For concreteness, suppose that we have a game in which there are 40 balls numbered 1 to 40 and six are drawn without replacement to determine the winning combination. A ticket purchase requires the customer to choose six different numbers from 1 to 40 and pay a fee. This game has $\binom{40}{6} = 3{,}838{,}380$ different winning combinations and the same number of possible tickets. One piece of advice often found in published lottery aids is not to choose the six numbers on your ticket too far apart. Many people tend to pick their six numbers uniformly spread out from 1 to 40, but the winning combination often has two consecutive numbers or at least two numbers very close together. Some of these "advisors" recommend that, since it is more likely that there will be numbers close together, players should bunch some of their six numbers close together. Such advice might make sense in order to avoid choosing the same numbers as other players in a parimutuel game (i.e., a game in which all winners share the jackpot). But the idea that any strategy can improve your chances of winning is misleading.

To see why this advice is misleading, let E be the event that the winning combination contains at least one pair of consecutive numbers. The reader can calculate $\Pr(E)$ in Exercise 13 in Section 1.12. For this example, $\Pr(E) = 0.577$. So the lottery aids are correct that E has high probability. However, by claiming that choosing a ticket in E increases your chance of winning, they confuse the probability of the event E with the probability of each outcome in E. If you choose the ticket $(5, 7, 14, 23, 24, 38)$, your probability of winning is only 1/3,828,380 just as it would be if you chose any other ticket. The fact that this ticket happens to be in E doesn't make your probability of winning equal to 0.577. The reason that $\Pr(E)$ is so big is that so many different combinations are in E. Each of those combinations still has probability 1/3,828,380 of winning, and you only get one combination on each ticket. The fact that there are so many combinations in E does not make each one any more likely than anything else.

1.12 Supplementary Exercises

1. Suppose that a coin is tossed seven times. Let A denote the event that a head is obtained on the first toss, and let B denote the event that a head is obtained on the fifth toss. Are A and B disjoint?

2. If A, B, and D are three events such that $\Pr(A \cup B \cup D) = 0.7$, what is the value of $\Pr(A^c \cap B^c \cap D^c)$?

3. Suppose that a certain precinct contains 350 voters, of which 250 are Democrats and 100 are Republicans. If 30 voters are chosen at random from the precinct, what is the probability that exactly 18 Democrats will be selected?

4. Suppose that in a deck of 20 cards, each card has one of the numbers 1, 2, 3, 4, or 5 and there are four cards with each number. If 10 cards are chosen from the deck at random, without replacement, what is the probability that each of the numbers 1, 2, 3, 4, and 5 will appear exactly twice?

5. Consider the contractor in Example 1.5.2 on page 16. He wishes to compute the probability that the total utility demand is high, meaning that the sum of water and electrical demand (in the units of Example 1.4.2) is at least 215. Draw a picture of this event on a graph like Fig. 1.5 or Fig. 1.8 and find its probability.

6. Suppose that a box contains r red balls and w white balls. Suppose also that balls are drawn from the box one at a time, at random, without replacement. **(a)** What is the probability that all r red balls will be obtained before any white balls are obtained? **(b)** What is the probability that all r red balls will be obtained before two white balls are obtained?

7. Suppose that a box contains r red balls, w white balls, and b blue balls. Suppose also that balls are drawn from the box one at a time, at random, without replacement. What is the probability that all r red balls will be obtained before any white balls are obtained?

8. Suppose that 10 cards, of which seven are red and three are green, are put at random into 10 envelopes, of which seven are red and three are green, so that each envelope contains one card. Determine the probability that exactly k envelopes will contain a card with a matching color ($k = 0, 1, \dots, 10$).

9. Suppose that 10 cards, of which five are red and five are green, are put at random into 10 envelopes, of which seven are red and three are green, so that each envelope contains one card. Determine the probability that exactly k envelopes will contain a card with a matching color ($k = 0, 1, \dots, 10$).

10. Suppose that the events A and B are disjoint. Under what conditions are A^c and B^c disjoint?

11. Let A_1, A_2, and A_3 be three arbitrary events. Show that the probability that exactly one of these three events will occur is

$$\Pr(A_1) + \Pr(A_2) + \Pr(A_3) - 2\Pr(A_1A_2) - 2\Pr(A_1A_3) - 2\Pr(A_2A_3) + 3\Pr(A_1A_2A_3).$$

12. Let $A_1, \dots, A_n$ be n arbitrary events. Show that the probability that exactly one of these n events will occur is

$$\sum_{i=1}^{n} \Pr(A_i) - 2\sum_{i<j} \Pr(A_iA_j) + 3\sum_{i<j<k} \Pr(A_iA_jA_k) - \cdots + (-1)^{n+1} n \Pr(A_1A_2\cdots A_n).$$

13. Consider a state lottery game in which each winning combination and each ticket consists of one set of k numbers chosen from the numbers 1 to n without replacement. We shall compute the probability that the winning combination contains at least one pair of consecutive numbers.

   a. Prove that if $n < 2k - 1$ every winning combination has at least one pair of consecutive numbers. For the rest of the problem assume that $n \ge 2k - 1$.

   b. Let $i_1 < \cdots < i_k$ be an arbitrary possible winning combination arranged in order from smallest to largest. For $s = 1, \dots, k$, let $j_s = i_s - (s - 1)$. That is, $j_1 = i_1$, $j_2 = i_2 - 1, \dots, j_k = i_k - (k - 1)$. Prove that $(i_1, \dots, i_k)$ contains at least one pair of consecutive numbers if and only if $(j_1, \dots, j_k)$ contains repeated numbers.

   c. Prove that $1 \le j_1 \le \cdots \le j_k \le n - k + 1$ and that the number of $(j_1, \dots, j_k)$ sets with no repeats is $\binom{n-k+1}{k}$.

   d. Find the probability that there is no pair of consecutive numbers in the winning combination.

   e. Find the probability of at least one pair of consecutive numbers in the winning combination.

Summary

The revised probability of an event A after learning that event B (with $\Pr(B) > 0$) has occurred is the conditional probability of A given B, denoted by $\Pr(A|B)$ and computed as $\Pr(AB)/\Pr(B)$. Often it is easy to assess a conditional probability, such as $\Pr(A|B)$, directly. In such a case, we can use the multiplication rule for conditional probabilities to compute $\Pr(AB) = \Pr(B)\Pr(A|B)$. All probability results have versions conditional on an event B with $\Pr(B) > 0$: just change *all* probabilities so that they are conditional on B in addition to anything else they were already conditional on. For example, the multiplication rule for conditional probabilities becomes $\Pr(A_1A_2|B) = \Pr(A_1|B)\Pr(A_2|A_1B)$.

EXERCISES

1. If $A \subset B$ with $\Pr(B) > 0$, what is the value of $\Pr(A|B)$?

2. If A and B are disjoint events and $\Pr(B) > 0$, what is the value of $\Pr(A|B)$?

3. If S is the sample space of an experiment and A is any event in that space, what is the value of $\Pr(A|S)$?

4. Each time a shopper purchases a tube of toothpaste, he chooses either brand A or brand B. Suppose that for each purchase after the first, the probability is 1/3 that he will choose the same brand that he chose on his preceding purchase and the probability is 2/3 that he will switch brands. If he is equally likely to choose either brand A or brand B on his first purchase, what is the probability that both his first and second purchases will be brand A and both his third and fourth purchases will be brand B?

5. A box contains r red balls and b blue balls. One ball is selected at random and its color is observed. The ball is then returned to the box and k additional balls of the same color are also put into the box. A second ball is then selected at random, its color is observed, and it is returned to the box together with k additional balls of the same color. Each time another ball is selected, the process is repeated. If four balls are selected, what is the probability that the first three balls will be red and the fourth ball will be blue?

6. A box contains three cards. One card is red on both sides, one card is green on both sides, and one card is red on one side and green on the other. One card is selected from the box at random, and the color on one side is observed. If this side is green, what is the probability that the other side of the card is also green?

7. Consider again the conditions of Exercise 2 of Section 1.10. If a family selected at random from the city subscribes to newspaper A, what is the probability that the family also subscribes to newspaper B?

8. Consider again the conditions of Exercise 2 of Section 1.10. If a family selected at random from the city subscribes to at least one of the three newspapers A, B, and C, what is the probability that the family subscribes to newspaper A?

9. Suppose that a box contains one blue card and four red cards, which are labeled A, B, C, and D. Suppose also that two of these five cards are selected at random, without replacement.
   a. If it is known that card A has been selected, what is the probability that both cards are red?
   b. If it is known that at least one red card has been selected, what is the probability that both cards are red?

10. Consider the following version of the game of craps: The player rolls two dice. If the sum on the first roll is 7 or 11, the player wins the game immediately. If the sum on the first roll is 2, 3, or 12, the player loses the game immediately. However, if the sum on the first roll is 4, 5, 6, 8, 9, or 10, then the two dice are rolled again and again until the sum is either 7 or 11 or the original value. If the original value is obtained a second time before either 7 or 11 is obtained, then the player wins. If either 7 or 11 is obtained before the original value is obtained a second time, then the player loses. Determine the probability that the player will win this game.

11. For any three events A, B, and D, such that $\Pr(D) > 0$, prove that $\Pr(A \cup B|D) = \Pr(A|D) + \Pr(B|D) - \Pr(AB|D)$.

7. Two students A and B are both registered for a certain course. Assume that student A attends class 80 percent of the time, student B attends class 60 percent of the time, and the absences of the two students are independent.
   a. What is the probability that at least one of the two students will be in class on a given day?
   b. If at least one of the two students is in class on a given day, what is the probability that A is in class that day?

8. If three balanced dice are rolled, what is the probability that all three numbers will be the same?

9. Consider an experiment in which a fair coin is tossed until a head is obtained for the first time. If this experiment is performed three times, what is the probability that exactly the same number of tosses will be required for each of the three performances?

10. The probability that any child in a certain family will have blue eyes is 1/4, and this feature is inherited independently by different children in the family. If there are five children in the family and it is known that at least one of these children has blue eyes, what is the probability that at least three of the children have blue eyes?

11. Consider the family with five children described in Exercise 10.
    a. If it is known that the youngest child in the family has blue eyes, what is the probability that at least three of the children have blue eyes?
    b. Explain why the answer in part (a) is different from the answer in Exercise 10.

12. Suppose that A, B, and C are three independent events such that $\Pr(A) = 1/4$, $\Pr(B) = 1/3$, and $\Pr(C) = 1/2$. **(a)** Determine the probability that none of these three events will occur. **(b)** Determine the probability that exactly one of these three events will occur.

13. Suppose that the probability that any particle emitted by a radioactive material will penetrate a certain shield is 0.01. If 10 particles are emitted, what is the probability that exactly one of the particles will penetrate the shield?

14. Consider again the conditions of Exercise 13. If 10 particles are emitted, what is the probability that at least one of the particles will penetrate the shield?

15. Consider again the conditions of Exercise 13. How many particles must be emitted in order for the probability to be at least 0.8 that at least one particle will penetrate the shield?

16. In the World Series of baseball, two teams A and B play a sequence of games against each other, and the first team that wins a total of four games becomes the winner of the World Series. If the probability that team A will win any particular game against team B is 1/3, what is the probability that team A will win the World Series?

17. Two boys A and B throw a ball at a target. Suppose that the probability that boy A will hit the target on any throw is 1/3 and the probability that boy B will hit the target on any throw is 1/4. Suppose also that boy A throws first and the two boys take turns throwing. Determine the probability that the target will be hit for the first time on the third throw of boy A.

18. For the conditions of Exercise 17, determine the probability that boy A will hit the target before boy B does.

19. A box contains 20 red balls, 30 white balls, and 50 blue balls. Suppose that 10 balls are selected at random one at a time, with replacement; that is, each selected ball is replaced in the box before the next selection is made. Determine the probability that at least one color will be missing from the 10 selected balls.

20. Suppose that $A_1, \ldots, A_k$ are a sequence of k independent events. Let $B_1, \ldots, B_k$ be another sequence of k events such that for each value of j ($j = 1, \ldots, k$), either $B_j = A_j$ or $B_j = A_j^c$. Prove that $B_1, \ldots, B_k$ are also independent events. *Hint:* Use an induction argument based on the number of events B_j for which $B_j = A_j^c$.

21. Prove Theorem 2.2.2 on page 61. *Hint:* The "only if" direction is direct from the definition of independence on page 58. For the "if" direction, use induction on the value of j in the definition of independence. Let $m = j - 1$ and let $\ell = 1$ with $j_1 = i_j$.

22. Prove Theorem 2.2.3 on page 63.

23. A programmer is about to attempt to compile a series of 11 similar programs. Let A_i be the event that the ith program compiles successfully for $i = 1, \ldots, 11$. When the programming task is easy, the programmer expects that 80 percent of programs should compile. When the programming task is difficult, she expects that only 40 percent of the programs will compile. Let B be the event that the programming task was easy. The programmer believes that the events $A_1, \ldots, A_{11}$ are conditionally independent given B and given B^c.
    a. Compute the probability that exactly 8 out of 11 programs will compile given B.
    b. Compute the probability that exactly 8 out of 11 programs will compile given B^c.

By continuing in this way, we finally arrive at the probability $\Pr(A_1A_2\cdots A_r)$ that the pictures of all r players are missing from the n packages. Of course, this probability is 0. Therefore, by Eq. (1.10.2) of Section 1.10,

$$\Pr\left(\bigcup_{i=1}^{r} A_i\right) = r\left(\frac{r-1}{r}\right)^n - \binom{r}{2}\left(\frac{r-2}{r}\right)^n + \cdots + (-1)^r\binom{r}{r-1}\left(\frac{1}{r}\right)^n$$

$$= \sum_{j=1}^{r-1}(-1)^{j+1}\binom{r}{j}\left(1-\frac{j}{r}\right)^n.$$

Since the probability p of obtaining a complete set of r different pictures is equal to $1-\Pr(\bigcup_{i=1}^{r} A_i)$, it follows from the foregoing derivation that p can be written in the form

$$p = \sum_{j=0}^{r-1}(-1)^j\binom{r}{j}\left(1-\frac{j}{r}\right)^n.$$

Summary

A collection of events is independent if and only if learning that some of them occur does not change the probabilities that any combination of the rest of them occurs. Equivalently, a collection of events is independent if and only if the probability of the intersection of every subcollection is the product of the individual probabilities. The concept of independence has a version conditional on another event. A collection of events is independent conditional on B if and only if the conditional probability of the intersection of every subcollection given B is the product of the individual conditional probabilities given B. Equivalently, a collection of events is conditionally independent given B if and only if learning that some of them (and B) occur does not change the conditional probabilities given B that any combination of the rest of them occur. The full power of conditional independence will become more apparent after we introduce partitions in the next section.

EXERCISES

1. If A and B are independent events and $\Pr(B) < 1$, what is the value of $\Pr(A^c|B^c)$?
2. Assuming that A and B are independent events, prove that the events A^c and B^c are also independent.
3. Suppose that A is an event such that $\Pr(A) = 0$ and that B is any other event. Prove that A and B are independent events.
4. Suppose that a person rolls two balanced dice three times in succession. Determine the probability that on each of the three rolls, the sum of the two numbers that appear will be 7.
5. Suppose that the probability that the control system used in a spaceship will malfunction on a given flight is 0.001. Suppose further that a duplicate, but completely independent, control system is also installed in the spaceship to take control in case the first system malfunctions. Determine the probability that the spaceship will be under the control of either the original system or the duplicate system on a given flight.
6. Suppose that 10,000 tickets are sold in one lottery and 5000 tickets are sold in another lottery. If a person owns 100 tickets in each lottery, what is the probability that she will win at least one first prize?

will have a positive reaction is 0.95 and the probability that the person will have a negative reaction is 0.05. If the test is applied to a person who does not have this type of cancer, the probability that the person will have a positive reaction is 0.05 and the probability that the person will have a negative reaction is 0.95. Suppose that in the general population, one person out of every 100,000 people has this type of cancer. If a person selected at random has a positive reaction to the test, what is the probability that he has this type of cancer?

9. In a certain city, 30 percent of the people are Conservatives, 50 percent are Liberals, and 20 percent are Independents. Records show that in a particular election, 65 percent of the Conservatives voted, 82 percent of the Liberals voted, and 50 percent of the Independents voted. If a person in the city is selected at random and it is learned that she did not vote in the last election, what is the probability that she is a Liberal?

10. Suppose that when a machine is adjusted properly, 50 percent of the items produced by it are of high quality and the other 50 percent are of medium quality. Suppose, however, that the machine is improperly adjusted during 10 percent of the time and that, under these conditions, 25 percent of the items produced by it are of high quality and 75 percent are of medium quality.

a. Suppose that five items produced by the machine at a certain time are selected at random and inspected. If four of these items are of high quality and one item is of medium quality, what is the probability that the machine was adjusted properly at that time?

b. Suppose that one additional item, which was produced by the machine at the same time as the other five items, is selected and found to be of medium quality. What is the new posterior probability that the machine was adjusted properly?

11. Suppose that a box contains five coins, and that for each coin there is a different probability that a head will be obtained when the coin is tossed. Let p_i denote the probability of a head when the ith coin is tossed $(i = 1, \ldots, 5)$, and suppose that $p_1 = 0$, $p_2 = 1/4$, $p_3 = 1/2$, $p_4 = 3/4$, and $p_5 = 1$.

a. Suppose that one coin is selected at random from the box and when it is tossed once, a head is obtained. What is the posterior probability that the ith coin was selected $(i = 1, \ldots, 5)$?

b. If the same coin were tossed again, what would be the probability of obtaining another head?

c. If a tail had been obtained on the first toss of the selected coin and the same coin were tossed again, what would be the probability of obtaining a head on the second toss?

12. Consider again the box containing the five different coins described in Exercise 11. Suppose that one coin is selected at random from the box and is tossed repeatedly until a head is obtained.

a. If the first head is obtained on the fourth toss, what is the posterior probability that the ith coin was selected $(i = 1, \ldots, 5)$?

b. If we continue to toss the same coin until another head is obtained, what is the probability that exactly three additional tosses will be required?

13. Prove the conditional version of the law of total probability Eq. (2.3.1).

14. Consider again the conditions of Example 2.3.8, in which the phenotype of an individual was observed and found to be the dominant trait. For which values of i $(i = 1, \ldots, 6)$ is the posterior probability that the parents have the genotypes of event B_i smaller than the prior probability that the parents have the genotyes of event B_i?

15. Suppose that in Example 2.3.8 the observed individual has the recessive trait. Determine the posterior probability that the parents have the genotypes of event B_4.

16. In the clinical trial in Examples 2.3.10 and 2.3.11, suppose that we have only observed the first five patients and three of the five had been successes. Use the two different sets of prior probabilities from Examples 2.3.10 and 2.3.11 to calculate two sets of posterior probabilities. Are these two sets of posterior probabilities as close to each other as were the two in Examples 2.3.10 and 2.3.11? Why or why not?

17. Suppose that a box contains one fair coin and one coin with a head on each side. Suppose that a coin is drawn at random from this box and that we begin to flip the coin. In Eqs. (2.3.6) and (2.3.7), we computed the conditional probability that the coin was fair given that the first two flips both produce heads.

a. Suppose that the coin is flipped a third time and another head is obtained. Compute the probability that the coin is fair given that all three flips produced heads.

b. Suppose that the coin is flipped a fourth time and the result is tails. Compute the posterior probability that the coin is fair.

18. Consider again the conditions of Exercise 23 in Section 2.2. Let A be the event that exactly 8 out of 11 programs compiled. Compute the conditional probability of B given A.

19. Use the prior probabilities in Example 2.3.11 for the events $B_1, \dots, B_{11}$. Let E_1 be the event that the first patient is a success. Compute the probability of E_1 and explain why it is so much less than the value computed in Example 2.3.10.

20. Consider again the conditions of Exercise 2. Suppose that several parts will be observed and that the different parts are conditionally independent given each of the three states of repair of the machine. If seven parts are observed and exactly one is defective, compute the posterior probabilities of the three states of repair.

*2.4 Markov Chains

A popular model for systems that change over time in a random manner is the Markov chain model. In a Markov chain, the conditional probability of each future state given the past states and the present state depends only on the present state.

Stochastic Processes

Suppose that a certain business office has five telephone lines and that any number of these lines may be in use at any given time. During a certain period of time, the telephone lines are observed at regular intervals of 2 minutes and the number of lines that are being used at each time is noted. Let X_1 denote the number of lines that are being used when the lines are first observed at the beginning of the period; let X_2 denote the number of lines that are being used when they are observed the second time, 2 minutes later; and in general, for $n = 1, 2, \dots$, let X_n denote the number of lines that are being used when they are observed for the nth time.

The sequence of observations $X_1, X_2, \dots$ is called a *stochastic process*, or *random process*, because the values of these observations cannot be predicted precisely beforehand but probabilities can be specified for each of the different possible values at any particular time. A stochastic process like the one just described is called a process with a *discrete time parameter* because the lines are observed only at discrete or separated points in time, rather than continuously in time.

In a stochastic process, the first observation X_1 is called the *initial state* of the process; and for $n = 2, 3, \dots$, the observation X_n is called the *state of the process at time n*. In the preceding example, the state of the process at any time is the number of lines being used at that time. Therefore, each state must be an integer between 0 and 5. In the remainder of this chapter we shall consider only stochastic processes for which there are just a finite number of possible states at any given time.

In a stochastic process with a discrete time parameter, the state of the process varies in a random manner from time to time. To describe a complete probability model for a particular process, it is necessary to specify a probability for each of the possible values of the initial state X_1 and also to specify for each subsequent state X_{n+1} $(n = 1, 2, \dots)$ every conditional probability of the following form:

$$\Pr(X_{n+1} = x_{n+1} | X_1 = x_1, X_2 = x_2, \dots, X_n = x_n).$$

In other words, for every time n, the probability model must specify the conditional probability that the process will be in state x_{n+1} at time $n + 1$, given that at times $1, \dots, n$, the process was in states $x_1, \dots, x_n$.