Extreme Value Theory for Space-Time Processes
With Heavy-Tailed Distributions

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Outline

- A Class of Space-Time Processes: \( X_t(s) = \sum_{i=0}^{\infty} \psi_i(s) Z_{t-i}(s) , \quad s \in [0, 1]^d \)
  - Dependence properties
- Preliminaries on Regular Variation on \( \mathbb{D}([0, 1]^d) \)
- Point Process Convergence
  - Basic properties
- Application
EVT for Space-Time Processes

Basic Set-up: 2 components, spatial and temporal.

**Spatial part.** Let $Z(s)$ be a random field on $[0, 1]^d$.

- Usually $d = 1$ (transect) or $d = 2$ (two-dimensional space).
- $Z(s)$ is value of the random field at location $s \in [0, 1]^d$.
- View $Z(s)$ as a random element of $\mathbb{D} = \mathbb{D}([0, 1]^d)$ of càdlàg functions.
  - $J_1$-topology; see Bickel and Wichura (1971).
- Will assume that $Z$ has *regularly varying tail probabilities*—to be described later.
EVT for Space-Time Processes

**Temporal part.** Build in serial dependence by filtering the random field at each location $s \in [0, 1]^d$. That is, set

$$X_t(s) = \sum_{i=0}^{\infty} \psi_i(s) Z_{t-i}(s), \quad s \in [0, 1]^d,$$

where

- $\{Z_t, t = 0, \pm 1, \pm 2, \ldots, \}$ is an iid sequence of random fields on $[0, 1]^d$,
- $\psi_i$’s are deterministic *continuous* real-valued fields on $[0, 1]^d$.

**Note:** For $s_1, \ldots, s_k$ fixed,

$$X_t := \begin{bmatrix} X_t(s_1) \\
\vdots \\
X_t(s_k) \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{\infty} \psi_i(s_1) Z_{t-i}(s_1) \\
\vdots \\
\sum_{i=0}^{\infty} \psi_i(s_k) Z_{t-i}(s_k) \end{bmatrix} = \sum_{i=0}^{\infty} A_i Z_{t-i}$$

is a multivariate linear time series.
Dependence structure of \((X_t)\)

Suppose the random field \(Z(s)\) is stationary with covariance function \(\gamma_Z(u)\),

\[
\text{Cov}(Z(s + u), Z(s)) = \gamma_Z(u).
\]

Spatial covariance of \(X_t\).

\[
\text{Cov}(X_t(s + u), X_t(s)) = \left( \sum_{j=0}^{\infty} \psi_j(s + u)\psi_j(s) \right) \gamma_Z(u),
\]

which is stationary in space (independent of \(s\)) if the \(\psi_j\)'s are constant functions. In this case,

\[
\text{Cov}(X_t(s + u), X_t(s)) = \left( \sum_{j=0}^{\infty} \psi_j^2 \right) \gamma_Z(u).
\]
Dependence structure of \((X_t)\)

**Time covariance function of** \(X_t(s)\). For each \(s \in [0, 1]^d\), the time series \(X_t(s)\) is a **linear process** with covariance function

\[
\text{Cov}(X_{t+h}(s), X_t(s)) = \left( \sum_{j=0}^{\infty} \psi_{j+h}(s)\psi_j(s) \right) \gamma_Z(0)
\]

If the \(\psi_j\)'s are constant functions, then the serial correlation does not depend on \(s\).

**Note:** In fact, the time series \(X_t\) defined on \([0, 1]^d\) is strictly stationary.

**Space-time covariance function of** \(X_t(s)\).

\[
\text{Cov}(X_{t+h}(s + u), X_t(s)) = \left( \sum_{j=0}^{\infty} \psi_{j+h}(s + u)\psi_j(s) \right) \gamma_Z(u)
\]
which, if the $\psi_j$'s are constant functions, is equal to

$$\gamma_X(h, u) = \text{Cov}(X_{t+h}(s + u), X_t(s))$$

$$= \left( \sum_{j=0}^{\infty} \psi_{j+h}\psi_j \right) \gamma_Z(u)$$

$$= \gamma_T(h) \gamma_Z(u)$$

Remarks:

(1) The filter functions $\psi_j$'s influence both the spatial and temporal covariances.

(2) If the $\psi_j$ are constant functions, then $X_t$ has a multiplicative covariance function, i.e.,

$$\gamma_X(h, u) = \text{Cov}(X_{t+h}(s + u), X_t(s))$$

$$= \gamma_T(h) \gamma_Z(u)$$
Examples and Applications

1. Maximum ozone levels. Suppose there exists a standard $L$ for annual maxima of ozone levels over the rectangular region $[0, 1]^2$. Set

$$X_t(s) = \text{maximum ozone level at site } s \text{ during year } t.$$ 

Then the probability the standard $L$ is not exceeded in $n$ consecutive years is

$$P\left( \max_{t=1,\ldots,n} X_t(s) \leq L, \text{ for all } s \in [0, 1]^2 \right).$$

2. Sea level (de Haan and Lin (2001)). Let $f(s)$ represent the height of a dyke off the Dutch coast at location $s$ and set

$$X_t(s) = \text{maximum sea level at site } s \text{ during day } t$$

The probability that the dyke is not breached along the coast for $n$ consecutive days is

$$P\left( \max_{t=1,\ldots,n} X_t(s) \leq f(s), \text{ for all } s \in [0, 1] \right).$$
Regular variation on $\mathbb{D}[0, 1]^d$—preliminaries:

Regular variation of $\mathbf{Z} = (Z_1, \ldots, Z_m)'$. There exists a random vector $\mathbf{\theta}$ defined on $\mathbb{S}^{m-1}$ such that

$$P(\|\mathbf{Z}\| > tz, \mathbf{Z}/\|\mathbf{Z}\| \in \cdot)/P(\|\mathbf{Z}\| > t) \xrightarrow{w} z^{-\alpha} P(\mathbf{\theta} \in \cdot),$$

as $t \to \infty$ where $\xrightarrow{w}$ is vague convergence on $\mathbb{S}^{m-1}$, the unit sphere in $\mathbb{R}^m$.

- $P(\mathbf{\theta} \in \cdot)$ is called the spectral measure.
- $\alpha$ is the index of regular variation.

**Equivalence:** There exists $a_n > 0$ such that for all $z > 0$

$$nP(\|\mathbf{Z}\| > a_n z, \mathbf{Z}/\|\mathbf{Z}\| \in \cdot) \xrightarrow{w} z^{-\alpha} P(\mathbf{\theta} \in \cdot)$$

or, equivalently,

$$nP(a_n^{-1} \mathbf{Z} \in \cdot) \xrightarrow{w} m(\cdot)$$

for some Radon measure $m$ on $\mathcal{B}(\overline{\mathbb{R}}^m \setminus \{0\})$. 
Examples of Regular Variation on $\mathbb{R}^2$:

1. If $Z_1 > 0$ and $Z_2 > 0$ are iid RV($\alpha$), then $Z = (Z_1, Z_2)$ is regularly varying with index $\alpha$ and spectral distribution

$$P(\theta = (0, 1)) = P(\theta = (1, 0)) = 0.5 \text{ (mass on axes).}$$

**Interpretation:** Unlikely that $Z_1$ and $Z_2$ are both large at the same time.

**Figure:** plot of $(Z_{t1}, Z_{t2})$ for realization of 10,000.
Examples of Regular Variation on $\mathbb{R}^2$:

2. If $Z_1 = Z_2 > 0$ and $\text{RV}(\alpha)$, then $Z = (Z_1, Z_2)$ is regularly varying with index $\alpha$ and spectral distribution

$$P(\theta = (1/\sqrt{2}, 1/\sqrt{2})) = 1.$$  

3. AR(1): $Z_t = .9Z_{t-1} + \epsilon_t$, $\epsilon_t \sim \text{IID symmetric stable (1.8)}$. Then $Z = (Z_1, Z_2)$ is RV(1.8) with spectral measure

$$\begin{cases} 
P(\theta = (1,.9)/\sqrt{1.81}) = .9898 \\
P(\theta = (0,1)) = .0102 
\end{cases}$$

**Figure:** plot of $(Z_t, Z_{t+1})$ for realization of 10,000.
Regular variation on $\mathbb{D}[0, 1]^d$

**Polar coordinate transformation:** For the càdlàg field $x \in \mathbb{D}\setminus\{0\}$

$$x \leftrightarrow (\|x\|_{\infty}, \tilde{x}), \quad \tilde{x} = x/\|x\|_{\infty},$$

where $\|x\|_{\infty}$ is the sup-norm of $x$, and 0 represents the zero function: We write

$$\overline{\mathbb{D}} = (0, \infty] \times \mathcal{S}, \text{ where } \mathcal{S} = \{\tilde{x} : x \in \mathbb{D}\setminus\{0\}\}.$$

Reg variation on $\mathbb{D} = \mathbb{D}([0, 1]^d)$ (de Haan and Lin '01; Hult and Lindskog '03).

$X$ is *regularly varying with spectral measure* $\sigma$ on $\mathcal{S}$ and index $\alpha > 0$, if there exists $a_n > 0$ such that for all $t > 0$,

$$n\ P(\|X\|_{\infty} > a_n, \tilde{X} \in \cdot) \overset{w}{\to} t^{-\alpha} \sigma(\cdot),$$

where $\overset{w}{\to}$ denotes weak convergence on $\mathcal{B}(\mathcal{S})$. This convergence is equivalent to (Hult and Lindskog (2003))

$$n\ P(a_n^{-1}X \in \cdot) \overset{\tilde{w}}{\to} m(\cdot).$$
Here \( \hat{w} \) denotes weak convergence of measures in the sense

\[
m_n(f) = \int f \, dm_n \to \int f \, dm = m(f)
\]

for all bounded continuous functions \( f \) on \( \mathbb{D}\backslash\{0\} \) which vanish outside a bounded set (see Appendix A2.6 in Daley and Vere-Jones (1988)), and \( m \) is a measure such that \( \mu(\overline{\mathbb{D}}\backslash\mathbb{D}) = 0 \);
Characterization of regular variation on $\mathbb{D}$

**Proposition 1.** (Hult and Lindskog (2003)) $Z$ is regularly varying if and only if there exist a $a_n > 0$ such that and a collection of Radon measures $m_{s_1, \ldots, s_k},\ s_i \in [0, 1]^d, \ i = 1, \ldots, k, \ k \geq 1$, not all of them being the null measure, with $m_{s_1, \ldots, s_k}(\mathbb{R}^k \setminus \mathbb{R}^k) = 0$, such that the following conditions hold:

1) Finite dimensional convergence:

$$n P(a_n^{-1}(Z(s_1), \ldots, Z(s_k)) \in \cdot) \xrightarrow{v} m_{s_1, \ldots, s_k}(\cdot).$$

2) **Tightness.** For any $\epsilon, \eta > 0$ there exist $\delta \in (0, 0.5)$ and $n_0$ such that for $n \geq n_0$,

$$n P(w''(Z, \delta) > a_n \epsilon) \leq \eta,$$

$$n P(w(Z, [0, 1]^d \setminus [\delta, 1-\delta]^d) > a_n \epsilon) \leq \eta.$$ 

**Note.** The measures $m_{s_1, \ldots, s_k}, \ s_i \in [0, 1]^d, \ i = 1, \ldots, k, \ k \geq 1$, determine the limiting measure $m$ in the definition of regular variation of $Z$. 

Application to Space-Time Processes

**Proposition 2.** Assume that \( \{Z_t\} \) is an iid sequence of random fields on \( \mathbb{D} \) such that \( Z \) is regularly varying with index \( \alpha \) and limiting measure \( m_Z \). Suppose \( (\psi_i) \) is a sequence of continuous fields with

\[
\sum_{i=0}^{\infty} \|\psi_i\|_{\infty}^{\min(1, \alpha - \epsilon)} < \infty
\]

for some \( \epsilon \in (0, \alpha) \). Then the infinite series

\[
X = \sum_{i=0}^{\infty} \psi_i Z_i
\]

converges a.s. in \( \mathbb{D} \) and is regularly varying with index \( \alpha \) and limiting measure

\[
m = \sum_{i=0}^{\infty} m_Z \circ \psi_i^{-1}.
\]
Application to Space-Time Processes

Main ideas behind proof:

- Show convergence by bounding the sup norm and using the fact that $\|Z_i\|_\infty$ is regularly varying.
- First establish regular variation for finite sums by checking conditions (fidi convergence and tightness) of Proposition 1.
- Extend to infinite sums by approximating the tail sums.
Point Process Convergence

Point process convergence for the $Z_t$'s. From Proposition 1, it follows that

$$I_n = \sum_{t=1}^{n} \varepsilon_{a_n^{-1}Z_t} \xrightarrow{d} I = \sum_{j=1}^{\infty} \varepsilon_{P_j}.$$ 

where $\xrightarrow{d}$ denotes convergence in distribution of point processes on the space $\widehat{M}(\overline{D}\setminus\{0\})$ and $\sum_{j=1}^{\infty} \varepsilon_{P_j}$ is a Poisson random measure on $\overline{D}\setminus\{0\}$ with intensity measure $m_\mathcal{Z}$.

Note: The space $\widehat{M}(\overline{D}^m\setminus\{0\})$ is the space of point measures on $\overline{D}^m\setminus\{0\}$ endowed with the topology generated by $\widehat{w}$-convergence.

Theorem.

$$N_n = \sum_{t=1}^{n} \varepsilon_{a_n^{-1}X_t} \xrightarrow{d} N = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \varepsilon_{\psi_i P_j}.$$ 

Remark: This theorem generalizes the Davis and Resnick (1985) point process convergence result for linear processes.
From the Theorem, we have

\[ P\left(a_n^{-1} \max_{t=1,\ldots,n} \|X_t\|_\infty \leq x \right) \rightarrow P\left(\sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \varepsilon_{\psi_i \psi_j} \|X_i\|_\infty (x, \infty) = 0\right) \]

\[ = \exp\left\{ -m_Z(B) \right\} \]

where

\[ B = \{ y : \|\psi_i y\| > x, \text{ for some } i = 0, 1, \ldots \} \]

If the \(\psi_i\)'s are constant functions, then

\[ B = \{ y : \|y\| > x/\psi_+ \} \]

and

\[ \exp\left\{ -m_Z(B) \right\} = \exp\left\{ -x^{-\alpha} \psi_+^\alpha \right\} \]

where \(\psi_+ = \max_j |\psi_j|\).

Extremal index \(= \frac{\psi_+^\alpha}{\sum_{i=0}^{\infty} |\psi_i|^\alpha}\).