**Professor Salzman**
**Statistics 1211 Section 4**
**Homework 8 Solutions**

**<u>Chapter 6.2.21</u>**

**a.** $E(X) = \beta \cdot \Gamma\left(1 + \frac{1}{\alpha}\right)$ and $E(X^2) = Var(X) + [E(X)]^2 = \beta^2 \Gamma\left(1 + \frac{2}{\alpha}\right)$, so the moment estimators $\hat{\alpha}$ and $\hat{\beta}$ are the solution to $\overline{X} = \hat{\beta} \cdot \Gamma\left(1 + \frac{1}{\hat{\alpha}}\right)$,

$\frac{1}{n}\sum X_i^2 = \hat{\beta}^2 \Gamma\left(1 + \frac{2}{\hat{\alpha}}\right)$. Thus $\hat{\beta} = \frac{\overline{X}}{\Gamma\left(1 + \frac{1}{\hat{\alpha}}\right)}$, so once $\hat{\alpha}$ has been determined

$\Gamma\left(1 + \frac{1}{\hat{\alpha}}\right)$ is evaluated and $\hat{\beta}$ then computed. Since $\overline{X}^2 = \hat{\beta}^2 \cdot \Gamma^2\left(1 + \frac{1}{\hat{\alpha}}\right)$,

$\frac{1}{n}\sum \frac{X_i^2}{\overline{X}^2} = \frac{\Gamma\left(1 + \frac{2}{\hat{\alpha}}\right)}{\Gamma^2\left(1 + \frac{1}{\hat{\alpha}}\right)}$, so this equation must be solved to obtain $\hat{\alpha}$.

**b.** From a, $\frac{1}{20}\left(\frac{16{,}500}{28.0^2}\right) = 1.05 = \frac{\Gamma\left(1 + \frac{2}{\hat{\alpha}}\right)}{\Gamma^2\left(1 + \frac{1}{\hat{\alpha}}\right)}$, so $\frac{1}{1.05} = .95 = \frac{\Gamma^2\left(1 + \frac{1}{\hat{\alpha}}\right)}{\Gamma\left(1 + \frac{2}{\hat{\alpha}}\right)}$, and

from the hint, $\frac{1}{\hat{\alpha}} = .2 \Rightarrow \hat{\alpha} = 5$. Then $\hat{\beta} = \frac{\bar{x}}{\Gamma(1.2)} = \frac{28.0}{\Gamma(1.2)}$.

## Chapter 6.2.24

We wish to take the derivative of $\ln\left[\binom{x+r-1}{x}p^r(1-p)^x\right]$ with respect to p, set it equal to zero, and solve for p: $\frac{d}{dp}\left[\ln\binom{x+r-1}{x}+r\ln(p)+x\ln(1-p)\right]=\frac{r}{p}-\frac{x}{1-p}$.

Setting this equal to zero and solving for p yields $\hat{p}=\frac{r}{r+x}$. This is the number of successes over the total number of trials, which is the same estimator for the binomial in exercise 6.20. The unbiased estimator from exercise 6.17 is $\hat{p}=\frac{r-1}{r+x-1}$, which is not the same as the maximum likelihood estimator.

## Chapter 6.29

**a.** The joint pdf (likelihood function) is

$$f(x_1,\ldots,x_n;\lambda,\theta)=\begin{cases}\lambda^n e^{-\lambda\Sigma(x_i-\theta)} & x_1\ge\theta,\ldots,x_n\ge\theta\\ 0 & otherwise\end{cases}$$

Notice that $x_1\ge\theta,\ldots,x_n\ge\theta$ iff $\min(x_i)\ge\theta$,

and that $-\lambda\Sigma(x_i-\theta)=-\lambda\Sigma x_i+n\lambda\theta$.

Thus likelihood = $\begin{cases}\lambda^n\exp(-\lambda\Sigma x_i)\exp(n\lambda\theta) & \min(x_i)\ge\theta\\ 0 & \min(x_i)<\theta\end{cases}$

Consider maximization wrt $\theta$. Because the exponent $n\lambda\theta$ is positive, increasing $\theta$ will increase the likelihood provided that $\min(x_i)\ge\theta$; if we make $\theta$ larger than $\min(x_i)$, the likelihood drops to 0. This implies that the mle of $\theta$ is $\hat{\theta}=\min(x_i)$. The log likelihood is now $n\ln(\lambda)-\lambda\Sigma(x_i-\hat{\theta})$. Equating the derivative wrt $\lambda$ to 0 and solving yields $\hat{\lambda}=\frac{n}{\Sigma(x_i-\hat{\theta})}=\frac{n}{\Sigma x_i-n\hat{\theta}}$.

**b.** $\hat{\theta}=\min(x_i)=.64$, and $\Sigma x_i=55.80$, so $\hat{\lambda}=\frac{10}{55.80-6.4}=.202$

**Chapter 6.32**

a. $F_Y(y) = P(Y \le y) = P(X_1 \le y, \ldots, X_n \le y) = P(X_1 \le y) .. P(X_n \le y) = \left(\frac{y}{\theta}\right)^n$

for $0 \le y \le \theta$, so $f_Y(y) = \frac{ny^{n-1}}{\theta^n}$.

b. $E(Y) = \int_0^\theta y \cdot \frac{ny^{n-1}}{n} dy = \frac{n}{n+1}\theta$. While $\hat{\theta} = Y$ is not unbiased, $\frac{n+1}{n}Y$ is, since

$$E\left[\frac{n+1}{n}Y\right] = \frac{n+1}{n}E(Y) = \frac{n+1}{n} \cdot \frac{n}{n+1}\theta = \theta .$$

**Chapter 6.35**

| $x_i + x_j$ | 23.5 | 26.3 | 28.0 | 28.2 | 29.4 | 29.5 | 30.6 | 31.6 | 33.9 | 49.3 |
|---|---|---|---|---|---|---|---|---|---|---|
| 23.5 | 23.5 | 24.9 | 25.75 | 25.85 | 26.45 | 26.5 | 27.05 | 27.55 | 28.7 | 36.4 |
| 26.3 | | 26.3 | 27.15 | 27.25 | 27.85 | 27.9 | 28.45 | 28.95 | 30.1 | 37.8 |
| 28.0 | | | 28.0 | 28.1 | 28.7 | 28.75 | 29.3 | 29.8 | 30.95 | 38.65 |
| 28.2 | | | | 28.2 | 28.8 | 28.85 | 29.4 | 29.9 | 31.05 | 38.75 |
| 29.4 | | | | | 29.4 | 29.45 | 30.0 | 30.5 | 30.65 | 39.35 |
| 29.5 | | | | | | 29.5 | 30.05 | 30.55 | 31.7 | 39.4 |
| 30.6 | | | | | | | 30.6 | 31.1 | 32.25 | 39.95 |
| 31.6 | | | | | | | | 31.6 | 32.75 | 40.45 |
| 33.9 | | | | | | | | | 33.9 | 41.6 |
| 49.3 | | | | | | | | | | 49.3 |

There are 55 averages, so the median is the 28th in order of increasing magnitude. Therefore, $\hat{\mu} = 29.5$

**Chapter 6.36**

With $\sum x = 555.86$ and $\sum x^2 = 15{,}490$, $s = \sqrt{s^2} = \sqrt{2.1570} = 1.4687$. The $|x_i - \tilde{x}|$'s are, in increasing order, .02, .02, .08, .22, .32, .42, .53, .54, .65, .81, .91, 1.15, 1.17, 1.30, 1.54, 1.54, 1.71, 2.35, 2.92, 3.50. The median of these values is $\frac{(.81 + .91)}{2} = .86$. The estimate based on the resistant estimator is then $\frac{.86}{.6745} = 1.275$. This estimate is in reasonably close agreement with s.