OPTIMAL EXECUTION OF DERIVATIVES, A TAYLOR EXPANSION APPROACH

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ABSTRACT. In this paper we derive the Markowitz optimal trading trajectory for a trader who wishes to sell a large position of K units on some contingent claim. To do so we first use a Taylor expansion of the derivative with respect to the price of the underlying asset at time zero. We then use up to the second order approximation to solve the mean-variance optimization problem.

1. INTRODUCTION

The problem of optimal execution is a very general problem in which a trader who wishes to buy or sell a *large* position K of a given asset S—for instance wheat, shares, derivatives, etc.—is confronted with the dilemma of executing slowly or as quick as possible. In the first case he/she would be exposed to volatility, and in the second to the laws of offer and demand. Thus the trader most hedge between the *market impact* (due to his trade) and the *volatility* (due to the market).

The main *aim* of this paper is to study and characterize the socalled Markowitz-optimal open-loop execution trajectory of contingent claims.

The problem of minimizing expected overall liquidity costs has been analyzed using different market models by Bertsimas and Lo (1998), Obizhaeva and Wang (2006), and Alfonsi et al. (2007a,2007b), just to mention a few. However, these approaches miss the volatility risk associated with time delay. Instead, Almgren and Chriss (1999,2000), suggested studying and solving a mean-variance optimization for sales revenues in the class of deterministic strategies. Further on Almgren and Lorenz (2007) allowed for intertemporal updating and proved that this can *strictly* improve the mean-variance performance. Nevertheless, in Schied and Schöneborn (2007), the authors study the original problem of expected utility maximization with CARA utility functions. Their main result states that for CARA investors there is surprisingly

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no added utility from allowing for intertemporal updating of strategies. Finally, we mention that the Hamilton-Jacobi-Bellman approach has also recently been studied in Forsyth (2011).

The paper is organized as follows in Section 2 we state the optimal execution contingent claim problem. Next, in Section 3 we provide its closed form solution. In Section 4 a numerical example is studied, finally we conclude in Section 5 with some final remarks and comments.

2. The problem

The model. A trader wishes to execute $K = k_0 + \cdots + k_n$ units of a *contingent claim* C with underlying S. The quantity to optimize is given by the so-called execution shortfall, defined as

$$Y = \sum_{j=0}^{n} k_j C_j - K C_0$$

and the problem is then to find k_0, \ldots, k_n such that attain the minimum

$$\min_{k_0,\ldots,k_n} \left(\mathbb{E}[Y] + \lambda \mathbb{V}[Y] \right),$$

for some $\lambda > 0$. Assuming the derivative C is smooth in terms of its underlying S, it follows from the Taylor series expansion that:

$$C_j = f(S_0) + f'(S_0)(\tilde{S}_j - S_0) + \frac{1}{2}f''(S_0)(\tilde{S}_j - S_0)^2 + R_3$$

where \tilde{S} is the effective price and R_3 is the remainder which is $o((\tilde{S}_j - S_0)^3)$. Hence

$$\sum_{j=0}^{n} k_j C_j = \sum_{j=0}^{n} k_j f(S_0) + \sum_{j=0}^{n} f'(S_0) k_j (\tilde{S}_j - S_0) + \frac{1}{2} f''(S_0) \sum_{j=0}^{n} k_j (\tilde{S}_j - S_0)^2 + \sum_{j=0}^{n} k_j R_3 = K C_0 + f'(S_0) \left(\sum_{j=0}^{n} k_j \tilde{S}_j - K S_0 \right) + \frac{1}{2} f''(S_0) \sum_{j=0}^{n} k_j (\tilde{S}_j - S_0)^2 + \sum_{j=0}^{n} k_j R_3$$

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That is

(1)

$$Y = \sum_{j=0}^{n} k_j C_j - K C_0$$

$$= f'(S_0) \left(\sum_{j=0}^{n} k_j \tilde{S}_j - K S_0 \right)$$

$$+ \frac{1}{2} f''(S_0) \sum_{j=0}^{n} k_j (\tilde{S}_j - S_0)^2 + \sum_{j=0}^{n} k_j R_3.$$

Note that if we use only the first order approximation, then our optimization problem has already been solved and corresponds to Almgren and Chriss (2000) trading trajectory.

3. Second order Taylor Approximation

In this section we extend Almgren and Chriss (2000) market impact model for the case of a contingent claim. We provide our main result which is the closed form objective function by adapting a second order Taylor approximation.

3.1. Effective price process. Given that ξ_1, ξ_2, \ldots is a sequence of i.i.d. Gaussian random variables with mean zero and variance 1. The price and 'effective' processes are respectively defined as:

$$S_{j} = S_{j-1} - \tau g\left(\frac{k_{j}}{\tau}\right) + \sigma \tau^{1/2} \xi_{j}$$
$$\tilde{S}_{j} = S_{j} - h\left(\frac{k_{j}}{\tau}\right)$$

the permanent and temporary market impact will be modeled, for simplicity, as

$$g\left(\frac{k_j}{\tau}\right) = \alpha \frac{k_j}{\tau} \qquad h\left(\frac{k_j}{\tau}\right) = \beta \frac{k_j}{\tau},$$

for some constant α and β . Hence, letting

$$x_j := K - \sum_{m=1}^{j} k_m$$
 and
 $W_j := \sum_{m=1}^{j} \xi_m$ i.e. $W_j \sim \mathbb{N}(0, j), \operatorname{Cov}(W_j, W_i) = \min(i, j)$

it follows that:

(2)
$$\tilde{S}_j - S_0 = \sigma \tau^{1/2} W_j - \alpha (K - x_j) - \frac{\beta}{\tau} k_j.$$

3.2. Second order approximation. From equations (1) and (2) the second order approximation of the execution shortfall Y is given by:

(3)
$$Y \approx f'(S_0) \sum_{j=0}^n k_j \left(\sigma \tau^{1/2} W_j - \alpha (K - x_j) - \frac{\beta}{\tau} k_j \right) + \frac{1}{2} f''(S_0) \sum_{j=0}^n k_j \left(\sigma \tau^{1/2} W_j - \alpha (K - x_j) - \frac{\beta}{\tau} k_j \right)^2.$$

Expanding the squared term

$$\left(\sigma\tau^{1/2}W_{j} - \alpha(K - x_{j}) - \frac{\beta}{\tau}k_{j}\right)^{2}$$

= $\sigma^{2}\tau W_{j}^{2} + \alpha^{2}(K - x_{j})^{2} + \frac{\beta^{2}}{\tau^{2}}k_{j}^{2} - 2\frac{\beta\sigma}{\tau^{1/2}}k_{j}W_{j}$
 $-2\alpha\sigma\tau^{1/2}(K - x_{j})W_{j} + 2\frac{\alpha\beta}{\tau}k_{j}(K - x_{j}),$

thus the expected value of Y is approximately

(4)

$$\mathbb{E}[Y] = f'(S_0) \sum_{j=0}^n k_j \left(-\alpha (K - x_j) - \frac{\beta}{\tau} k_j \right) + \frac{1}{2} f''(S_0) \sum_{j=0}^n k_j \left[\sigma^2 \tau j + \alpha^2 (K - x_j)^2 + \frac{\beta^2}{\tau^2} k_j^2 + 2 \frac{\alpha \beta}{\tau} k_j (K - x_j) \right]$$

to compute the variance \mathbb{V} of Y we rearrange (3) as

$$Y \approx \sum_{j=0}^{n} \nu_j k_j W_j + \sum_{j=0}^{n} \eta_j k_j W_j^2 + D$$

where D are all the deterministic terms and

$$\nu_j := f'(S_0)\sigma\tau^{1/2} - f''(S_0) \left[\alpha\sigma\tau^{1/2}(K - x_j) + \frac{\beta\sigma}{\tau^{1/2}}k_j\right]$$

$$\eta_j := \frac{1}{2}f''(S_0)\sigma^2\tau.$$

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It follows that the variance of Y is

$$\mathbb{V}[Y] = \mathbb{V}\left[\sum_{j=0}^{n} \nu_{j} k_{j} W_{j}\right] + \mathbb{V}\left[\sum_{j=0}^{n} \eta_{j} k_{j} W_{j}^{2}\right]$$
$$+ 2 \operatorname{Cov}\left(\sum_{j=0}^{n} \nu_{j} k_{j} W_{j}, \sum_{j=0}^{n} \eta_{j} k_{j} W_{j}^{2}\right)$$
$$(5) = \sum_{j=0}^{n} v_{j}^{2} k_{j}^{2} j + 2 \sum_{0 \le i < j \le n} v_{i} k_{i} v_{j} k_{j} i$$
$$+ \sum_{j=0}^{n} \eta_{j}^{2} k_{j}^{2} \cdot 2j^{2} + 2 \sum_{0 \le i < j \le n} \eta_{i} k_{i} \eta_{j} k_{j} \cdot 2i^{2}$$

and the last term equals zero.

3.3. Optimal trading schedule for the second order approximation. To find the optimal trading schedule for the second order approximation of Y we need find the sequence of k_0, \ldots, k_n such that

$$\mathbb{E}[Y] + \lambda \mathbb{V}[Y]$$

is minimized for a given λ and where $\mathbb{E}[Y]$ and $\mathbb{V}[Y]$ are as in (4) and (5) respectively. After some simplification

$$\begin{split} \mathbb{E}[Y] &+ \lambda \mathbb{V}[Y] \\ &= f'(S_0) \sum_{j=0}^n k_j \left[\alpha(x_j - K) - \frac{\beta}{\tau} k_j \right] \\ &+ \frac{1}{2} f''(S_0) \sum_{j=0}^n k_j \left[\sigma^2 \tau j + \alpha^2 (K - x_j)^2 + \frac{\beta^2}{\tau^2} k_j^2 + \frac{2\alpha\beta}{\tau} (K - x_j) k_j \right] \\ &+ \lambda \sum_{j=0}^n j k_j \left[v_j^2 k_j + 2j k_j \eta_j^2 + 2v_j \sum_{m=j+1}^n v_m k_m + 4j \eta_j \sum_{m=j+1}^n \eta_m k_m \right]. \end{split}$$

4. NUMERICAL SOLUTION

For Y as in (3), the optimization problem we aim to solve is

(6)
$$\min_{k_0,k_1,\dots,k_n} \mathbb{E}[Y] + \lambda \mathbb{V}[Y]$$

subject to

(7)
$$\sum_{j=0}^{n} k_j = K$$

We solve the problem using *fmincon* in the **Matlab**.

Example 4.1. Considering the example:

$$n = 2; K = 1000; \alpha = 0.1; \beta = 0.5; \lambda = 0.4; \tau = 1$$

$$\delta = f'(S_0) = 0.5; \gamma = f'(S_0) = 0.2; \sigma = 0.5$$

The optimal trading strategy is

 $k_0 = 333.3348, k_1 = 333.3336; k_2 = 333.3316$

and the optimal objective function is 5.5736×10^8 .

Remark 4.2. The trading trajectory has a downward trend. Intuitively, and on contrast to executing a large size at a single transaction our result suggests to split the overall position in almost even trades. The linear assumption that we made on the temporary and the permanent impacts seems to explain the almost equal execution quantities.

5. Concluding Remarks

In this work we study the Markowitz-optimal execution trajectory of contingent claims. In order to do so, we use a second order Taylor approximation with respect to the contingent claim C evaluated at the initial value of the underlying S. We obtain the closed form objective function given a risk averse criterion. Our approach allows us to obtain the explicit numerical solution and we provide an example.

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