ON CHANGES OF MEASURE AND REPRESENTATIONS OF THE FIRST HITTING TIME OF A BESSEL PROCESS

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Abstract. In this work we relate systems of coupled backward and forward Kolmogorov equations. These in turn are used to derive changes of measure and/or dual representations of SDEs. The first coupled system of PDEs allows us to transform problems of SDEs with drift into problems of SDEs without drift, under an appropriate change of measure. The second, in turn, allows us to obtain a dual representation of problems of SDEs with drift, under an appropriate change of measure.

The results are illustrated by revisiting the problem of finding the density of the first time that a 5-D Brownian motion hits a ball, of radius a, from inside. As opposed to the standard literature, we do not make use of a Laplace-Gegenbauer transform. Furthermore we find the probability that a Bessel process of order negative -1 hits level $a > 0$ before reaching zero. This in turn gives analytical insight as to why the Bessel processes of order 5, 3, and -1 together with a squared one-dimensional Brownian motion are intrinsically related. As a final example we study the non-homogeneous Ornstein-Uhlenbeck process.

1. Introduction

Alternative representations of a problem are often used in Stochastic Analysis. Such is the case of Girsanov's theorem—removal of a drift [see Section 3.5 in Karatzas and Shreve [10]]—Doob's h-transform [see Chapter 6 in Rogers and Williams [15]], or the absolute continuity property for the laws of the Bessel processes with different indices [see Chapter XI in Revuz and Yor [14]].

In this work we derive three systems of coupled Kolmogorov backward and forward PDEs which are used to derive changes of measure and/or dual representations of SDEs. [see Hernandez-del-Valle [6] for an application of similar techniques in hitting densities of Brownian motion, Hernandez-del-Valle [7] for alternative results on the procedure described within, or Hernandez-del-Valle [8] for an application in the 3-D Brownian motion case]. Applications of the techniques described in this work is finding the transition probabilities of, at first sight, "strange" SDEs [see Section 5]. For instance, we find the density of a one-dimensional Wiener process started within a range $[0, a]$ and that hits level a for the first time at time $v > 0$ (before ever reaching level 0).

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As an application of one of the coupled PDE systems, we obtain (for a special case) Bessel's absolute continuity property, as well as a PDE interpretation of the so-called "Bessel process with negative dimension" [see Chapter XI, Section 1 in Revuz and Yor [14], and Göing-Jaeschke and Yor [5]]. We also illustrate the results by deriving the density of the first time that a 5-D Brownian motion hits a ball from within: (1) in terms of a functional of a one-dimensional Brownian motion absorbed at 0 and a, and (2) in terms of the mean of a cubed Bessel process with dimension $d = -1$, absorbed at 0 or a.

The paper is organized as follows. In Sections 2 and 3, equivalent systems of backward and forward PDEs are derived. These in turn are used to derive changes of measure and/or dual representations of SDEs. Next, in Sections 4 and 5, the results in Section 2 are used to derive: (a) the transition density of a Brownian motion, started at y (for $0 \leq y \leq a$), which is absorbed at 0 and that reaches level a at a time u, and (b) an *alternative* derivation of the density of a 5-D Wiener process hitting a ball from within.

In Section 6, we relate Bessel process of order 5 and -1 and find the density of the first time that a Bessel -1 hits level $a > 0$ before reaching zero. To this end, we make use of the ideas developed Section in 5. We conclude in Section 7 with some final comments.

2. Coupled systems of PDEs I

Remark 2.1. *(On notation.)* In our results below we consider real valued differentiable functions h, u, v, and w, depending on variables $(t, x; s, y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}$. As in the analysis of diffusion processes, PDEs with derivatives with respect to (t, x) are called backward equations, whereas PDEs with derivatives in (s, y) are called forward equations.

In this section we relate systems of coupled backward and forward Kolmogorov equations, through a pair of backward equations [see (2.5.a) and (2.7.a), or equivalently (2.9.a) and (2.11.a)]. This in turn allows us to transform problems of diffusions with drift (or convective mass transfer) into problems of diffusions without drift (or with potential), under a suitable change of measure. In particular, the drift is at most a sum of two functions which in turn are solutions to backward heat equations.

We will clarify the last sentence of the previous paragraph with the following example.

Example 2.2. (i) (Bessel process of order 3.) First note that $h(t, x) = x$ is a solution to the backward heat equation

$$-h_t = \frac{1}{2} h_{xx}.$$

Let us suppose that the process X satisfies that

$$dX_t = \frac{1}{X_t} dt + dW_t, \qquad X_0 = x > 0. \tag{2.1}$$

Since $h_x/h = 1/x$ we may express (2.1) as

$$dX_t = \frac{h_x(t, X_t)}{h(t, X_t)} dt + dW_t, \qquad X_0 = x > 0.$$

(ii) (Bessel process of order 5.) Suppose the process Y is such that

$$dY_t = \frac{2}{Y_t} dt + dW_t, \qquad Y_0 = y > 0, \tag{2.2}$$

and set $k(t, x) = h(t, x) = x$. Then we may express (2.2) as

$$dY_t = \left[\frac{h_y(t, Y_t)}{h(t, Y_t)} + \frac{k_y(t, Y_t)}{k(t, Y_t)} \right] dt + dW_t, \qquad Y_0 = y > 0$$

(iii) (Brownian bridge.) Suppose Z has the dynamics

$$dZ_t = -\frac{Z_t}{s-t} dt + dW_t, \qquad t \leq s, \tag{2.3}$$

and set

$$q(t, x) = \frac{1}{\sqrt{2\pi(s-t)}} \exp\left\{ -\frac{x^2}{2(s-t)} \right\}.$$

Since $q_x/q = -x/(s-t)$, it follows that we may express (2.3) as

$$dZ_t = \frac{q_z(t, Z_t)}{q(t, Z_t)} dt + dW_t, \qquad t \leq s.$$

Proposition 2.3. *Let $\sigma : \mathbb{R}^+ \to \mathbb{R}$, and h, v, w and k be of class $C^{1,2}$. Suppose that v, w, and h satisfy the following identity*

$$v(t, x) = \frac{w(t, x)}{h(t, x)} \tag{2.4}$$

and $h(t, x) \neq 0$ for some strip in $\mathbb{R}^+ \times \mathbb{R}$. If h and v satisfy respectively (a) and (b)

$$\begin{cases} -h_t = \frac{1}{2}\sigma^2 h_{xx} & (a) \\ -v_t = \frac{1}{2}\sigma^2 v_{xx} + \sigma^2 \left[\frac{h_x}{h} + \frac{k_x}{k} \right] v_x & (b) \\ -w_t + \sigma^2 \left[\frac{k_x}{k} \frac{h_x}{h} \right] w = \frac{1}{2}\sigma^2 w_{xx} + \sigma^2 \frac{k_x}{k} w_x, & (c) \end{cases} \tag{2.5}$$

then (c) holds.

Proof. From (2.4) and (2.5.b) we have

$$\begin{aligned} \frac{h_t w - h w_t}{h^2} &= \frac{1}{2}\sigma^2 \frac{w_{xx} h - w h_{xx}}{h^2} - \sigma^2 \frac{h_x (w_x h - w h_x)}{h^3} \\ &\quad + \sigma^2 \frac{h_x (w_x h - w h_x)}{h^3} + \sigma^2 \frac{k_x}{k} \left(\frac{h w_x - h_x w}{h^2} \right). \end{aligned}$$

Equivalently

$$\frac{h_t w - h w_t}{h^2} = \frac{1}{2}\sigma^2 \frac{w_{xx} h - w h_{xx}}{h^2} + \sigma^2 \frac{k_x}{k} \left(\frac{h w_x - h_x w}{h^2} \right).$$

In turn, from (2.5.a) it follows that

$$-\frac{w_t}{h} = \frac{1}{2}\sigma^2 \frac{w_{xx}}{h} + \sigma^2 \frac{k_x}{k}\frac{w_x}{h} - \sigma^2 \frac{k_x}{k}\frac{h_x}{h^2} w.$$

□

Proposition 2.4. *Let $\sigma : \mathbb{R}^+ \to \mathbb{R}$, and u, w and k be of class $C^{1,2}$. Suppose that w, u, and k satisfy the following identity*

$$w(t,x) = \frac{u(t,x)}{k(t,x)} \tag{2.6}$$

and $k(t,x) \neq 0$ for some strip in $\mathbb{R}^+ \times \mathbb{R}$. If k and w satisfy respectively (a) and (b)

$$\begin{cases} -k_t = \frac{1}{2}\sigma^2 k_{xx} & (a) \\ -u_t + \sigma^2 \left[\frac{k_x}{k}\frac{h_x}{h}\right] u = \frac{1}{2}\sigma^2 u_{xx} & (b) \\ -w_t + \sigma^2 \left[\frac{k_x}{k}\frac{h_x}{h}\right] w = \frac{1}{2}\sigma^2 w_{xx} + \sigma^2 \frac{k_x}{k} w_x & (c) \end{cases} \tag{2.7}$$

then (c) holds.

From Propositions 2.3 and 2.4 we have.

Corollary 2.5. *Let h, v, k and u be of class $C^{1,2}$ as well as solutions to (2.5.a), (2.5.b), (2.7.a), and (2.7.b), respectively. Then they satisfy the identity*

$$v(t,x) = \frac{u(t,x)}{k(t,x) \cdot h(t,x)}$$

given that $k(t,x) \cdot h(t,x) \neq 0$ for some strip in $\mathbb{R}^+ \times \mathbb{R}$.

Remark 2.6. Propositions 2.3 and 2.4 suggest that for $j = 1, \ldots, n$ and given

$$\delta_j(t,x) = \frac{h^j_x(t,x)}{h^j(t,x)}$$

where each h^j is a solution to

$$-h^j_t = \frac{1}{2} h^j_{xx}.$$

A backward equation of the following type

$$-v_t = \frac{1}{2} v_{xx} + \left[\sum_{j=1}^{n} \delta_j(t,x)\right] v_x$$

may be solved using the procedure described above.

Proposition 2.7. *Let h, v and w be of class $C^{1,2}$. Suppose that v, h, and w satisfy the following identity*

$$v(s,y) = h(s,y) \cdot w(s,y) \tag{2.8}$$

and $h(t,x)\cdot k(t,x)\neq 0$ for some strip in $\mathbb{R}^+\times\mathbb{R}$. If h and v satisfy respectively (a) and (b)

$$\begin{cases} -h_s=\frac{1}{2}h_{yy} & (a)\\ v_s=\frac{1}{2}v_{yy}-\frac{\partial}{\partial y}\left[\left(\frac{h_y}{h}+\frac{k_y}{k}\right)v\right] & (b)\\ w_s+\left[\frac{k_y}{k}\frac{h_y}{h}\right]w=\frac{1}{2}w_{yy}-\frac{\partial}{\partial y}\left[\frac{k_y}{k}w\right], & (c)\end{cases} \tag{2.9}$$

then (c) holds.

Proof. From (2.8) and (2.9.b) we have

$$\begin{aligned} h_sw+hw_s &= \frac{1}{2}h_{yy}w+h_yw_y+\frac{1}{2}hw_{yy}\\ &\quad -\frac{\partial}{\partial y}\left[\left(\frac{h_y}{h}+\frac{k_y}{k}\right)hw\right]\\ &= \frac{1}{2}h_{yy}w+h_yw_y+\frac{1}{2}hw_{yy}\\ &\quad -\frac{\partial}{\partial y}\left[h_yw\right]-\frac{\partial}{\partial y}\left[\frac{k_y}{k}hw\right]\\ &= \frac{1}{2}h_{yy}w+h_yw_y+\frac{1}{2}hw_{yy}\\ &\quad -h_{yy}w-h_yw_y-\frac{\partial}{\partial y}\left[\frac{k_y}{k}w\right]h-\frac{k_y}{k}h_yw\\ &= -\frac{1}{2}h_{yy}w+\frac{1}{2}hw_{yy}-\frac{\partial}{\partial y}\left[\frac{k_y}{k}w\right]h-\frac{k_y}{k}h_yw. \end{aligned}$$

Equivalently, from (2.9.a),

$$hw_s+\frac{k_y}{k}h_yw = \frac{1}{2}hw_{yy}-\frac{\partial}{\partial y}\left[\frac{k_y}{k}w\right]h.$$

□

Proposition 2.8. *Let k, u and w are of class $C^{1,2}$. Suppose that w, k, and u satisfy the following identity*

$$w(s,y)=k(s,y)\cdot u(s,y) \tag{2.10}$$

and $k(t,x)\cdot h(t,x)\neq 0$ for some strip in $\mathbb{R}^+\times\mathbb{R}$. If k and u satisfy respectively (a) and (b)

$$\begin{cases} -k_s=\frac{1}{2}k_{yy} & (a)\\ u_s+\left[\frac{k_y}{k}\frac{h_y}{h}\right]u=\frac{1}{2}u_{yy} & (b)\\ w_s+\left[\frac{k_y}{k}\frac{h_y}{h}\right]w=\frac{1}{2}w_{yy}-\frac{\partial}{\partial y}\left[\frac{k_y}{k}w\right], & (c)\end{cases} \tag{2.11}$$

then (c) holds.

From Propositions 2.7 and 2.8 we have.

Corollary 2.9. *Let h, v, k and u be of class $C^{1,2}$, as well as solutions to (2.9.a), (2.9.b), (2.11.a), and (2.11.b), respectively. Then they satisfy the identity*

$$v(s,y) = h(s,y)\cdot k(s,y)\cdot u(s,y).$$

Theorem 2.10. *Let h and k be of class $C^{1,2}(\mathbb{R}^+\times\mathbb{R})$ as well as solutions to the backward equations (2.5.a) and (2.9.a) [or equivalently (2.7.a) and (2.11.a)] respectively. Furthermore, consider processes X, Y, and Z, which respectively satisfy (at least in the weak sense), the following equations (each under their corresponding measures $\mathbb{P}$, $\tilde{\mathbb{Q}}$, and $\mathbb{Q}$),*

$$\begin{aligned}
(\mathbb{P}) \quad dX_t &= \left[\frac{h_x(t,X_t)}{h(t,X_t)} + \frac{k_x(t,X_t)}{k(t,X_t)}\right]dt + dB_t \\
(\tilde{\mathbb{Q}}) \quad dZ_t &= \left[\frac{k_x(t,Z_t)}{k(t,Z_t)}\right]dt + dB_t \\
(\mathbb{Q}) \quad dY_t &= dB_t.
\end{aligned} \tag{2.12}$$

Moreover, suppose that

$$f(\tau,z) = \frac{k_z(\tau,z)}{k(\tau,z)}\frac{h_z(\tau,z)}{h(\tau,z)} \tag{2.13}$$

and $k(\tau,z)\cdot h(\tau,z)\neq 0$ for some strip in $\mathbb{R}^+\times\mathbb{R}$. Then the following identities hold

$$\begin{aligned}
&\mathbb{P}_{t,x}(X_s\in A) \\
&\quad = \mathbb{E}_z^{\tilde{\mathbb{Q}}}\left[\frac{h(s,Z_s)}{h(t,z)}\exp\left\{-\int_t^s f(\tau,Z_\tau)d\tau\right\}\mathbb{I}_{(Z_s\in A)}\right] \\
&\quad = \mathbb{E}_y^{\mathbb{Q}}\left[\frac{k(s,Y_s)}{k(t,y)}\frac{h(s,Y_s)}{h(t,y)}\exp\left\{-\int_t^s f(\tau,Y_\tau)d\tau\right\}\mathbb{I}_{(Y_s\in A)}\right].
\end{aligned} \tag{2.14}$$

Proof. Let X be a process with dynamics as in (2.12.$\mathbb{P}$). Hence, its transition density G is simultaneously a solution of (2.5.b), in the backward variables (t,x), and a solution of (2.9.b), in the forward variables (s,y). In turn, from (2.4) and (2.8) we have that

$$\begin{aligned}
G(t,x;s,y) &=: v(t,x) & G(t,x;s,y) &:= \mathsf{v}(s,y) \\
&= \frac{w(t,x)}{h(t,x)} & &= \mathsf{h}(s,y)\cdot\mathsf{w}(s,y)
\end{aligned}$$

respectively. Next, if we let H be simultaneously

$$\mathsf{H}(t,x;s,y) =: w(t,x) \qquad \text{and} \qquad \mathsf{H}(t,x;s,y) =: \mathsf{w}(s,y)$$

we have that

$$G(t,x;s,y) = \frac{\mathrm{h}(s,y)}{h(t,x)}\mathsf{H}(t,x;s,y).$$

That is, if G is the transition density of process Z, which has dynamics as in (2.12.$\tilde{\mathbb{Q}}$) it follows

$$\begin{aligned}
\mathbb{P}_{t,x}(X_s \in A) &= \int_A G(t,x;s,y)dy \\
&= \int_A \frac{\mathrm{h}(s,y)}{h(t,x)} \mathsf{H}(t,x;s,y)dy \\
&= \int_A \frac{\mathrm{h}(s,y)}{h(t,x)} \frac{\mathsf{H}(t,x;s,y)}{\mathsf{G}(t,x;s,y)} \mathsf{G}(t,x;s,y)dy \\
&= \mathbb{E}^{\tilde{\mathbb{Q}}}_{t,x}\left[\frac{h(s,Z_s)}{h(s,x)} \mathcal{Z}_s \mathbb{I}_{(Z_s \in A)}\right].
\end{aligned}$$

Our goal is to characterize $\mathcal{Z}$ under $\tilde{\mathbb{Q}}$. To this end, and given that f is as in (2.13), we have from Ito's lemma

$$\begin{aligned}
&d\left[h(s,Z_s)\exp\left\{-\int_t^s f(\tau,Z_\tau)d\tau\right\}\right] \\
&= \exp\left\{-\int_t^s f(\tau,Z_\tau)d\tau\right\}\left(\left[h_s - fh + \frac{1}{2}h_{zz} + h_z\frac{k_z}{k}\right]ds + h_z dB_s\right) \\
&= \exp\left\{-\int_t^s f(\tau,Z_\tau)d\tau\right\} \\
&\quad \times \left(\left[h_s - \frac{k_z h_z}{kh}h + \frac{1}{2}h_{zz} + h_z\frac{k_z}{k}\right]ds + h_z dB_s\right) \\
&= \exp\left\{-\int_t^s f(\tau,Z_\tau)d\tau\right\} h_z dB_s.
\end{aligned}$$

hence $h(\cdot, Z_\cdot)\exp\{-\int_t^\cdot f(\tau,Z_\tau)d\tau\}$ is a $\tilde{\mathbb{Q}}$-martingale, and such that

$$\mathbb{E}^{\tilde{\mathbb{Q}}}_{t,x}\left[\frac{h(s,Z_s)}{h(t,z)}\exp\left\{-\int_t^s f(\tau,Z_\tau)d\tau\right\}\right] = 1.$$

This yields

$$\mathcal{Z}_s = \exp\left\{-\int_t^s f(\tau,Z_\tau)d\tau\right\}.$$

The second identity follows using similar arguments. □

3. Coupled systems of PDEs II

In this section we relate systems of coupled backward and forward Kolmogorov equations, through a backward equation [see (3.5.a), or equivalently (3.7.a)]. This in turn allows us to obtain a dual representation of SDE problems under a proper change of measure.

To clarify the last sentence of the previous paragraph with the following example.

Example 3.1. Let process X and Y be such that

$$\begin{aligned} dX_t &= \frac{2}{X_t}dt + dB_t \\ dY_t &= -\frac{1}{Y_t}dt + dB_t. \end{aligned}$$

It is known that these two processes are related; X is a Bessel process of order 5, and Y of order -1; see Göing-Jaeschke and Yor [5]. We will give a PDE interpretation of how these two processes are linked. To do so, let us consider the backward Kolmogorov equations associated to the processes X and Y, namely,

$$(X) \quad -v_t = \frac{1}{2}v_{xx} + \frac{2}{x}v_x \tag{3.1}$$

$$(Y) \quad -w_t = \frac{1}{2}w_{xx} - \frac{1}{x}w_x. \tag{3.2}$$

Next note that the sum of the drifts of X and Y, which add up to

$$\frac{2}{x} + \frac{1}{x} = \frac{3}{x},$$

can be expressed in terms of functions h and p, which are solutions of

$$\begin{aligned} -h_t &= \frac{1}{2}h_{xx} - \frac{1}{x}h_x, \quad \text{and} \\ -p_t &= \frac{1}{2}p_{xx} \end{aligned} \tag{3.3}$$

respectively. To do so, let $p = x$ and $h = x^3$, *i.e.*

$$\frac{2}{x} + \frac{p_x}{p} = \frac{h_x}{h}.$$

In particular, in this section we will show, that for real valued functions v, w, and h which satisfy respectively (3.1), (3.2), and (3.3) the following identity holds

$$h \cdot v = w.$$

In turn, the previous relationship will allow us to give a systematic procedure to transform a process X (with drift) into another process Y (with drift) under a suitable change of measure.

Proposition 3.2. *Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a real valued function, and h, v and w be of class $C^{1,2}$. Suppose that v, w, and h satisfy the following identity*

$$v(t,x) = \frac{w(t,x)}{h(t,x)} \tag{3.4}$$

and $h(t,x) \neq 0$ for some strip in $\mathbb{R}^+ \times \mathbb{R}$. If h and v satisfy respectively (a) and (b)

$$\begin{cases} -h_t = \frac{1}{2}h_{xx} + \alpha(x)h_x & (a) \\ -v_t = \frac{1}{2}v_{xx} + \left[\alpha(x) + \frac{h_x}{h}\right] v_x & (b) \\ -w_t = \frac{1}{2}w_{xx} + \alpha(x)w_x, & (c) \end{cases} \tag{3.5}$$

then (c) holds.

Proof. From (3.4) and (3.5.b) we have

$$\begin{aligned}\frac{h_t w - h w_t}{h^2} &= \frac{1}{2}\frac{w_{xx}h - w h_{xx}}{h^2} - \frac{h_x(w_x h - w h_x)}{h^3} \\ &\quad + \frac{h_x(w_x h - w h_x)}{h^3} + \alpha(x)\frac{w_x h - w h_x}{h^2},\end{aligned}$$

alternatively

$$h_t w - h w_t = \frac{1}{2} w_{xx} h - \frac{1}{2} w h_{xx} + \alpha(x) w_x h - \alpha(x) w h_x.$$

Rearranging terms it follows that

$$-h\left[w_t + \frac{1}{2}w_{xx} + \alpha(x)w_x\right] = -w\left[h_t + \frac{1}{2}h_{xx} + \alpha(x)h_x\right].$$

□

Proposition 3.3. *Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a real valued function, and h, v and w be of class $C^{1,2}$. Suppose that v, w, and h satisfy the following identity*

$$v(s,y) = w(s,y) \cdot h(s,y). \tag{3.6}$$

and $h(t,x) \neq 0$ for some strip in $\mathbb{R}^+ \times \mathbb{R}$. If h and v satisfy respectively (a) and (b)

$$\begin{cases} -h_s = \frac{1}{2}h_{yy} + \alpha(y)h_y & (a) \\ v_s = \frac{1}{2}v_{yy} - \frac{\partial}{\partial y}\left[\left(\alpha(y) + \frac{h_y}{h}\right)v\right] & (b) \\ w_s = \frac{1}{2}w_{yy} - \frac{\partial}{\partial y}\left[\alpha(y)w\right] & (c) \end{cases} \tag{3.7}$$

then (c) holds.

Proof. From (3.6) and (3.7.b) we have

$$\begin{aligned} w_s h + w h_s &= \frac{1}{2}w_{yy}h + w_y h_y + \frac{1}{2}w h_{yy} \\ &\quad - \frac{\partial}{\partial y}\left[\alpha(y)wh\right] - \frac{\partial}{\partial y}\left[h_y w\right] \\ &= \frac{1}{2}w_{yy}h + w_y h_y + \frac{1}{2}w h_{yy} \\ &\quad - \frac{\partial}{\partial y}\left[\alpha(y)w\right]h - w\alpha(y)h_y - h_{yy}w - h_y w_y \\ &= \frac{1}{2}w_{yy}h - \frac{1}{2}w h_{yy} - \frac{\partial}{\partial y}\left[\alpha(y)w\right]h - w\alpha(y)h_y. \end{aligned}$$

Equivalently, rearranging terms we have

$$h\left(w_s - \frac{1}{2}w_{yy} + \frac{\partial}{\partial y}\left[\alpha(y)w\right]\right) = -w\left(h_s + \frac{1}{2}h_{yy} + \alpha(y)h_y\right).$$

□

Theorem 3.4. *For a given $\alpha : \mathbb{R} \to \mathbb{R}$ let h be a solution to (3.5.a) [or equivalently (3.7.a)]. Furthermore, let processes X and Y be well defined (at least in the weak sense), with the following dynamics (each under their corresponding measures $\mathbb{P}$, and $\mathbb{Q}$),*

$$\begin{aligned} (\mathbb{P}) \quad dX_t &= \left[\alpha(X_t) + \frac{h_x(t, X_t)}{h(t, X_t)}\right] dt + dB_t \\ (\mathbb{Q}) \quad dY_t &= \alpha(Y_t)dt + dB_t. \end{aligned}$$

Then the following identity holds

$$\mathbb{P}_{t,x}(X_s \in A) = \mathbb{E}_y^{\mathbb{Q}} \left[\frac{h(s, Y_s)}{h(t, y)} \mathbb{I}_{(Y_s \in A)}\right].$$

Proof. The proof is carried out in the same way as the proof of Theorem 2.10. That is, It follows from Kolmogorv's backward and forward equations together with Propositions 3.2 and 3.3. □

Corollary 3.5. *Under the assumptions of Theorem 3.4, the process $h(\cdot, Y.)$ is a $\mathbb{Q}$ martingale.*

Proof. This corollary follows from Ito's lemma, the fact that h is a solution to (3.5.a) and

$$dY_t = \alpha(Y_t)dt + dB_t.$$

□

4. Bessel process I

In this section, making use of the ideas presented in Section 3, we revisit the problem of finding the density of the first time that a 5-D Brownian motion hits a ball from within. Our derivation, however is *new* and does not make use of a Laplace-Gegenbauer transform as in, Wendel [17], Hsu [9], Yin [18], and Betz & Gzyl [1, 2]. Recent results on the subject can also be found in Byczkowki *et. al* [4] and references therein. Standard references in the theory of Bessel processes are Knight [11], Ray [13], Lamperti [12] or Revuz and Yor [14].

Remark 4.1. In the next two sections we will make use of the following notation:

$$\begin{aligned} p(t; y, z) &:= \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{(y-z)^2}{2t}\right\}, \qquad (4.1) \\ p_-(t; y, z) &:= p(t; y, z) - p(t; y, -z). \end{aligned}$$

In addition, if T and T_0 are respectively the first time that a one-dimensional standard Brownian motion B started at y reaches level a and level 0 respectively, let

$$\begin{aligned} \mathbb{P}_{t,y}(T \in s, T_0 > s) &:= \frac{1}{\sqrt{2\pi(s-t)^3}} \sum_{n=-\infty}^{\infty} \Bigg[(2na + a - y) \\ &\qquad \times \exp\left\{-\frac{(2na + a - y)^2}{2(s-t)}\right\}\Bigg], \qquad (4.2) \end{aligned}$$

and

$$\mathbb{P}_{t,y}(B_s \in dx, T \wedge T_0 > s) \quad := \quad \sum_{n=-\infty}^{\infty} p_-(s-t; y, x+2na)dx. \tag{4.3}$$

Which is the probability of a Brownian motion B started at $y \in (0,a)$ before reaching levels 0 or a.

Remark 4.2. Let X be a Bessel process such that

$$dX_t \quad = \quad \frac{2}{X_t}dt + dB_t.$$

Next, note that $h(t,x) = k(t,x) = x$ are solutions to the backward heat equation (2.5.a) and (2.7.a) respectively (with $\sigma^2 = 1$). Hence the latter SDE for X can be expressed, in terms of h and k, as

$$dX_t \quad = \quad \left[\frac{h_x(t,X_t)}{h(t,X_t)} + \frac{k_x(t,X_t)}{k(t,X_t)}\right]dt + dB_t. \tag{4.4}$$

From Theorem 2.10 it follows that process X is a $\mathbb{Q}$-Wiener process

$$dY_t = dB_t \tag{4.5}$$

absorbed at 0.

For X as in (4.4) and $X_0 < a$, let

$$T := \inf\{t \geq 0 | X_t = a\}. \tag{4.6}$$

From (2.14), in Theorem 2.10, the probability of hitting a, starting from within the ball, satisfies the identity

$$\mathbb{P}(T < t) = \mathbb{E}_y^{\mathbb{Q}}\left[\frac{Y_s^2}{y^2}\exp\left\{-\int_0^s \frac{1}{Y_u^2}du\right\}\mathbb{I}_{(T_0>s, T<s)}\right]$$

where Y is as in (4.5) and, under $\mathbb{Q}$,

$$T_0 := \inf\{t \geq 0 | Y_t = 0\}. \tag{4.7}$$

Lemma 4.3. *Let Y be a process with $\mathbb{Q}$-dynamics as in (4.5) (an absorbed Wiener process at zero). Then*

$$Y_s^2 \cdot \exp\left\{-\int_t^s \frac{1}{Y_u^2}du\right\}$$

is a $\mathbb{Q}$-martingale.

Proof. Set

$$R_t \quad = \quad \exp\left\{-\int_0^t \frac{1}{Y_u^2}du\right\},$$

i.e.,

$$dR_t \quad = \quad -\frac{R_t}{Y_t^2}dt.$$

By Ito's lemma

$$\begin{aligned} d\left(Y_t^2 \cdot R_t\right) &= 2Y_t R_t dY_t + R_t dt - Y_t^2 \left(\frac{R_t}{Y_t^2}\right) dt \\ &= 2R_t Y_t dY_t. \end{aligned}$$

□

Proposition 4.4. *For Y, as in (4.5), T_0 as in (4.7), and*

$$(\mathbb{Q}) \qquad T := \inf\{t \geq 0 | Y_t = a\}$$

we have

$$\begin{aligned} &\mathbb{E}_y^{\mathbb{Q}}\left[\frac{Y_s^2}{y^2}\left\{-\int_0^s \frac{1}{Y_u^2}du\right\}\mathbb{I}_{(T_0>s,T<s)}\right] \\ &= \int_0^t \frac{a^2}{y^2}\mathbb{E}_y^{\mathbb{Q}}\left[\exp\left\{-\int_0^u \frac{1}{\tilde{Y}_u^2}du\right\}\right]\mathbb{P}_y\left(T \in du, T_0 > u\right). \end{aligned}$$

where $\tilde{Y}$ is a process which starts at y, is absorbed at zero, and reaches a, for the first time at $t = u$.

Proof. Let $M = Y^2 \cdot R$.

$$\begin{aligned} &\mathbb{E}_y^{\mathbb{Q}}\left[\frac{Y_s^2}{y^2}\left\{-\int_0^s \frac{1}{Y_u^2}du\right\}\mathbb{I}_{(T_0>s)}\mathbb{I}_{(T<s)}\right] \\ &= \mathbb{E}_y^{\mathbb{Q}}\left[\frac{M_s}{y^2}\mathbb{I}_{(T_0>s)}\mathbb{I}_{(T<s)}\right] \\ &= \mathbb{E}_y^{\mathbb{Q}}\left[\mathbb{E}_y^{\mathbb{Q}}\left[\frac{M_s}{y^2}\middle|\mathcal{F}_{s\wedge T_0\wedge T}\right]\mathbb{I}_{(T_0>s)}\mathbb{I}_{(T<s)}\right] \end{aligned}$$

(by the optional sampling theorem)

$$\begin{aligned} &= \mathbb{E}_y^{\mathbb{Q}}\left[\frac{M_{s\wedge T_0\wedge T}}{y^2}\mathbb{I}_{(T_0>s)}\mathbb{I}_{(T<s)}\right] \\ &= \mathbb{E}_y^{\mathbb{Q}}\left[\frac{M_T}{y^2}\mathbb{I}_{(T_0>s)}\mathbb{I}_{(T<s)}\right] \\ &= \mathbb{E}_y^{\mathbb{Q}}\left[\frac{a^2}{y^2}\exp\left\{-\int_0^T \frac{1}{Y_u^2}du\right\}\mathbb{I}_{(T_0>s)}\mathbb{I}_{(T<s)}\right] \\ &= \mathbb{E}_y^{\mathbb{Q}}\left[\frac{a^2}{y^2}\exp\left\{-\int_0^T \frac{1}{Y_u^2}du\right\}\mathbb{I}_{(T_0>s,T<s)}\right] \end{aligned}$$

(conditioning with respect to T)

$$\begin{aligned} &= \int_0^t \frac{a^2}{y^2}\mathbb{E}_y^{\mathbb{Q}}\left[\exp\left\{-\int_0^v \frac{1}{Y_u^2}du\right\}\middle|T = v, T_0 > v\right] \\ &\qquad \times\mathbb{P}_y\left(T \in dv, T_0 > v\right) \\ &= \int_0^t \frac{a^2}{y^2}\mathbb{E}_y^{\mathbb{Q}}\left[\exp\left\{-\int_0^v \frac{1}{\tilde{Y}_u^2}du\right\}\right]\mathbb{P}_y\left(T \in dv, T_0 > v\right). \end{aligned}$$

□

5. Dynamics of the process $\tilde{Y}$

In this section we derive the dynamics of process $\tilde{Y}$.

Proposition 5.1. *Let*

$$\mathbb{P}_{t,x}(T \in s, T_0 > s) \tag{5.1}$$

be the probability density of a one-dimensional standard Brownian motion absorbed at the origin defined in (4.2). Since (5.1) is a function of only $s-t$ and x let

$$h^a(s-t,x) := \mathbb{P}_{t,x}(T \in s, T_0 > s).$$

Then $\tilde{Y}$, which is a process absorbed at 0 and that reaches a for the first time at $t = s$ (and given that h_y stands for differentiation with respect to y) satisfies that

$$\begin{aligned} d\tilde{Y}_t &= \frac{h^a_y(s-t,\tilde{Y}_t)}{h^a(s-t,\tilde{Y}_t)}dt + dW_t, \quad 0 < t < s \\ \tilde{Y}_s &= a. \end{aligned} \tag{5.2}$$

In particular it has the following transition density

$$G(t,y;\tau,x) = \frac{h^a(s-\tau,x)}{h^a(s-t,y)}\mathbb{P}_{t,y}\left(Y_\tau \in dx, T \wedge T_0\right), \tag{5.3}$$

where $\mathbb{P}_{t,y}\left(Y_\tau \in dx, T \wedge T_0\right)$ is defined in (4.3).

Proof. We will construct the finite dimensional distributions of an absorbed Brownian motion at 0, and that reaches level a for the first time at $t = s$.

Given $t_0 < t_1 < \cdots < t = s$, we shall compute

$$\begin{aligned} &\mathbb{P}_y(Y_{t_1} \in dy_1, Y_{t_2} \in dy_2, \ldots, Y_{t_n} \in dy_n | T = s, T_0 > s) \\ &= \frac{\mathbb{P}_y(Y_{t_1} \in dy_1, Y_{t_2} \in dy_2, \ldots, Y_{t_n} \in dy_n, T \in ds, T_0 > s)}{\mathbb{P}_y(T \in ds, T_0 > s)}. \end{aligned} \tag{5.4}$$

By the independence of increments of Brownian motion Y, the numerator of the previous expression is

$$\begin{aligned} &\mathbb{P}_{t_0,y}(Y_{t_1} \in dy_1, T \wedge T_0 > t_1)\mathbb{P}_{t_1,y_1}(Y_{t_2} \in dy_2, T \wedge T_0 > t_2) \times \cdots \\ &\cdots \times \mathbb{P}_{t_{n-1},y_{n-1}}(Y_{t_n} \in dy_n, T \wedge T_0 > t_n)\mathbb{P}_{t_n,y_n}(T \in ds, T_0 > s). \end{aligned}$$

This in turn implies that (5.4) equals

$$\frac{\mathbb{P}_{t_n,y_n}(T \in ds, T_0 > s)}{\mathbb{P}_y(T \in ds, T_0 > s)} \prod_{j=1}^{n} \mathbb{P}_{t_{j-1},y_{j-1}}(Y_{t_j} \in dy_j, T \wedge T_0 > t_j). \tag{5.5}$$

However, since

$$\begin{aligned}
&\frac{\mathbb{P}_{t_n,y_n}(T\in ds, T_0>s)}{\mathbb{P}_y(T\in ds, T_0>s)}\\
&\quad=\frac{\mathbb{P}_{t_n,y_n}(T\in ds, T_0>s)}{\mathbb{P}_y(T\in ds, T_0>s)}\cdot\frac{\mathbb{P}_{t_{n-1},y_{n-1}}(T\in dt_n, T_0>t_n)}{\mathbb{P}_{t_{n-1},y_{n-1}}(T\in dt_n, T_0>t_n)}\cdots\\
&\qquad\cdots\frac{\mathbb{P}_{t_1,y_1}(T\in ds, T_0>s)}{\mathbb{P}_{t_1,y_1}(T\in ds, T_0>s)}\\
&\quad=\prod_{j=1}^{n}\frac{\mathbb{P}_{t_j,y_j}(T\in s, T_0>s)}{\mathbb{P}_{t_{j-1},y_{j-1}}(T\in s, T_0>s)}
\end{aligned}$$

(5.5) equals

$$\prod_{j=1}^{n}\frac{\mathbb{P}_{t_j,y_j}(T\in s, T_0>s)}{\mathbb{P}_{t_{j-1},y_{j-1}}(T\in s, T_0>s)}\mathbb{P}_{t_{j-1},y_{j-1}}(Y_{t_j}\in dy_j, T\wedge T_0>t_j).$$

On the other hand, to show that equation (5.3) is the transition density of $\tilde{Y}$, with dynamics as in (5.2), we make use of Propositions 2.4 and 2.8 by noting that h^a is a solution to the backward equation (2.7.a) [or equivalently (2.11.a)] (with $\sigma=1$ and $h=0$). □

Proposition 5.2. *Let $\tilde{Y}$ be a process with dynamics as in (5.2). Then*

$$u(t,y)=\mathbb{E}^{\mathbb{Q}}_{t,y}\left[\exp\left\{-\int_t^s\frac{1}{\tilde{Y}_u^2}du\right\}\right] \tag{5.6}$$

is equivalent to the following Cauchy problem

$$\begin{aligned}
-u_t+\frac{1}{y^2}u &= \frac{1}{2}u_{yy}+\frac{h^a_y(s-t,y)}{h^a(s-t,y)}u_y\\
u(s,y) &= 1.
\end{aligned} \tag{5.7}$$

Furthermore,

$$u=\frac{v}{h^a}$$

where

$$-v_t+\frac{1}{y^2}v = \frac{1}{2}v_{yy} \tag{5.8}$$

Proof. The equivalence between (5.6) and (5.7) follows from the Feynman-Kac theorem [see for instance pp. 366–367, Theorem 5.7.6 in Karatzas and Shreve [10]]. On the other hand equation (5.8) follows from Proposition 2.4 and by noting that h^a is a solution of the backward equation (2.7.a) [or equivalently (2.11.a)] (with $\sigma=1$ and $h=0$). □

Theorem 5.3. *Let X be as in (4.4) and T as in (4.6), then the probability that a 5-D Brownian motion started at 0 hits a ball of radius a is*

$$\begin{aligned}\mathbb{P}(T \in dt)/dt \\ &= \sum_{j=1}^{\infty} \frac{2}{3}\frac{\lambda_j^3}{a^2} \exp\left\{-\frac{1}{2}\frac{\lambda_j^2}{a^2}u\right\} \\ &\qquad \times \left[\frac{(1-\lambda_j^2)\sin(\lambda_j) - \lambda_j\cos(\lambda_j)}{2\lambda_j^2 + \lambda_j\sin(2\lambda_j) + 2\cos(2\lambda_j) - 2}\right],\end{aligned} \tag{5.9}$$

where λ_j is the j-th root of the Bessel function $J_{3/2}$ of the first kind.

Proof. Given a variable τ which satisfies $0 \le t \le \tau \le s$, let $\delta = \tau - t$. Then equation (5.8) becomes

$$v_\delta = \frac{1}{2}v_{yy} - \frac{1}{y^2}v. \tag{5.10}$$

For constants $\mu > 0$ and $A \in \mathbb{R}$, a solution to (5.10) is given by

$$v(\delta, y) = Ae^{-\frac{1}{2}\mu^2\delta}\sqrt{x}J_{3/2}(\mu y).$$

In fact (in terms of t) the following linear combination is also a solution

$$v(t, y) = \sum_{j=1}^{\infty} c_j e^{-\frac{1}{2}\frac{\lambda_j^2}{a^2}(\tau - t)}\sqrt{y}J_{3/2}\left(\lambda_j\frac{y}{a}\right).$$

Let the previous function satisfy the following boundary condition at $t = \tau$

$$u(\tau, y) = h^a(s - \tau, y).$$

From the two previous expressions we have that

$$h^a(s - \tau, y) = \sum_{j=1}^{\infty} c_j\sqrt{y}J_{3/2}\left(\lambda_j\frac{y}{a}\right).$$

Hence, by the orthogonality properties of the Bessel function $J_{3/2}$ we have that

$$\int_0^a \sqrt{x}\cdot h^a(s-\tau, x)J_{3/2}\left(\lambda_j\frac{x}{a}\right)dx = c_j\int_0^a x\cdot J_{3/2}^2\left(\lambda_j\frac{x}{a}\right)dx.$$

This implies that

$$c_j = \frac{\int_0^a \sqrt{x}h^a(s-\tau, x)J_{3/2}\left(\lambda_j\frac{x}{a}\right)dx}{\int_0^a x\cdot J_{3/2}^2\left(\lambda_j\frac{x}{a}\right)dx}.$$

Since we will only study the case when $y = 0$, we will make use of the following fact in the sequel

$$\lim_{y\to 0} y^{-3/2}J_{3/2}(\mu y) = \frac{1}{3}\mu^{3/2}\sqrt{\frac{2}{\pi}}.$$

If we let $b_j = \lambda_j/a$

$$\begin{aligned} p_j(x) &:= \frac{d}{dx}\left[\sqrt{x}J_{3/2}(b_jx)\right] \\ &= \sqrt{\frac{2}{\pi}}\frac{1}{b_j^{3/2}x^2}\left[(b_j^2x^2-1)\sin(b_jx)+b_jx\cos(b_jx)\right]. \end{aligned}$$

Given that p is as in (4.1), we need to find the limit as $\tau \to s$ of

$$\begin{aligned} &\int_0^a \sqrt{x}\cdot h^a(s-\tau,x)J_{3/2}\left(\lambda_j\frac{x}{a}\right)dx \\ &= \sum_{n=-\infty}^{\infty}\int_0^a \sqrt{x}\cdot J_{3/2}\left(\lambda_j\frac{x}{a}\right)dp(s-\tau;2na+a,x) \\ &= -\sum_{n=-\infty}^{\infty}\int_0^a p_j(x)p(s-\tau;2na+a,x)dx, \end{aligned}$$

The second line of the previous expression follows from the integration by parts formula and by noting that $J_{3/2}(\lambda_j)=0$, and on the other extreme we have $\sqrt{0}$ for all n. Next we make the following change of variable

$$z = \frac{2na+a-x}{\sqrt{s-\tau}}.$$

That is, for each p_j

$$-\int_{\frac{2na+a}{\sqrt{s-\tau}}}^{\frac{2na}{\sqrt{s-\tau}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}p_j(2na+a-z\sqrt{s-\tau})dz.$$

However notice that as $\tau \to s$ the only term that survives is when $n=0$ in which case we have

$$-\int_{\frac{2na+a}{\sqrt{s-\tau}}}^{0}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}p_j(a-z\sqrt{s-\tau})dz = \frac{1}{2}p_j(a).$$

Equivalently,

$$\frac{1}{2}p_j(a) = \frac{1}{\sqrt{2\pi a}}\frac{1}{\lambda_j^{3/2}}\left[(\lambda_j^2-1)\sin(\lambda_j)+\lambda_j\cos(\lambda_j)\right].$$

Since

$$\begin{aligned} \int_0^a x\left[J_{3/2}(b_jx)\right]^2dx &= \frac{2a^2b_j^2+ab_j\sin(2ab_j)+2\cos(2ab_j)-2}{2\pi ab_j^3} \\ &= a^2\cdot\frac{2\lambda_j^2+\lambda_j\sin(2\lambda_j)+2\cos(2\lambda_j)-2}{2\pi\lambda_j^3} \end{aligned}$$

we conclude that

$$c_j = \frac{\sqrt{2\pi}\lambda_j^{3/2}}{a^{5/2}}\left[\frac{(1-\lambda_j^2)\sin(\lambda_j)-\lambda_j\cos(\lambda_j)}{2\lambda_j^2+\lambda_j\sin(2\lambda_j)+2\cos(2\lambda_j)-2}\right],$$

which in turn yields

$$\sum_{j=1}^{\infty} \frac{2}{3}\frac{\lambda_j^3}{a^4} \exp\left\{-\frac{1}{2}\frac{\lambda_j^2}{a^2}u\right\}\left[\frac{(1-\lambda_j^2)\sin(\lambda_j)-\lambda_j\cos(\lambda_j)}{2\lambda_j^2+\lambda_j\sin(2\lambda_j)+2\cos(2\lambda_j)-2}\right]$$

or after multiplying the previous expression by a^2

$$\sum_{j=1}^{\infty} \frac{2}{3}\frac{\lambda_j^3}{a^2} \exp\left\{-\frac{1}{2}\frac{\lambda_j^2}{a^2}u\right\}\left[\frac{(1-\lambda_j^2)\sin(\lambda_j)-\lambda_j\cos(\lambda_j)}{2\lambda_j^2+\lambda_j\sin(2\lambda_j)+2\cos(2\lambda_j)-2}\right],$$

which is (5.9). □

6. Bessel process II

In this section we will make use of the ideas presented in Section 4 to find the probability that a Bessel process with negative order hits level $a > 0$ before hitting zero.

Theorem 6.1. *Consider X and Y satisfy that*

$$\begin{aligned} (\mathbb{P}) \qquad dX_t &= \frac{2}{X_t}dt + dB_t, \\ (\mathbb{Q}) \qquad dY_t &= -\frac{1}{Y_t}dt + dB_t. \end{aligned}$$

Let also T be as in (4.6) and, under $\mathbb{Q}$,

$$T_0 = \inf\{t \geq 0 | Y_t = 0\}.$$

Then

$$\begin{aligned} \mathbb{P}_y(T < t) &= \mathbb{E}_y^{\mathbb{Q}}\left[\frac{h(t,Y_t)}{h(0,Y_0)}\mathbb{I}_{(T<t,T_0>t)}\right] \\ &= \mathbb{E}_y^{\mathbb{Q}}\left[\frac{Y_t^3}{y^3}\mathbb{I}_{(T<t,T_0>t)}\right]. \end{aligned}$$

In particular

$$\mathbb{P}_y(T < t) = \frac{a^3}{y^3}\mathbb{Q}\left(T < t, T_0 > t\right).$$

Proof. Let $\alpha(x) = -1/x$. Then from equation (3.5) in Proposition 3.2 we have that

$$\begin{cases} -h_t = \frac{1}{2}h_{xx} - \frac{1}{x}h_x \\ -v_t = \frac{1}{2}v_{xx} + \left[-\frac{1}{x} + \frac{h_x}{h}\right]v_x \\ -w_t = \frac{1}{2}w_{xx} - \frac{1}{x}w_x. \end{cases}$$

In particular a solution to (3.5.a) is $h(x) = x^3$. Hence

$$\begin{cases} -h_t = \frac{1}{2}h_{xx} - \frac{1}{x}h_x \\ -v_t = \frac{1}{2}v_{xx} + \left[\frac{2}{x}\right]v_x \\ -w_t = \frac{1}{2}w_{xx} - \frac{1}{x}w_x. \end{cases}$$

From (3.4) we know that

$$\begin{aligned} v &= \frac{w}{h} \\ &= \frac{w}{x^3} \end{aligned}$$

Alternatively, from Theorem 3.4, we have that the processes X and Y that satisfy

$$\begin{aligned} (\mathbb{P}) \qquad dX_t &= \frac{2}{X_t}dt + dB_t \\ (\mathbb{Q}) \qquad dY_t &= -\frac{1}{Y_t}dt + dB_t \end{aligned}$$

are related by

$$\begin{aligned} \mathbb{P}_y(T > t) &= \mathbb{E}_y^{\mathbb{Q}}\left[\frac{h(t, Y_t)}{h(0, Y_0)}\mathbb{I}_{(T>t, T_0>t)}\right] \\ &= \mathbb{E}_y^{\mathbb{Q}}\left[\frac{Y_t^3}{y^3}\mathbb{I}_{(T>t, T_0>t)}\right]. \end{aligned}$$

From Ito's Lemma

$$\begin{aligned} dY_t^3 &= 3Y_t^2 dY_t + 3Y_t dt \\ &= -3Y_t^2 dB_t. \end{aligned}$$

Thus Y^3 is a martingale and hence, from the optional sampling theorem

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left[Y_t^3\mathbb{I}_{(T<t, T_0>t)}\right] &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[Y_t^3\mathbb{I}_{(T<t, T_0>t)}|\mathcal{F}_{t\wedge T_0\wedge T}\right]\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[Y^3_{T_0\wedge T\wedge t}\mathbb{I}_{(T<t, T_0>t)}\right] \\ &= a^3\mathbb{Q}\left(T < t, T_0 > t\right). \end{aligned}$$

□

Example 6.2. In Salminen [16] the author derives the density of a process X with dynamics

$$dX_t = \left[bX_t + \frac{1}{X_t}\right]dt + dB_t.$$

This process X can be interpreted as the Euclidean norm of three independent copies of an Ornstein-Uhlenbeck process with dynamics

$$dU_t = bU_t + dW_t.$$

Making use of Theorem 3.4 in Section 3 we arrive at the same conclusion. Namely, given processes X and Y with dynamics

$$\begin{aligned} (\mathbb{P}) \qquad dX_t &= \left[bX_t + \frac{1}{X_t}\right]dt + dB_t \\ (\mathbb{Q}) \qquad dY_t &= bY_t + dB_t \end{aligned}$$

the following identity holds

$$\mathbb{P}_{t,x}(X_s \in A) = \mathbb{E}_y^{\mathbb{Q}}\left[Y_t e^{-b(s-t)}\mathbb{I}_{(Y_s\in A, T_0>s)}\right],$$

where

$$T_0 := \inf\{t \geq 0 | Y_t = 0\}.$$

To do so, in (3.5)

$$\begin{cases} -h_t = \frac{1}{2}h_{xx} + \alpha(x)h_x \\ -v_t = \frac{1}{2}v_{xx} + \left[\alpha(x) + \frac{h_x}{h}\right] v_x \\ -w_t = \frac{1}{2}w_{xx} + \alpha(x)w_x \end{cases} \tag{6.1}$$

set $\alpha(x) = bx$ and as a solution of

$$-h_t = \frac{1}{2}h_{xx} + bxh_x$$

choose

$$h(t,x) = xe^{-bt}.$$

Hence $h_x/h = 1/x$ and thus

$$\begin{aligned} -v_t &= \frac{1}{2}v_{xx} + \left[\alpha(x) + \frac{h_x}{h}\right] v_x \\ &= -v_t = \frac{1}{2}v_{xx} + \left[bx + \frac{1}{x}\right] v_x. \end{aligned}$$

7. Concluding remarks

In this work we derive systems of coupled backward and forward Kolmogorov equations which in turn are used to derive dual representations of SDEs under a proper change of measure. In particular, the local drift of the corresponding process must be a linear combination of solutions of backward heat equations. Examples of process with this property are: Brownian bridge, 3-D Bessel bridge, Bessel processes of order 1 and 2. We also derive, making use of the techniques described within, the density of a Brownian motion started within some range $[0, a]$, which hits level a at time $v > 0$ before ever reaching zero.

We also derive, without making use of a Laplace-Gegenbauer transform, the density of the first time that a Bessel process of order 5 hits level $a > 0$. Our function is the same as the one appearing in equation (2.0.2) p. 398 in Borodin and Salminen [3]. Next, we relate the previous result, with the density of the first time that a Bessel of order -1 hits level a before reaching zero. Regarding a Bessel process of negative order see Göing-Jaeschke and Yor [5], and Revuz and Yor [14].

The coupled systems of one-dimensional backward and forward Kolmogorov equations, described in this paper, seem to have a "nice" and useful analytical structure. The extension of these results to d-dimensional PDEs as well as its implementation is work in progress.

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