Fast low-SNR high-dimensional optimal filtering, applied to inference of dynamic receptive fields.

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Representation of the spatial environment in the brain

- **Place Field** $\leftrightarrow$ neurons in the rodent hippocampus respond selectively depending on the animal’s **current location**.

- In many situations, e.g. learning, the place field is **time varying**.

\[
\begin{align*}
n_t & \sim f(x_t, t) + \text{noise} \\
 f(x, t) & \sim \text{time varying place field} \\
 B_t & \sim N \times N\text{-pixel indicating the current location} \\
 q_t & \sim N \times N\text{-pixel time varying place field}
\end{align*}
\]

Figure 1: Trajectory of a rat through a square environment is shown in black. Red dots indicate locations at which the particular entorhinal cell being examined fired.
Dynamic receptive field estimation

The activity of a neuron in a sensory brain region depends on the linear projection of the stimulus into the time varying receptive field.

\[ n_t \sim \langle B_t, q_t \rangle + \text{noise} \]
\[ B_t \sim N \times N\text{-pixel time varying visual stimuli} \]
\[ q_t \sim N \times N\text{-pixel time varying receptive field} \]

Main question: How to estimate the time varying receptive field?
One common problem:

- Understanding the dynamics of **large systems** for which **limited** and **noisy** observations are available.
- Classical solutions include **state space** models. See [1, 2, 3].
- Standard implementations of the Kalman filter require $O(\text{dim}(q)^3)$ **time** and $O(\text{dim}(q)^2)$ **memory** per time step, and are therefore impractical for applications involving very high-dimensional ($\text{dim}(q) \sim 100 \times 100$) systems.
Fast low-SNR optimization

- When there are no observations the uncertainty reflects our prior belief such as smoothness and/or boundedness of the receptive/place fields.

- Observations decrease the uncertainty.

- The decrease in the uncertainty due to low snr observation is small in magnitude and only changes our uncertainty in one direction.

- The effect of previous observations decays exponentially fast.

- The difference between the uncertainty of no observation and low snr observation is effectively a low rank matrix, i.e. $C_t = C_0 + U_t D_t U_t^T$.

- All computations are fast: optimal smoother requires $O(n^3 + n \dim(q) \log \dim(q))$ time and $O(n \dim(q))$ space; $n = \text{rank}(U_t)$.

- Can be used for fast experimental design. See [4, 5]
The model

- Smoothness along the **temporal** and **spatial** dimensions:

  \[ q_{t+1} = A q_t + \epsilon_t \quad q_t \sim \text{receptive/place field} \quad \epsilon \sim \mathcal{N}(0, V) \]

  \[ A \sim \text{temporal correlation} \quad V \sim \text{spatial correlation} \]

  Three independent samples \( \epsilon_t \) drawn from the Gaussian prior with covariance matrix \( V \).

- Noisy low dimensional **observations**:

  \[ y_{t+1} = B_t q_t + \eta_t \quad B_t \sim \text{visual/spatial stimuli} \quad \eta_t \sim \mathcal{N}(0, W_t) \]
Standard Kalman recursion

\[ \mu_t = \mathbb{E}[q_t | y_{1:t}] \quad C_t = \text{cov}[q_t | y_{1:t}] \]

- no observation, equilibrium covariance: \( AC_0 A + V = C_0 \) or \( C_0 = V(I - AA^T)^{-1} \).

\[ \begin{align*}
\mu_t &= C_t \left( (AC_{t-1}A^T + V)^{-1} A\mu_{t-1} + B^T W^{-1} y_t \right) \\
C_t &= \left[ (AC_{t-1}A^T + V)^{-1} + B^T W^{-1} B \right]^{-1}
\end{align*} \]

- computational difficulty \( \rightarrow C_t \) costs \( O(\text{dim}(q)^3) \) time (\( O(\text{dim}(q)^2) \) is \( B \) is low rank), and \( O(\text{dim}(q)^3) \) space
Low snr observation

- no observation: $C_t = C_0 = V(I - AA^T)^{-1}$
- single observation at $t = 1$ and no observation for $t > 1$:

$$C_1 = \left[ C_0^{-1} + B_1^T W^{-1} B_1 \right]^{-1} = C_0 - C_0 B_1^T (B_1 C_0^{-1} B_1^T + W^{-1})^{-1} B_1 C_0$$

$$= C_0 + U_1 D_1 U_1^T \quad \text{rank}(U_1) = \text{rank}(B_1)$$

similarly $C_{t+1} = C_0 + A^t U_1 D_1 (A^t U_1)^T$.

Since $A$ is stable, the perturbation to $C_{t+1}$ around the equilibrium covariance $C_0$ caused by a lag $t$ observation decays exponentially in $t$. 
Fast methods

- Approximating $C_t \sim C_0 + U_t D_t U_t^T$ where $U_t$ is low rank, i.e.
  $n := \text{rank}(U_t) \ll \text{dim}(q)$ allows us to perform fast efficient recursion:
- Updating $U_t$ and $D_t$ costs $O(n^3 + nN \log N)$ time and $O(nN)$ space.

Figure 2: $C_t$ is fairly close to $C_0$; in particular, $I - C_0^{-1}C_t$ has low effective rank. Left: true $C_t$. Middle: $C_0$. 
The superimposed black trace in all but the lower left panel indicates the simulated path $x_t$ of the animal. Upper left: true simulated place field $q_t(x)$ is shown in color. Top middle and right panels: estimated place fields, forward ($E(q_t | Y_{1:t})$) and forward-backward ($E(q_t | Y_{1:T})$), respectively. Bottom middle and right panels: marginal variance of the estimated place fields, forward ($\text{var}(q_t | Y_{1:t})$) and forward-backward ($\text{var}(q_t | Y_{1:T})$), respectively. Lower left panel: effective rank of $C_0 - C_t^S$ as a function of $t$ in the forward-backward smoother; the effective rank is largest when $x_t$ samples many locations in a short time period.
Comparison of the true vs. approximate covariance. Left panel: true covariance. Middle panel: approximate covariance. The maximal pointwise error between these two matrices is about 1%. Right panel: true and approximate mean $\mu_t$. The black trace indicates the true mean and the red trace (barely visible) the approximate mean.
Tracking a time-varying one-dimensional receptive field

Figure 3: Second panel: the stimulus $B_t$ was chosen to be spatiotemporal white Gaussian noise. Third panel: simulated output observed according to the Gaussian model $n_t = B_t q^t + \eta_t$. 
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References


