Heavy Tails and Financial Time Series Models

Richard A. Davis
Colorado State University
www.stat.colostate.edu/~rdavis

Thomas Mikosch
Bojan Basrak
Outline

Financial time series modeling
  • General comments
  • Characteristics of financial time series
  • Examples (exchange rate, Amazon)
  • Multiplicative models for log-returns (GARCH, SV)

Regular variation
  • univariate case
  • multivariate case

Applications of regular variation
  • Stochastic recurrence equations (GARCH)
  • Stochastic volatility
  • Time-reversibility
  • Point process convergence
  • Extremes and extremal index
  • Limit behavior of sample correlations

Wrap-up
One possible goal: Develop models that capture essential features of financial data.

Strategy: Formulate families of models that at least exhibit these key characteristics. (e.g., GARCH and SV)

Linkage with goal: Do fitted models actually capture the desired characteristics of the real data?

Answer wrt to GARCH and SV models: Yes and no. Answer may depend on the features.

Stărică’s paper: “Is GARCH(1,1) Model as Good a Model as the Nobel Accolades Would Imply?”

This paper discusses inadequacy of GARCH(1,1) model as a “data generating process” for the data.
Goal of this talk: compare and contrast some of the features of GARCH and SV models.

- Regular-variation of finite dimensional distributions
- Time-reversibility
- Point process convergence
- Extreme value behavior
- Sample ACF
Characteristics of financial time series

Define $X_t = \ln (P_t) - \ln (P_{t-1})$ (log returns)

- heavy tailed
  $$P(|X_1| > x) \sim RV(-\alpha), \quad 0 < \alpha < 4.$$  
- uncorrelated
  $$\hat{\rho}_X(h) \text{ near 0 for all lags } h > 0$$
- $|X_t|$ and $X_t^2$ have slowly decaying autocorrelations
  $$\hat{\rho}_{|X|}(h) \text{ and } \hat{\rho}_{X^2}(h) \text{ converge to 0 slowly as } h \text{ increases.}$$
- process exhibits ‘volatility clustering’.
Example: Pound-Dollar Exchange Rates
Example: Pound-Dollar Exchange Rates
Hill’s estimate of alpha (Hill Horror plots-Resnick)
15 realizations from GARCH model fitted to exchange rates + exchange rate data. Which one is the real data?
Stărică Plots for Pound-Dollar Exchange Rates

ACF of the squares from the 15 realizations from the GARCH model on previous slide.
Stărică Plots for Pound-Dollar Exchange Rates

15 realizations from SV model fitted to exchange rates + real data. Which one is the real data?
Example: Amazon-returns
Hill’s estimate of alpha (Hill Horror plots-Resnick)
Stărică Plots for the Amazon Data

15 realizations from GARCH model fitted to Amazon + exchange rate data. Which one is the real data?
Stărică Plots for Amazon

ACF of the squares from the 15 realizations from the GARCH model on previous slide.
Multiplicative models for log(returns)

**Basic model**

\[ X_t = \ln (P_t) - \ln (P_{t-1}) \quad \text{(log returns)} \]

\[ = \sigma_t Z_t , \]

where

- \( \{Z_t\} \) is IID with mean 0, variance 1 (if exists). (e.g. \( N(0,1) \) or a \( t \)-distribution with \( \nu \) df.)
- \( \{\sigma_t\} \) is the volatility process
- \( \sigma_t \) and \( Z_t \) are independent.

**Properties:**

- \( \text{EX}_t = 0, \ Cov(X_t, X_{t+h}) = 0, \ h>0 \) (uncorrelated if \( \text{Var}(X_t) < \infty \))
- conditional heteroscedastic (condition on \( \sigma_t \)).
Multiplicative models for log(returns)-cont

\[ X_t = \sigma_t Z_t \]  (observation eqn in state-space formulation)

Two classes of models for volatility:

(i) GARCH(p,q) process (General AutoRegressive Conditional Heteroscedastic-observation-driven specification)

\[ \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_p X_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_q \sigma_{t-q}^2. \]

Special case: ARCH(1):

\[
X_t^2 = (\alpha_0 + \alpha_1 X_{t-1}^2)Z_t^2 \\
= \alpha_1 Z_t^2 X_{t-1}^2 + \alpha_0 Z_t^2 \\
= A_t X_{t-1}^2 + B_t \quad \text{(stochastic recurrence eqn)}
\]

\[ \rho_{X^2}(h) = \alpha_1^h, \text{ if } \alpha_1^2 < 1/3. \]
Multiplicative models for log(returns)-cont

GARCH(2,1): \( X_t = \sigma_t Z_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \alpha_2 X_{t-2}^2 + \beta_1 \sigma_{t-1}^2. \)

Then \( Y_t = (\sigma_t^2, X_{t-1}^2)' \) follows the SRE given by

\[
\begin{bmatrix}
\sigma_t^2 \\
X_{t-1}^2 \\
Z_{t-1}^2 \\
\end{bmatrix} = \begin{bmatrix}
\alpha_1 & \beta_1 \\
\alpha_2 & 0 \\
0 & 0 \\
\end{bmatrix} \begin{bmatrix}
\sigma_{t-1}^2 \\
X_{t-2}^2 \\
\end{bmatrix} + \begin{bmatrix}
\alpha_0 \\
0 \\
\end{bmatrix}
\]

Questions:

- Existence of a unique stationary solution to the SRE?
- Regular variation of the joint distributions?
Multiplicative models for log(returns)-cont

\( X_t = \sigma_t Z_t \) (observation eqn in state-space formulation)

(ii) stochastic volatility process (parameter-driven specification)

\[
\log \sigma_t^2 = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}, \quad \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty, \{\varepsilon_t\} \sim \text{IIDN}(0,\sigma^2)
\]

\[
\rho_{\chi^2}(h) = \text{Cor}(\sigma_t^2, \sigma_{t+h}^2) / EZ_1^4
\]

Question:

• Joint distributions of process regularly varying if distr of \( Z_1 \) is regularly varying?
GARCH(1,1):

\[ X_t = \sigma_t Z_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \quad \{Z_t\} \sim \text{IID}(0,1) \]

Stochastic Volatility:

\[ X_t = \sigma_t Z_t, \quad \log \sigma_t^2 = \phi_0 + \phi_1 \log \sigma_{t-1}^2 + \varepsilon_t, \quad \{\varepsilon_t\} \sim \text{IID N}(0,\sigma^2) \]

Main question:

What intrinsic features in the data (if any) can be used to discriminate between these two models?
Def: The random variable $X$ is *regularly varying with index* $\alpha$ if

$$\frac{P(|X| > t \cdot x)}{P(|X| > t)} \to x^{-\alpha} \quad \text{and} \quad \frac{P(X > t)}{P(|X| > t)} \to p,$$

or, equivalently, if

$$\frac{P(X > t \cdot x)}{P(|X| > t)} \to px^{-\alpha} \quad \text{and} \quad \frac{P(X < -t \cdot x)}{P(|X| > t)} \to qx^{-\alpha},$$

where $0 \leq p \leq 1$ and $p+q=1$.

Equivalence: $X$ is RV($\alpha$) *if and only if* $P(X \in t \bullet) / P(|X| > t) \to_v \mu(\bullet)$

($\to_v$ vague convergence of measures on $\mathbb{R}\{0\}$). In this case,

$$\mu(dx) = \left( p \alpha x^{-\alpha-1} I(x>0) + q \alpha (-x)^{-\alpha-1} I(x<0) \right) dx$$

Note: $\mu(tA) = t^{-\alpha} \mu(A)$ for every $t$ and $A$ bounded away from 0.
Another formulation (polar coordinates):

Define the ± 1 valued rv θ, \( P(\theta = 1) = p, P(\theta = -1) = 1 - p = q \).

Then

X is RV(α) if and only if

\[
\frac{P(|X| > t, X/|X| \in S)}{P(|X| > t)} \xrightarrow{v} x^{-\alpha} P(\theta \in S)
\]

or

\[
\frac{P(|X| > t, X/|X| \in \bullet)}{P(|X| > t)} \xrightarrow{v} x^{-\alpha} P(\theta \in \bullet)
\]

(\( \xrightarrow{v} \) vague convergence of measures on \( S^0 = \{-1, 1\} \)).
Multivariate regular variation of $X=(X_1, \ldots, X_m)$: There exists a random vector $\theta \in S^{m-1}$ such that

$$P(|X| > t x, \frac{X}{|X|} \in \cdot )/P(|X| > t) \rightarrow_v x^{-\alpha} P(\theta \in \cdot )$$

($\rightarrow_v$ vague convergence on $S^{m-1}$, unit sphere in $\mathbb{R}^m$).

- $P(\theta \in \cdot )$ is called the spectral measure
- $\alpha$ is the index of $X$.

**Equivalence:**

$$\frac{P( X \in t\cdot )}{P(|X| > t )} \rightarrow_v \mu(\cdot)$$

$\mu$ is a measure on $\mathbb{R}^m$ which satisfies for $x > 0$ and $A$ bounded away from 0,

$$\mu(xB) = x^{-\alpha} \mu(xA).$$
Examples:

1. If $X_1 > 0$ and $X_2 > 0$ are iid $RV(\alpha)$, then $X = (X_1, X_2)$ is multivariate regularly varying with index $\alpha$ and *spectral distribution*

   $$P(\theta = (0,1)) = P(\theta = (1,0)) = 0.5 \text{ (mass on axes).}$$

   Interpretation: Unlikely that $X_1$ and $X_2$ are very large at the same time.

   **Figure:** plot of $(X_{t1}, X_{t2})$ for realization of 10,000.
2. If \( X_1 = X_2 > 0 \), then \( \mathbf{X} = (X_1, X_2) \) is multivariate regularly varying with index \( \alpha \) and spectral distribution

\[
P( \mathbf{\theta} = (1/\sqrt{2}, 1/\sqrt{2}) ) = 1.
\]

3. AR(1): \( X_t = 0.9 X_{t-1} + Z_t \), \( \{Z_t\} \sim \text{IID symmetric stable (1.8)} \)

Distr of \( \mathbf{\theta} \):

\[
\pm(1,0.9)/\sqrt{1.81}, \text{ W.P. } 0.9898 \\
\pm(0,1), \quad \text{ W.P. } 0.0102
\]

**Figure:** plot of \((X_t, X_{t+1})\) for realization of 10,000.
Applications of multivariate regular variation

- Domain of attraction for *sums of iid random vectors* (Rvaceva, 1962). That is, when does the partial sum
  \[ a_n^{-1} \sum_{t=1}^{n} X_t \]
  converge for some constants \( a_n \)?
- *Spectral measure* of multivariate stable vectors.
- *Domain of attraction* for componentwise maxima of iid random vectors (Resnick, 1987). Limit behavior of
  \[ a_n^{-1} \bigvee_{t=1}^{n} X_t \]
- Weak convergence of *point processes* with iid points.
- Solution to *stochastic recurrence equations*, \( Y_t = A_t Y_{t-1} + B_t \)
- Weak convergence of *sample autocovariances*. 
Applications of multivariate regular variation (cont)

**Linear combinations:**

\( \mathbf{X} \sim \text{RV}(\alpha) \Rightarrow \) all linear combinations of \( \mathbf{X} \) are regularly varying

i.e., there exist \( \alpha \) and slowly varying fcn \( L(.) \), s.t.

\[
P(c^\top \mathbf{X} > t)/(t^\alpha L(t)) \to w(c), \text{ exists for all real-valued } c,
\]

where

\[
w(tc) = t^{-\alpha}w(c).
\]

Use vague convergence with \( A_c = \{ \mathbf{y} : c^\top \mathbf{y} > 1 \} \), i.e.,

\[
\frac{P(\mathbf{X} \in tA_c)}{t^{-\alpha} L(t)} = \frac{P(c^\top \mathbf{X} > t)}{P(|\mathbf{X}| > t)} \to \mu(A_c) =: w(c),
\]

where \( t^{-\alpha} L(t) = P(|\mathbf{X}| > t) \).
Applications of multivariate regular variation (cont)

Converse?

\( X \sim RV(\alpha) \iff \) all linear combinations of \( X \) are regularly varying?

There exist \( \alpha \) and slowly varying fcn \( L(.) \), s.t.

\[
\text{(LC)} \quad P(c^TX > t)/(t^{\alpha}L(t)) \to w(c), \text{ exists for all real-valued } c.
\]

Theorem (Basrak, Davis, Mikosch, `02). Let \( X \) be a random vector.

1. If \( X \) satisfies (LC) with \( \alpha \) non-integer, then \( X \) is \( RV(\alpha) \).

2. If \( X > 0 \) satisfies (LC) for non-negative \( c \) and \( \alpha \) is non-integer, then \( X \) is \( RV(\alpha) \).

3. If \( X > 0 \) satisfies (LC) with \( \alpha \) an odd integer, then \( X \) is \( RV(\alpha) \).
Applications of multivariate regular variation (cont)

There exist $\alpha$ and slowly varying fcn $L(.)$, s.t.

\[(LC) \quad \frac{P(c^T X > t)}{(t^\alpha L(t))} \rightarrow w(c)\], exists for all real-valued $c$.

1. If $X$ satisfies (LC) with $\alpha$ non-integer, then $X$ is RV($\alpha$).
2. If $X > 0$ satisfies (LC) for non-negative $c$ and $\alpha$ is non-integer, then $X$ is RV($\alpha$).
3. If $X > 0$ satisfies (LC) with $\alpha$ an odd integer, then $X$ is RV($\alpha$).

Remark: Hult and Lindskog (2005) show that .

- 1 cannot be extended to integer $\alpha$.
- 2 cannot be extended to integer $\alpha$.
- It is unknown if 3 can be extended to even integers.
1. Kesten (1973). Under general conditions, (LC) holds with \( L(t) = 1 \) for stochastic recurrence equations of the form

\[ Y_t = A_t Y_{t-1} + B_t, \quad (A_t, B_t) \sim \text{IID}, \]

\[ A_t \text{ } d \times d \text{ random matrices, } B_t \text{ random } d\text{-vectors}. \]

It follows that the distributions of \( Y_t \), and in fact all of the finite dim’l distrs of \( Y_t \) are regularly varying (if \( \alpha \) is non-even).

2. GARCH processes. Since squares of a GARCH process can be embedded in a SRE, the finite dimensional distributions of a GARCH are regularly varying.
Example of ARCH(1): \( X_t = (\alpha_0 + \alpha_1 X_{t-1}^2)^{1/2} Z_t, \quad \{Z_t\} \sim \text{IID}. \)

\( \alpha \) found by solving \( \mathbb{E}|_{\alpha_1} Z^2|^{\alpha/2} = 1. \)

\[
\begin{array}{c|cccc}
\alpha_1 & .312 & .577 & 1.00 & 1.57 \\
\hline
\alpha & 8.00 & 4.00 & 2.00 & 1.00
\end{array}
\]

Distr of \( \theta \):

\[
P(\theta \in \bullet) = \mathbb{E}\{||(B,Z)||^\alpha I(\text{arg}((B,Z)) \in \bullet)\}/ \mathbb{E}|(B,Z)||^\alpha
\]

where

\[
P(B = 1) = P(B = -1) = .5
\]
Example of ARCH(1): \( \alpha_0=1, \alpha_1=1, \alpha=2 \), \( X_t=(\alpha_0+\alpha_1 X_{t-1}^2)^{1/2}Z_t, \) \( \{Z_t\} \sim \text{IID} \)

**Figures:** plots of \((X_t, X_{t+1})\) and estimated distribution of \(\theta\) for realization of 10,000.
Excursion to time-reversibility

Reversibility. A stationary sequence of random variables \( \{X_t\} \) is **time-reversible** if \( (X_1, \ldots, X_n) \sim_d (X_n, \ldots, X_1) \) for all \( n > 1 \).

Results: i) IID sequences \( \{Z_t\} \) are time-reversible.

ii) Linear time series (with a couple obvious exceptions) are time-reversible iff **Gaussian**. (Breidt and Davis `91)

Application: If plot of time series does not look time-reversible, then it **cannot** be modeled as IID or a Gaussian process. Use the “flip and compare” inspection test!
Reversibility. *Does the following series look time-reversible?*
Example of ARCH(1): \( \alpha_0 = 1, \alpha_1 = 1, \alpha = 2, X_t = (\alpha_0 + \alpha_1 X_{t-1}^2)^{1/2}Z_t, \{Z_t\} \sim \text{IID} \)

Is this process time-reversible?

**Figures:** plots of \((X_t, X_{t+1})\) and \((X_{t+1}, X_t)\) implies *non-reversible.*
Example: SV model $X_t = \sigma_t Z_t$

Suppose $Z_t \sim RV(\alpha)$ and

$$X_t = \sigma_t Z_t, \quad \log \sigma_t^2 = \phi_0 + \phi_1 \log \sigma_{t-1}^2 + \epsilon_t, \quad \{\epsilon_t\} \sim \text{IID } N(0, \sigma^2)$$

Then $Z_n = (Z_1, \ldots, Z_n)'$ is regularly varying with index $\alpha$ and so is $X_n = (X_1, \ldots, X_n)' = \text{diag}(\sigma_1, \ldots, \sigma_n) Z_n$

with spectral distribution concentrated on $(\pm 1,0)$, $(0, \pm 1)$.

**Figure:** plot of $(X_t, X_{t+1})$ for realization of 10,000.
Example: SV model $X_t = \sigma_t Z_t$

- SV processes are time-reversible if log-volatility is Gaussian.
- Asymptotically time-reversible if log-volatility is non-Gaussian.
Theorem (Davis & Hsing `95, Davis & Mikosch `97). Let \( \{X_t\} \) be a stationary sequence of random \( m \)-vectors. Suppose

(i) finite dimensional distributions are jointly regularly varying (let \((\theta_{-k}, \ldots, \theta_k)\) be the vector in \( S^{(2k+1)m-1} \) in the definition).

(ii) mixing condition \( A(\mathbf{a}_n) \) or strong mixing.

(iii) \( \lim \limsup_{k \to \infty} \lim_{n \to \infty} P(\bigvee_{k \leq |t| \leq n} |X_t| > a_n y | X_0 | > a_n y) = 0. \)

Then

\[
\gamma = \lim_{k \to \infty} E(|\theta_0^{(k)}|^\alpha - \sqrt[k]{\sum_{j=1}^{k} |\theta_j^{(k)}|^\alpha}) / E |\theta_0^{(k)}|^\alpha
\]

(extrimal index)

exists. If \( \gamma > 0 \), then

\[
N_n := \sum_{t=1}^{n} \varepsilon_{X_t/a_n} \overset{d}{\to} N := \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varepsilon_{P_{ij}},
\]
Point process convergence (cont)

- \((P_i)\) are points of a Poisson process on \((0, \infty)\) with intensity function 
  \[ \nu(dy) = \gamma \alpha y^{-\alpha - 1} dy. \]

- \(\sum_{i=1}^{\infty} \varepsilon_{Q_i} \), \(i \geq 1\), are iid point process with distribution \(Q\), and \(Q\) is the weak limit of

\[
\lim_{k \to \infty} E \left( \left| \theta_0^{(k)} \right|^\alpha - \sum_{j=1}^{k} \left| \theta_j^{(k)} \right| \right)_+ I \left( \sum_{|i| \leq k} \varepsilon_{\theta_{j}^{(k)}} \right) / E \left( \left| \theta_0^{(k)} \right|^\alpha - \sum_{j=1}^{k} \left| \theta_j^{(k)} \right| \right)_+. 
\]

Remarks:

1. GARCH and SV processes satisfy the conditions of the theorem.

2. Limit distribution for sample extremes and sample ACF follows from this theorem.
Extremes for GARCH and SV processes

**Setup**

- $X_t = \sigma_t Z_t$, \( \{Z_t\} \sim \text{IID (0,1)} \)
- $X_t$ is RV (\( \alpha \))
- Choose \( \{b_n\} \) s.t. \( nP(X_t > b_n) \to 1 \)

Then

\[ P^n(b_n^{-1} X_1 \leq x) \to \exp\{-x^{-\alpha}\}. \]

Then, with \( M_n = \max\{X_1, \ldots, X_n\} \),

(i) **GARCH:**

\[ P(b_n^{-1} M_n \leq x) \to \exp\{-\gamma x^{-\alpha}\}, \]

\( \gamma \) is extremal index \( 0 < \gamma < 1 \).

(ii) **SV model:**

\[ P(b_n^{-1} M_n \leq x) \to \exp\{-x^{-\alpha}\}, \]

extremal index \( \gamma = 1 \) no clustering.
Remarks about extremal index.

(i) $\gamma < 1$ implies clustering of exceedances

(ii) Numerical example. Suppose $c$ is a threshold such that

$$P^n(b^{-1}_n X_1 \leq c) \sim .95$$

Then, if $\gamma = .5$, $P(b^{-1}_n M_n \leq c) \sim (.95)^5 = .975$

(iii) $1/\gamma$ is the mean cluster size of exceedances.

(iv) Use $\gamma$ to discriminate between GARCH and SV models.

(v) Even for the light-tailed SV model (i.e., $\{Z_t\} \sim$ IID $N(0,1)$, the extremal index is 1 (see Breidt and Davis `98 )
Extremes for GARCH and SV processes (cont)

Absolute values of ARCH
Extremes for GARCH and SV processes (cont)

Absolute values of SV process
Extremal Index Estimates for Amazon

\[ \gamma_1 = \text{block method} \]
\[ \gamma_2 = \frac{1}{\text{mean cluster size}} \]
\[ \gamma_3 = \text{interval method (Ferro and Segers)} \]
\[ \gamma_4 = \text{interval method (Ferro and Segers)} \]
Summary of results for ACF of GARCH(p,q) and SV models

**GARCH(p,q)**

\( \alpha \in (0,2) \):

\[
(\hat{\rho}_X(h))_{h=1,\ldots,m} \overset{d}{\longrightarrow} (V_h/V_0)_{h=1,\ldots,m},
\]

\( \alpha \in (2,4) \):

\[
\left(n^{1-2/\alpha} \hat{\rho}_X(h)\right)_{h=1,\ldots,m} \overset{d}{\longrightarrow} \gamma_X^{-1}(0)(V_h)_{h=1,\ldots,m}.
\]

\( \alpha \in (4,\infty) \):

\[
\left(n^{1/2} \hat{\rho}_X(h)\right)_{h=1,\ldots,m} \overset{d}{\longrightarrow} \gamma_X^{-1}(0)(G_h)_{h=1,\ldots,m}.
\]

**Remark:** Similar results hold for the sample ACF based on \(|X_t|\) and \(X_t^2\).
Summary of results for ACF of GARCH(p,q) and SV models (cont)

SV Model

\( \alpha \in (0, 2) : \)

\[
\left( \frac{n}{\ln n} \right)^{1/\alpha} \hat{\rho}_X(h) \xrightarrow{d} \frac{\| \sigma_1 \sigma_{h+1} \|_\alpha}{\| \sigma_1 \|_\alpha^2} \frac{S_h}{S_0}.
\]

\( \alpha \in (2, \infty) : \)

\[
\left( n^{1/2} \hat{\rho}_X(h) \right)_{h=1, \ldots, m} \xrightarrow{d} \gamma_X^{-1}(0) (G_h)_{h=1, \ldots, m}.
\]
Sample ACF for GARCH and SV Models (1000 reps)

(a) GARCH(1,1) Model, n=10000

(b) SV Model, n=10000
Sample ACF for Squares of GARCH (1000 reps)

(a) GARCH(1,1) Model, n=10000

(b) GARCH(1,1) Model, n=100000
Sample ACF for Squares of SV (1000 reps)

(c) SV Model, n=10000

(d) SV Model, n=100000
Amazon returns (GARCH model)

GARCH(1,1) model fit to Amazon returns:

$$\alpha_0 = 0.0002493, \alpha_1 = 0.0385, \beta_1 = 0.957, X_t = (\alpha_0 + \alpha_1 X^2_{t-1})^{1/2} Z_t, \quad \{Z_t\} \sim \text{IID } t(3.672)$$

Simulation from GARCH(1,1) model
Amazon returns (SV model)

Stochastic volatility model fit to Amazon returns:

[Graphs showing ACF of squares and ACF of abs values]
Wrap-up

• *Regular variation* is a flexible tool for modeling both *dependence* and *tail heaviness*.

• Useful for establishing *point process convergence* of heavy-tailed time series.

• *Extremal index* $\gamma < 1$ for GARCH and $\gamma = 1$ for SV.

• ACF has faster convergence for SV.