# ON THE FIRST TIME THAT A 3-D BESSEL BRIDGE HITS A BOUNDARY 

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#### Abstract

In this work we derive the density of the first time that a 3-D Bessel bridge started at $y>0$, reaches a linear level $a+b \cdot t$ for $t \geq 0$ from above, $y>a$. As well as from below $y<a$. The reason as to why a distinction has to be made will be clarified.

Next, a sort of equivalence relationship between the 3-D Bessel bridge, the 3-D Bessel process, and the Brownian bridge is derived. In order to do so, we first introduce a new classification of SDEs in terms of solutions of Burgers' equation. For instance, these processes fall within the same class. This follows since one can show that the local drift of each is a solution to Burgers' equation. However, the local drift of the 3-D Bessel bridge, is also the sum of two solutions of Burgers' equation.

In general, linear combinations of solutions of Burgers' equation are not solutions themselves.


## 1. Introduction

In Gikhman (1957) and Kiefer (1959) the authors derive the distribution of the running maximum of a Bessel bridge $M_{k}$ of integer order $k$, which in turn is related to the Kolmogorov-Smirnov and Cramérvon Mises tests. The distribution is expressed in terms of a series of modified Bessel functions. In Pitman and Yor (1999) not only do they provide several characterizations of the distribution of $M_{k}$, they also show that Gikhman and Kiefer representation is valid for all real $k>0$. For recent applications of the 3-D Bessel bridge in credit risk see for instance Davis and Pistorius (2010).

There are a number of recent papers on the closely related topic of hitting times of Bessel process: e.g. Salminen and Yor (2011) or Alili and Patie (2010).

[^0]In this note we derive the density of the first time that a 3-D Bessel bridge $X$ started at $y \geq 0$ reaches a fixed level $a$ both from above as well as from below. The case in which the process starts at zero and reaches a fixed level $a>0$ corresponds to the distribution derived by Gikhman and Kiefer in the special case in which $k=3$. However our series expansions is in terms of the distribution of a one-dimensional standard Brownian motion absorbed at zero and level $a>0$. The density is also derived in the case in which the process starts above a linear barrier, which in turn remains strictly positive during the lifetime of the process. To do so we make use of an $h$ transform derived in Hernandez-del-Valle (2011) [regarding the $h$-transform see for instance Rogers and Williams (2000)].

Next we derive a sort of equivalence relationship between the 3-D Bessel bridge, the 3-D Bessel process, and the Brownian bridge. In order to so we classify processes whose local drift can be expressed as linear combinations of solutions to Burgers' equation. Under this classification these processes are within the same class. This follows since their local drift are all solutions to Burgers' equation. However in the case of the 3-D Bessel bridge its local drift is also a sum of two solutions of Burgers' equation. This property in general does not hold.

The paper is organized as follows in Section 2 an $h$-transform is derived without proof [for more details see Hernandez-del-Valle (2011) or Hernandez-del-Valle (2012)]. Next, in Section 3 the density of hitting a fixed boundary $a$ starting from above is derived, a concrete example is provided. This result is extended, in Section 4, in the case in which the barrier is linear. This result is a direct consequence of the fixed boundary case, Girsanov's theorem, and Appell's transform. Next, in Section 5, the problem of hitting the boundary from below is revisited. However, as pointed out above our representation involves the density of a Brownian motion absorbed at zero and level $a$. In Section 6 we develop a relationship class between the 3-D Bessel bridge, the 3-D Bessel process, and the Brownian bridge. We conclude in Section 7 with some final comments and remarks.

## 2. An $h$-TRANSFORM

Remark 2.1. (On notation.) In our results below we consider real valued differentiable functions $h, u, v$, and $w$, depending on variables $(t, x ; s, y) \in \mathbb{R}^{+} \times \mathbb{R} \times \mathbb{R}^{+} \times \mathbb{R}$. As in the analysis of diffusion processes, PDEs with derivatives with respect to $(t, x)$ are called backward equations, whereas PDEs with derivatives in $(s, y)$ are called forward equations.

Furthermore, through out this work (i) $B=\left\{B_{t}, \mathcal{F}_{t}\right\}_{t \geq 0}$ stands for one-dimensional standard Brownian motion. (ii) For a given function, say $w$, partial differentiation with respect to a given variable, say $x$, will be denoted as $w_{x}$.

In this section we relate, in Propositions 2.2 and 2.3, systems of coupled backward and forward Kolmogorov equations, through a pair of backward equations. This in turn allows us, in Theorem 2.4, to transform problems of diffusions with drift (or convective mass transfer) into problems of diffusions without drift (or with potential), under a suitable change of measure.

For the proof of the following statements, which have been omitted for ease of exposition, the reader may consult Hernandez-del-Valle (2011).

Proposition 2.2. For $\sigma: \mathbb{R}^{+} \times \mathbb{R} \rightarrow \mathbb{R}$, and $h$, $v$, and $w$ be of class $C^{1,2}$. Suppose that $v, w$, and $h$ satisfy the following identity

$$
\begin{equation*}
v(\tau, y)=w(\tau, y) \cdot h(\tau, y) \tag{1}
\end{equation*}
$$

If $h$ and $v$ satisfy respectively (a) and (b)

$$
\left\{\begin{array}{l}
w_{\tau}=\frac{1}{2} \sigma^{2} w_{y y}+\sigma_{y}^{2} w_{y}+\frac{1}{2} \sigma_{y y}^{2} w  \tag{2}\\
h_{\tau}=-\frac{1}{2} \sigma^{2} h_{y y} \\
v_{\tau}=\frac{1}{2} \frac{\partial^{2}}{\partial y^{2}}\left[\sigma^{2} v\right]-\frac{\partial}{\partial y}\left[\frac{h_{y}}{h} \sigma^{2} v\right]
\end{array}\right.
$$

then (c) holds.
Proposition 2.3. Let $\sigma: \mathbb{R}^{+} \times \mathbb{R} \rightarrow \mathbb{R}$, and $h$, $v$, and $w$ be of class $C^{1,2}$. Suppose that $v, w$, and $h$ satisfy the following identity

$$
\begin{equation*}
v(t, x)=\frac{w(t, x)}{h(t, x)} \tag{3}
\end{equation*}
$$

and $h(t, x) \neq 0$ for some strip in $\mathbb{R}^{+} \times \mathbb{R}$. If $w$ and $h$ satisfy respectively (a) and (b) satisfy the following system of partial differential equations

$$
\left\{\begin{array}{l}
-w_{t}=\frac{1}{2} \sigma^{2} w_{x x}  \tag{4}\\
-h_{t}=\frac{1}{2} \sigma^{2} h_{x x} \\
-v_{t}=\frac{1}{2} \sigma^{2} v_{x x}+\sigma^{2} \frac{h_{x}}{h} v_{x}
\end{array}\right.
$$

then (c) holds.
The following theorem follows from the previous propositions.
Theorem 2.4. Let $h$ be of class $C^{1,2}\left(\mathbb{R}^{+} \times \mathbb{R}\right)$ as well as a solution to the backward heat equation (2). Furthermore, consider processes $X$, and $Y$ which respectively satisfy (at least in the weak sense), the
following equations (each under their corresponding measures $\mathbb{P}$, and $\mathbb{Q})$,

$$
\begin{equation*}
(\mathbb{P}) \quad d X_{t}=\frac{h_{x}\left(t, X_{t}\right)}{h\left(t, X_{t}\right)} \sigma^{2}\left(t, X_{t}\right) d t+\sigma\left(t, X_{t}\right) d B_{t} \tag{5}
\end{equation*}
$$

$$
(\mathbb{Q}) \quad d Y_{t}=\sigma\left(t, Y_{t}\right) d B_{t} .
$$

Moreover, suppose that $f$ and $\sigma$ are real valued and integrable. Then the following identity holds

$$
\begin{equation*}
\mathbb{E}_{t, x}^{\mathbb{P}}\left[f\left(X_{\tau}\right)\right]=\mathbb{E}_{t, x}^{\mathbb{Q}}\left[\frac{h\left(\tau, Y_{\tau}\right)}{h(t, x)} f\left(Y_{\tau}\right)\right] . \tag{7}
\end{equation*}
$$

## 3. Hitting a fixed ball from outside

Throughout the remainder of this work $X$ denotes a 3-D Bessel bridge, which has the following dynamics:

$$
\begin{equation*}
d X_{t}=\left\{\frac{1}{X_{t}}-\frac{X_{t}}{s-t}\right\} d t+d B_{t}, \quad X_{0}=y>0 \tag{8}
\end{equation*}
$$

Furthermore let

$$
\begin{equation*}
T=\inf \left\{t \geq 0 \mid X_{t}=a\right\}, \quad a>0 \tag{9}
\end{equation*}
$$

be the first passage time of $X$ to level $a$ (unless otherwise stated).
Making use of Theorem 2.4 and assuming that $X$ starts outside a ball of radius $a$, that is $0<a \leq y$. The probability that a 3-D Bessel bridge process, absorbed at zero at time $s$, hits for the first time a ball of radius $a$ is the following.

Theorem 3.1. Let $X$ be a process with $\mathbb{P}$-dynamics as in (8). Then for $0 \leq t \leq s$ and $0<a \leq y$

$$
\mathbb{P}_{y}(T<t)=e^{\frac{y^{2}}{2 s}} \int_{0}^{t} \frac{s^{3 / 2} \cdot a}{(s-u)^{3 / 2} \cdot y} e^{-\frac{a^{2}}{2(s-u)}} \frac{y-a}{\sqrt{2 \pi u^{3}}} e^{-\frac{(a-y)^{2}}{2 u}} d u
$$

Proof. For $t \leq s, x \in \mathbb{R}$, the function

$$
h(t, x)=\frac{x}{\sqrt{2 \pi(s-t)^{3}}} \exp \left\{-\frac{x^{2}}{2(s-t)}\right\}
$$

is a solution to the backward heat equation. Alternatively since

$$
\frac{h_{x}}{h}=\left\{\frac{1}{x}-\frac{x}{(s-t)}\right\}
$$

then the dynamics of the 3-D Bessel bridge (8), for $0 \leq t \leq s$, may be written in terms of $h$ as follows

$$
\begin{aligned}
d X_{t} & =\left\{\frac{1}{X_{t}}-\frac{X_{t}}{(s-t)}\right\} d t+d B_{t} \\
& =\frac{h_{x}\left(t, X_{t}\right)}{h\left(t, X_{t}\right)} d t+d B_{t}, \quad X_{0}=y
\end{aligned}
$$

Given $T$ as in (9), and a process $Y$, with dynamics

$$
\begin{equation*}
d Y_{t}=d B_{t}, \quad Y_{0}=y, \quad t \geq 0 \tag{10}
\end{equation*}
$$

it follows, as a consequence of Theorem 2.4 that

$$
\begin{aligned}
\mathbb{P}_{y}(T<t) & =\mathbb{E}_{y}\left[\mathbb{I}_{(T<t)}\right] \\
& =\mathbb{E}_{y}^{\mathbb{Q}}\left[\frac{h\left(t, Y_{t}\right)}{h\left(0, Y_{0}\right)} \mathbb{I}_{(T<t)}\right] \\
& =e^{\frac{y^{2}}{2 s}} \mathbb{E}_{y}^{\mathbb{Q}}\left[\frac{s^{3 / 2} \cdot Y_{t}}{(s-t)^{3 / 2} \cdot y} \exp \left\{-\frac{Y_{t}^{2}}{2(s-t)}\right\} \mathbb{I}_{(T<t)}\right]
\end{aligned}
$$

From the following two facts: (i) $h$ is a solution to the backward heat equation then process $h(\cdot, Y$.) is a $\mathbb{Q}$ martingale, and (ii) $T \wedge t$ is a bounded stopping time w.r.t to the filtration generated by $\tilde{Y}$. Then, from the optional sampling theorem

$$
\begin{aligned}
& =e^{\frac{y^{2}}{2 s}} \mathbb{E}_{y}^{\mathbb{Q}}\left[\frac{s^{3 / 2} \cdot a}{(s-T)^{3 / 2} \cdot y} \exp \left\{-\frac{a^{2}}{2(s-T)}\right\} \mathbb{I}_{(T<t)}\right] \\
& =e^{\frac{y^{2}}{2 s}} \int_{0}^{t} \frac{s^{3 / 2} \cdot a}{(s-u)^{3 / 2} \cdot y} e^{-\frac{a^{2}}{2(s-u)}} \frac{y-a}{\sqrt{2 \pi u^{3}}} e^{-\frac{(a-y)^{2}}{2 u}} d u
\end{aligned}
$$

Example 3.2. In Figures 1 and 2, we have the true and empirical densities and distributions of the first time that a 3-D Bessel bridge process started at $y=3$ and absorbed at time $s=4$, reaches level $a=1$.

## 4. Hitting a moving ball from outside

In this section we find the probability of the first time that a 3D Bessel bridge absorbed at zero at time $s$, hits a ball from outside. However, now the inner ball is expanding or contracting at a linear speed. That is for $a<y$ let

$$
\begin{equation*}
T:=\inf \left\{0 \leq t \leq s \mid X_{t}=a+b t\right\} \tag{11}
\end{equation*}
$$

be the first passage time of $X$ to a linear moving boundary. Furthermore we assume that the linear boundary is strictly positive on $[0, s]$.

Theorem 4.1. Choose $0<a \leq y$ and let $f$ be the linear boundary

$$
f(t):=a+b t, \quad b \in \mathbb{R}, \quad t \in \mathbb{R}^{+} .
$$

If we assume that $f$ remains strictly positive on $[0, s]$ then for $t<s$

$$
\mathbb{P}_{y}(T<t)=e^{\frac{y^{2}}{2 s}} \int_{0}^{t} \frac{s^{3 / 2} \cdot(a+b u)}{(s-u)^{3 / 2} \cdot y} e^{-\frac{(a+b u)^{2}}{2(s-u)}} \frac{y-a}{\sqrt{2 \pi u^{3}}} e^{-\frac{(y-a+b t)^{2}}{2 a}} d u
$$

Proof. We follow the proof of Theorem 3.1. Let $T$ be as in (11)

$$
\begin{aligned}
\mathbb{P}_{y}(T<t) & =\mathbb{E}_{y}\left[\mathbb{I}_{(T<t)}\right] \\
& =\mathbb{E}_{y}^{\mathbb{Q}}\left[\frac{h\left(t, Y_{t}\right)}{h\left(0, Y_{0}\right)} \mathbb{I}_{(T<t)}\right] .
\end{aligned}
$$

Next observe that under $\mathbb{Q}$, and given process $Y$ with dynamics as in (10), the stopping time $T$ becomes

$$
\begin{aligned}
T & =\inf \left\{0 \leq t \leq s \mid Y_{t}=a+b t\right\} \\
& =\inf \left\{0 \leq t \leq s \mid Y_{t}-b t=a\right\} \\
& =\inf \left\{0 \leq t \leq s \mid \tilde{Y}_{t}=a\right\}
\end{aligned}
$$

That is process process $\tilde{Y}$ has the following $\mathbb{Q}$-dynamics

$$
d \tilde{Y}_{t}=-b d t+d B_{t}, \quad \tilde{Y}_{0}=y
$$

As a direct consequence of Girsanov's theorem we have

$$
\mathbb{P}_{y}(T<t)=\mathbb{E}_{y}^{\tilde{\mathbb{Q}}}\left[e^{-b \tilde{Y}_{t}+b y-\frac{1}{2} b^{2} t} \frac{h\left(t, \tilde{Y}_{t}+b t\right)}{h\left(0, \tilde{Y}_{0}\right)} \mathbb{I}_{(T<t)}\right]
$$

From the following two facts: (i) given that $h$ is a solution to the backward heat equation, we know that

$$
\exp \left\{-b x-\frac{1}{2} b^{2} t\right\} h(t, x+b t), \quad \mathbb{R}^{+} \times \mathbb{R}
$$

is also a solution to the backward heat equation as well (Appell's transform). This implies that process

$$
\exp \left\{-b \tilde{Y} \cdot-\frac{1}{2} b^{2} \cdot\right\} h(\cdot, Y \cdot+b \cdot)
$$

is a $\tilde{\mathbb{Q}}$-martingale, and (ii) $T \wedge t$ is a bounded stopping time w.r.t. the filtration generated by $\tilde{Y}$. Then, from the optional sampling theorem

$$
\begin{aligned}
& =\mathbb{E}_{y}^{\tilde{\mathbb{Q}}}\left[e^{-b(a-y)-\frac{1}{2} b^{2} T} \frac{h(T, a+b T)}{h\left(0, \tilde{Y}_{0}\right)} \mathbb{I}_{(T<t)}\right] \\
& =e^{\frac{y^{2}}{2 s}} \int_{0}^{t} \frac{s^{3 / 2} \cdot(a+b u)}{(s-u)^{3 / 2} \cdot y} e^{-\frac{(a+b u)^{2}}{2(s-u)}} \frac{y-a}{\sqrt{2 \pi u^{3}}} e^{-\frac{(y-a+b t)^{2}}{2 u}} d u .
\end{aligned}
$$

## 5. Hitting a ball from inside

In this section we derive the density of the first time that a 3-D Bessel process started at $y$ hits a fixed level $a$, given that $y<a$. That is, we find the probability of the first time that a ball with initial radius $y$. Which will contract by time $s$, will ever hit a fixed outer ball of radius $a$.

In order to do so, we will make use of the following. Given that $Y$ is a one-dimensional Wiener process started at $y$, let us define the following two stopping times

$$
\begin{aligned}
T_{0} & =\inf \left\{t \geq 0 \mid Y_{t}=0\right\} \\
T & =\inf \left\{t \geq 0 \mid Y_{t}=a\right\}, \quad 0<y<a
\end{aligned}
$$

Alternatively

$$
\begin{align*}
\mathbb{P}_{y}\left(T \wedge T_{0} \in d t\right):= & \frac{1}{\sqrt{2 \pi t^{3}}} \sum_{n=-\infty}^{\infty}\left[(2 n a+y) \exp \left\{-\frac{(2 n a+y)^{2}}{2 t}\right\}\right.  \tag{12}\\
& \left.+(2 n a+a-y) \exp \left\{-\frac{(2 n a+a-y)^{2}}{2 t}\right\}\right] d t  \tag{13}\\
12) & \\
13) \mathbb{P}_{y}\left(T_{0} \in d t\right):= & \frac{y}{\sqrt{2 \pi t^{3}}} \exp \left\{-\frac{y^{2}}{2 t}\right\} d t .
\end{align*}
$$

See for instance Chapter 2, Section 8 in Karatzas and Shreve (1991).
Theorem 5.1. Let process $X$ have dynamics as in (8) and $T$ be as in (9). We have for $0 \leq t \leq s$ and $0<y \leq a$

$$
\begin{aligned}
& \mathbb{P}_{y}(T<t)=e^{\frac{y^{2}}{2 s}} \int_{0}^{t} \frac{s^{3 / 2} \cdot a}{(s-u)^{3 / 2} \cdot y} e^{-\frac{a^{2}}{2(s-u)}} \\
& \quad \times\left(\mathbb{P}_{y}\left(T \wedge T_{0} \in d u\right)-\mathbb{P}_{y}\left(T_{0} \in d u\right)\right),
\end{aligned}
$$

where $\mathbb{P}_{y}\left(T \wedge T_{0} \in d t\right)$, and $\mathbb{P}_{y}\left(T_{0} \in d t\right)$ are as in (12), and (13) respectively.

Proof. We follow the proofs of Theorems 3.1 and 4.1. However, we must take into account the fact that the $\mathbb{Q}$-Wiener process $Y$ is absorbed at
zero. Given that $T$ is as in (9)

$$
\begin{aligned}
\mathbb{P}_{y}(T<t) & =\mathbb{E}_{y}\left[\mathbb{I}_{(T<t)} \mathbb{I}_{\left(T_{0}>t\right)}\right] \\
& =\mathbb{E}_{y}^{\mathbb{Q}}\left[\frac{h\left(t, Y_{t}\right)}{h\left(0, Y_{0}\right)} \mathbb{I}_{\left(T<t, T_{0}>t\right)}\right] \\
& =\mathbb{E}_{y}^{\mathbb{Q}}\left[\frac{h(T, a)}{h\left(0, Y_{0}\right)} \mathbb{I}_{\left(T<t, T_{0}>t\right)}\right],
\end{aligned}
$$

where the last line follows from the optional sampling theorem. Finally we recall the identity

$$
\begin{aligned}
\mathbb{P}_{y}\left(T<t, T_{0}>t\right) & =\mathbb{P}_{y}\left(T_{0}>t\right)-\mathbb{P}_{y}\left(T>t, T_{0}>t\right) \\
& =\mathbb{P}_{y}\left(T_{0}>t\right)-\mathbb{P}_{y}\left(T \wedge T_{0}>t\right) \\
& =\mathbb{P}_{y}\left(T \wedge T_{0}<t\right)-\mathbb{P}_{y}\left(T_{0}<t\right)
\end{aligned}
$$

Alternatively, in the case in which the boundary is linear, but remains strictly positive during the lifetime of the 3-D Bessel bridge, we have the following result:

Theorem 5.2. Choose $0 \leq y \leq a$ and let $f$ be the linear boundary

$$
f(t):=a+b t, \quad b \in \mathbb{R}, \quad t \in \mathbb{R}^{+} .
$$

If we assume that $f$ remains strictly positive on $[0, s]$ then for $t<s$

$$
\begin{aligned}
& \mathbb{P}_{y}(T<t)=\int_{0}^{t} e^{-a b+b y-\frac{1}{2} b^{2} u} \frac{h(u, a+b u)}{h(0, y)} \\
& \times\left(\mathbb{P}\left(T \wedge T_{0} \in d u\right)-\mathbb{P}_{y}\left(T_{0} \in d u\right)\right),
\end{aligned}
$$

where $\mathbb{P}_{y}\left(T \wedge T_{0} \in d t\right)$, and $\mathbb{P}_{y}\left(T_{0} \in d t\right)$ are as in (12), and (13) respectively.

Proof. We follow the proof of Theorem 5.1 together with Girsanov's rule as used in the proof of Theorem 4.1

$$
\begin{aligned}
& \mathbb{P}_{y}(T<t)= \mathbb{E}_{y}\left[\mathbb{I}_{(T<t)} \mathbb{I}_{\left(T_{0}>t\right)}\right] \\
&= \mathbb{E}_{y}^{\mathbb{Q}}\left[\frac{h\left(t, Y_{t}\right)}{h\left(0, Y_{0}\right)} \mathbb{I}_{(T<t)} \mathbb{I}_{\left(T_{0}>t\right)}\right] \\
&= \mathbb{E}_{y}^{\tilde{\mathbb{Q}}}\left[e^{-b \tilde{Y}_{t}+b y-\frac{1}{2} b^{2} t} \frac{h\left(t, \tilde{Y}_{t}+b t\right)}{h\left(0, \tilde{Y}_{0}\right)} \mathbb{I}_{(T<t)} \mathbb{I}_{\left(T_{0}>t\right)}\right] \\
&= \mathbb{E}_{y}^{\tilde{\mathbb{Q}}}\left[e^{-b a+b y-\frac{1}{2} b^{2} T} \frac{h(T, a+b T)}{h(0, y)} \mathbb{I}_{(T<t)} \mathbb{I}_{\left(T_{0}>t\right)}\right] \\
&= \int_{0}^{t} e^{-a b+b y-\frac{1}{2} b^{2} u} \frac{h(u, a+b u)}{h(0, y)} \\
& \quad \times\left(\mathbb{P}\left(T \wedge T_{0} \in d u\right)-\mathbb{P}_{y}\left(T_{0} \in d u\right)\right) .
\end{aligned}
$$

6. Equivalence of 3-D Bessel bridge, 3-D Bessel process, and Brownian bridge

In this section we derive a sort of equivalence relationship between the 3-D Bessel bridge, the 3-D Bessel process and the Brownian bridge. To this end we will make use of a more general version of Theorem 2.4.

Theorem 6.1. Let $g$ and $k$ be of class $C^{1,2}\left(\mathbb{R}^{+} \times \mathbb{R}\right)$ as well as solutions to the backward heat equation (2.b) [or equivalently (4.b)]. Furthermore, consider processes $X, Y$, and $Z$, which respectively satisfy (at least in the weak sense), the following equations (each under their corresponding measures $\mathbb{P}, \tilde{\mathbb{Q}}$, and $\mathbb{Q})$,

$$
\begin{align*}
& (\mathbb{P}) \quad d X_{t}=\left[\frac{k_{x}\left(t, X_{t}\right)}{k\left(t, X_{t}\right)}+\frac{g_{x}\left(t, X_{t}\right)}{g\left(t, X_{t}\right)}\right] d t+d B_{t} \\
& (\tilde{\mathbb{Q}}) \quad d Z_{t}=\left[\frac{g_{x}\left(t, Z_{t}\right)}{g\left(t, Z_{t}\right)}\right] d t+d B_{t}  \tag{14}\\
& (\mathbb{Q}) \quad d Y_{t}=d B_{t} .
\end{align*}
$$

Moreover, suppose that

$$
\begin{equation*}
u(\sigma, \phi):=\frac{k_{x}(\sigma, \phi)}{k(\sigma, \phi)} \frac{g_{x}(\sigma, \phi)}{g(\sigma, \phi)} \tag{15}
\end{equation*}
$$

and $k(\sigma, \phi) \cdot g(\sigma, \phi) \neq 0$ for some strip in $\mathbb{R}^{+} \times \mathbb{R}$. Then the following identities hold

$$
\begin{align*}
\mathbb{P}_{t, x} & \left(X_{\tau} \in A\right) \\
& =\mathbb{E}_{z}^{\tilde{\mathbb{Q}}}\left[\frac{h\left(\tau, Z_{\tau}\right)}{h(t, z)} \exp \left\{-\int_{t}^{\tau} u\left(\sigma, Z_{\sigma}\right) d \sigma\right\} \mathbb{I}_{\left(Z_{\tau} \in A\right)}\right] \\
& =\mathbb{E}_{y}^{\mathbb{Q}}\left[\frac{k\left(\tau, Y_{\tau}\right)}{k(t, y)} \frac{h\left(\tau, Y_{\tau}\right)}{h(t, y)} \exp \left\{-\int_{t}^{\tau} u\left(\sigma, Y_{\sigma}\right) d \sigma\right\} \mathbb{I}_{\left(Y_{\tau} \in A\right)}\right] . \tag{16}
\end{align*}
$$

Proof. See the proof of Theorem 2.10 in Hernandez-del-Valle (2011).

We will also introduce the following class.
Definition 6.2. We will say that process $X$, which satisfies the following equation

$$
d X_{t}=\mu\left(t, X_{t}\right) d t+d B_{t}
$$

is of class $\mathcal{B}^{n}$ if

$$
\begin{equation*}
\mu(t, x)=\sum_{j=1}^{n} \frac{h_{x}^{j}(t, x)}{h^{j}(t, x)} \tag{17}
\end{equation*}
$$

Where each $h^{j}$ is a solution of the backward heat equation (2.b) [or equivalently (4.b)].

Remark 6.3. Given that $h$ is a solution to the backward heat equation (2.b). Then function $\mu$, defined in (17), is a solution to (the backwards) Burgers' equation

$$
-\mu_{t}=\frac{1}{2} \mu_{x x}+\mu \cdot \mu_{x}
$$

. Furthermore note that if $\mu_{1}$ and $\mu_{2}$ are solutions to Burgers' equation, in general, $\mu_{1}+\mu_{2}$ is not.

Remark 6.4. Processes $X_{1}, X_{2}$, and $X_{3}$ which respectively satisfy the following equations

$$
\begin{cases}\left(\mathbb{P}^{1}\right) & d X_{1}(t)=\frac{1}{X_{1}(t)} d t+d B_{t}  \tag{18}\\ \left(\mathbb{P}^{2}\right) & d X_{2}(t)=-\frac{X_{2}(t)}{s-t} d t+d B_{t} \\ \left(\mathbb{P}^{3}\right) & d X_{3}(t)=\left[\frac{1}{X_{3}(t)}-\frac{X_{3}(t)}{s-t}\right] d t+d B_{t}\end{cases}
$$

are of class $\mathcal{B}^{1}$. That is, the 3-D Bessel process $X_{1}$, the Brownian bridge $X_{2}$, and the 3-D Bessel bridge have a local drift which is a solution to

Burgers' equation. This statement is verified by using the following solutions of the heat equation correspondingly

$$
\left\{\begin{array}{l}
k(t, x)=x, \quad g(t, x)=\frac{1}{\sqrt{2 \pi(s-t)}} \exp \left\{-\frac{x^{2}}{2(s-t)}\right\}  \tag{19}\\
h(t, x)=\frac{x}{\sqrt{2 \pi(s-t)^{3}}} \exp \left\{-\frac{x^{2}}{2(s-t)}\right\}
\end{array}\right.
$$

However, the local drift of the 3-D Bessel bridge can also be written as a sum of two solutions of burgers equations. That is.

Proposition 6.5. The 3-D Bessel bridge process $X$, which solves (8) is $\mathcal{B}^{1}$ and $\mathcal{B}^{2}$.

Proof. it follows by verifying that for $k, g$, and $h$ as in (19) the following identity holds

$$
\frac{h_{x}}{h}=\frac{k_{x}}{k}+\frac{g_{x}}{g}
$$

Example 6.6. The Bessel process of order 5 is $\mathcal{B}^{2}$ but not $\mathcal{B}^{1}$. See Hernandez-del-Valle (2011).

In the previous Sections we have studied the 3-D Bessel bridge process $X$ making use of Theorem 2.4. However since $X$ is also $\mathcal{B}^{2}$ we can also use Theorem 6.1 to obtain the following.

Theorem 6.7. Let processes $X_{1}, X_{2}$, and $X_{3}$ be as in (18). Furthermore, let functions $k, g$, and $h$ be as in (19) and process $Y$ be a solution of (14.Q). For a real-valued function $f$ the following identities hold

$$
\begin{align*}
\mathbb{E}_{t, x_{3}}^{\mathbb{P}_{3}}\left[f\left(X_{3}(t)\right)\right] & =\mathbb{E}_{t, x_{1}}^{\mathbb{P}_{1}}\left[\frac{k\left(\tau, X_{1}(\tau)\right)}{k\left(t, x_{1}\right)} f\left(X_{1}(\tau)\right) \frac{s-t}{s-\tau}\right] \\
& =\mathbb{E}_{t, x_{2}}^{\mathbb{P}_{2}}\left[\frac{g\left(\tau, X_{2}(\tau)\right)}{g\left(t, x_{2}\right)} f\left(X_{2}(\tau)\right) \frac{s-t}{s-\tau}\right]  \tag{20}\\
& =\mathbb{E}_{t, y}^{\mathbb{Q}}\left[\frac{g(\tau, Y(\tau))}{g(t, y)} \frac{k(\tau, Y(\tau))}{k(t, z)} f(Y(\tau)) \frac{s-t}{s-\tau}\right] .
\end{align*}
$$

Proof. For $k, g$, and $h$ as in (19) the following identity holds

$$
\frac{h_{x}}{h}=\frac{k_{x}}{k}+\frac{g_{x}}{g} .
$$

It follows that the dynamics of $X_{3}$, equation $\left(18 \cdot \mathbb{P}^{3}\right)$, can be written as

$$
\left(\mathbb{P}^{3}\right) \quad d X_{3}(t)=\left[\frac{k_{x}\left(t, X_{3}(t)\right)}{k\left(t, X_{3}(t)\right)}+\frac{g_{x}\left(t, X_{3}(t)\right)}{g\left(t, X_{3}(t)\right)}\right] d t+d B_{t}
$$

Moreover from Theorem 6.1, (18) and (14.Q), we have

$$
\begin{aligned}
\left(\mathbb{P}^{1}\right) \quad d X_{1}(t) & =\left[\frac{k_{x}\left(t, X_{1}(t)\right)}{k\left(t, X_{1}(t)\right)}\right] d t+d B_{t} \\
(\mathbb{Q}) & d Y_{t}
\end{aligned}=d B_{t} .
$$

and

$$
\begin{aligned}
u(\sigma, \phi) & :=\frac{k_{x}(\sigma, \phi)}{k(\sigma, \phi)} \frac{g_{x}(\sigma, \phi)}{g(\sigma, \phi)} \\
& =-\frac{1}{s-\sigma} \\
\Rightarrow-\int_{t}^{\tau} u(\sigma, z) d \sigma & =\ln \left[\frac{s-t}{s-\tau}\right]
\end{aligned}
$$

which yield lines one and three in equation (20). The second line follows by applying Theorem 6.1 first to process $X_{2}$.

## 7. Concluding Remarks

In this work we derive the density of the first time that a 3-D Bessel bridge: (i) hits a fixed level $a$ from above, (ii) hits a fixed level $a$ from below, and (iii) and hits a linear boundary from above. Keeping in mind applications of the 3-D Bessel process in Credit risk [see Davis and Pistorius (2010)].

Next, a relationship between the 3-D Bessel bridge, the 3-D Bessel process, and the Brownian bridge is introduced. General results of processes in class $\mathcal{B}^{1}$ is work in progress.

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Figure 1. (Example 3.2). The graph is plotted in R. The upper left graph is the histogram of the (simulated) first time that a 3-D Bessel bridge started at $y=3$, and absorbed at $s=4$, reaches level $a=1$. In the upper right frame we have the its theoretical density. In the lower left we have the simulated distribution, and finally on its right we have its theoretical counterpart.


Figure 2. (Example 3.2). The graph is plotted in R. The dotted line is the theoretical probability. The hard line is a simulation with $n=4500$.


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