Gamma-driven CAR(1)

Suppose $L$ is a standardized gamma process, i.e., $L(t)$ has the gamma density $f_{L(t)}$ with exponent $\gamma t$, and scale-parameter $\gamma^{-1/2}$, mean $\gamma^{1/2}t$ and variance $t$. The Laplace transform of $L(t)$ is

$$\tilde{f}_{L(t)}(s) = E \exp(-sL(t)) = \exp\{-t\Phi(s)\}, \quad R(s) \geq 0,$$

where $\Phi(s) = \gamma \log(1 + \beta s)$, $\beta = \gamma^{-1/2}$, and $\gamma > 0$.

**Theorem 1.** For the gamma-driven CAR(1) process, we have $\hat{a}_N^{(h)} \to a$ a.s. and

$$\lim_{N \to \infty} P \left[ (-h)e^{-ah}k_N^{-1}(\hat{a}_N^{(h)} - a)c_\alpha \leq x \right] = G_\alpha(x)$$

where $\alpha = \gamma h$,

$$k_N^{-1} \sim (\sigma \beta)^{-1}[\Gamma(\gamma h + 1)]^{-1/(\gamma h)}e^{5ah}N^{1/(\gamma h)},$$

and $c_\alpha$ is computed numerically using a result of Brockwell and Brown ‘78.
Examining the normalizing constant $k_N^{-1}$, we find that

$$\lim_{h \to 0} \lim_{N \to \infty} \frac{k_N^{-1}}{N^{1/(\gamma h)}} = (\sigma \beta)^{-1} e^{\gamma E}$$

where $\gamma_E$ is the Euler-Mascheroni constant.

Convergence is thus extremely fast for large $N$ and small $h$. 

Example. We now illustrate the estimation procedure with a simulated example. The gamma-driven CAR(1) process defined by,

\[ DY(t) + 0.6Y(t) = DL(t), \quad t \in [0, 5000], \quad (7) \]

was simulated at times 0, 0.001, 0.002, \ldots, 5000, using an Euler approximation. The parameter \( \gamma \) of the standardized gamma process was 2. The process was then sampled at intervals \( h = 0.01, \ h = 0.1 \) and \( h = 1 \) by selecting every 10\(^{th}\), 100\(^{th}\) and 1000\(^{th}\) observation respectively. We generated 100 such realizations of the process and applied the above estimation procedure to generate 100 independent estimates, for each \( h \), of the parameters \( a \) and \( \sigma \). The sample means and standard deviations of these estimators are shown in Table 1, which illustrates the remarkable accuracy of the estimators.
Table 1. Estimated parameters based on 100 replicates on $[0, 5000]$ of the gamma-driven CAR(1) process (7) with $\gamma = 2$, observed at times $nh, n = 0, \ldots, [T/h]$.

<table>
<thead>
<tr>
<th>Spacing</th>
<th>Parameter</th>
<th>Gamma increments</th>
<th>Sample mean of estimators</th>
<th>Sample std deviation of estimators</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h=1$</td>
<td>$a$</td>
<td>0.59269</td>
<td>0.00381</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\sigma$</td>
<td>0.99796</td>
<td>0.01587</td>
<td></td>
</tr>
<tr>
<td>$h=0.1$</td>
<td>$a$</td>
<td>0.59999</td>
<td>0.00000</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\sigma$</td>
<td>1.00011</td>
<td>0.01281</td>
<td></td>
</tr>
<tr>
<td>$h=0.01$</td>
<td>$a$</td>
<td>0.60000</td>
<td>0.00000</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\sigma$</td>
<td>0.99990</td>
<td>0.01175</td>
<td></td>
</tr>
</tbody>
</table>
In order to suggest an appropriate parametric model for $L$ and to estimate the parameters, it is important to recover an approximation to $L$ from the observed data. If the CAR(1) process is observed continuously on $[0, T]$, we have

$$L(t) = \sigma^{-1} \left[ Y(t) - Y(0) + a \int_0^t Y(s) ds \right].$$
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$$L(t) = \sigma^{-1} \left[ Y(t) - Y(0) + a \int_0^t Y(s) ds \right].$$

Replacing the CAR(1) parameters by their estimators and the integral by a trapezoidal approximation, we obtain the estimator for the Lévy increments $\Delta L_n^{(h)} := L(nh) - L((n-1)h)$ on the interval $((n-1)h, nh]$, given by

$$\Delta \hat{L}_n^{(h)} = \hat{\sigma}^{-1}_N \left[ Y_n^{(h)} - Y_{n-1}^{(h)} + \hat{a}_N^{(h)} h(Y_n^{(h)} + Y_{n-1}^{(h)})/2 \right].$$

(6)
Figure 1: The probability density of the increments per unit time of the standardized Lévy process and the histogram of the estimated increments from a realization of the CAR(1) process (7).
## Table 2. Estimated parameter of the standardized driving Lévy process.

<table>
<thead>
<tr>
<th>Spacing</th>
<th>Parameter</th>
<th>Sample mean of estimators</th>
<th>Sample std deviation of estimators</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h = 1$</td>
<td>$\gamma$</td>
<td>1.99598</td>
<td>0.05416</td>
</tr>
<tr>
<td>$h = 0.1$</td>
<td>$\gamma$</td>
<td>2.00529</td>
<td>0.03226</td>
</tr>
<tr>
<td>$h = 0.01$</td>
<td>$\gamma$</td>
<td>2.00547</td>
<td>0.02762</td>
</tr>
</tbody>
</table>
For a continuously observed realization on \([0, T]\) of a CAR(1) process driven by a non-decreasing Lévy process with drift \(m = 0\), the value of \(a\) can be identified exactly with probability 1. This contrast strongly with the case of a Gaussian CAR(1) process. This result is a corollary of the following theorem.

**Theorem 2.** If the CAR(1) process \(\{Y(t), t \geq 0\}\) is driven by a non-decreasing Lévy process \(L\) with drift \(m\) and Lévy measure \(\nu\), then for each fixed \(t\),

\[
\frac{Y(t + h) - Y(t)}{h} + aY(t) \to m \quad \text{a.s. as } h \downarrow 0.
\]
Corollary. If \( m = 0 \) in the Theorem 2 (this is the case if the point zero belongs to the closure of the support of \( L(1) \)), then with probability 1,

\[
a = \sup_{0 \leq s < t \leq T} \frac{\log Y(s) - \log Y(t)}{t - s}.
\]

For observations available at times \( \{nh : n = 0, 1, 2, \ldots, [T/h]\} \), our estimator can be expressed as

\[
\hat{a}(h)_T = \sup_{0 \leq n < [T/h]} \frac{\log Y(nh) - \log Y((n + 1)h)}{h}.
\]

The analogous estimator, based on closely but irregularly spaced observations at times \( t_1, t_2, \ldots, t_N \) such that \( 0 \leq t_1 < t_2 < \cdots < t_N \leq T \), is

\[
\hat{a}_T = \sup_n \frac{\log Y(t_n) - \log Y(t_{n+1})}{t_{n+1} - t_n}.
\]
**Remark.** We have shown that, if the drift of the driving Lévy process is zero and $T$ is any finite positive time, both estimators,

$$
\hat{a}^{(h)}_T = \sup_{0 \leq n < [T/h]} \frac{\log Y(nh) - \log Y((n+1)h)}{h}.
$$

and

$$
\hat{a}_T = \sup_n \frac{\log Y(t_n) - \log Y(t_{n+1})}{t_{n+1} - t_n}.
$$

converge almost surely to $a$ as the maximum spacing between successive observations converges to zero.
We found a highly efficient method, based on observations at times $0, h, 2h, \ldots, Nh$, for estimating the parameters of a stationary Ornstein-Uhlenbeck process $\{Y(t)\}$ driven by a non-decreasing Lévy process.
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Under specific conditions on the driving Lévy process, the asymptotic behavior of the estimators can be determined.
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Under specific conditions on the driving Lévy process, the asymptotic behavior of the estimators can be determined.

If the sample spacing h is small, we used a discrete approximation to the exact integral representation of \(L(t)\) in terms of \(\{Y(s), s \leq t\}\) to estimate the increments of the driving Lévy process, and hence to estimate the parameters of the Lévy process.
Conclusions

- We found a highly efficient method, based on observations at times $0, h, 2h, \ldots, Nh$, for estimating the parameters of a stationary Ornstein-Uhlenbeck process $\{Y(t)\}$ driven by a non-decreasing Lévy process.

- Under specific conditions on the driving Lévy process, the asymptotic behavior of the estimators can be determined.

- If the sample spacing $h$ is small, we used a discrete approximation to the exact integral representation of $L(t)$ in terms of $\{Y(s), s \leq t\}$ to estimate the increments of the driving Lévy process, and hence to estimate the parameters of the Lévy process.

- Examples suggest extremely good performance of the estimates.
If the driving Lévy process has no drift, then CAR(1) coefficient $a$ is determined almost surely by a continuously observed realization of $Y$ on any interval $[0, T]$. 
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The expression for $a$ suggests an estimator based on discrete observations of $Y$ which, for uniformly spaced observations, is the same as the estimator developed above and establishes the almost sure convergence of our estimator for any fixed $T$ as $h \to 0$. 
Conclusions (cont)

- If the driving Lévy process has no drift, then CAR(1) coefficient $a$ is determined almost surely by a continuously observed realization of $Y$ on any interval $[0, T]$.

- The expression for $a$ suggests an estimator based on discrete observations of $Y$ which, for uniformly spaced observations, is the same as the estimator developed above and establishes the almost sure convergence of our estimator for any fixed $T$ as $h \to 0$.

- Analogous procedures for non-negative Lévy-driven continuous-time ARMA processes are currently being investigated.