Laplace's theories of cognitive illusions, heuristics, and biases*

Joshua B. Miller† Andrew Gelman‡

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Abstract

In his book from the early 1800s, *Essai Philosophique sur les Probabilités*, the mathematician Pierre-Simon de Laplace anticipated many ideas developed in the 1970s in cognitive psychology and behavioral economics, explaining human tendencies to deviate from norms of rationality in the presence of probability and uncertainty. A look at Laplace's theories and reasoning is striking, both in how modern they seem and in how much progress he made without the benefit of systematic experimentation. We argue that this work points to these theories being more fundamental and less contingent on recent experimental findings than we might have thought.

1. Heuristics and biases, two hundred years ago

> *One sees in this essay that the theory of probabilities is basically only common sense reduced to a calculus. It makes one estimate accurately what right-minded people feel by a sort of instinct, often without being able to give a reason for it.* (Laplace, 1825, p. 124)[1]

A century and a half before the papers of Kahneman and Tversky that kickstarted the "heuristics and biases" approach to the psychology of judgment and decision making and the rise of behavioral economics, the celebrated physicist, mathematician, and statistician Pierre-Simon de Laplace wrote a chapter in his *Essai Philosophique sur les Probabilités* that anticipated many of these ideas which have been so influential, from psychology and economics to business and sports.[2]

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†Department of Economics, University of Alicante, Spain.

‡Department of Statistics and Department of Political Science, Columbia University, New York.

[1] The quote continues: "It leaves nothing arbitrary in the choice of opinions and of making up one's mind, every time one is able, by this means, to determine the most advantageous choice. Thereby, it becomes the most happy supplement to ignorance and to the weakness of the human mind. If one considers the analytical methods to which this theory has given rise, the truth of the principles that serve as the groundwork, the subtle and delicate logic needed to use them in the solution of the problems, the public-benefit businesses that depend on it, and the extension that it has received and may still receive from its application to the most important questions of natural philosophy and the moral sciences; if one observes also that even in matters which cannot be handled by the calculus, it gives the best rough estimates to guide us in our judgments, and that it teaches us to guard ourselves from the illusions which often mislead us, one will see that there is no science at all more worthy of our consideration, and that it would be a most useful part of the system of public education."

[2] Laplace's essay began as an attempt to communicate to a general audience the practical insights of the probability theory that he developed in *Théorie Analytique des Probabilités*: "This philosophical Essay is an expanded version of a lecture on probability that I gave in 1795 at the Écoles Normales, whither I had been called as professor of mathematics with Lagrange by decree of the National Convention. I have recently published, on the same subject, a work entitled *Théorie Analytique des Probabilités.* Here I shall present, without using Analysis, the principles and general results of the *Théorie*, applying them to the most important questions of life, which are indeed, for the most part, only problems in probability." (Laplace, 1825, p. 1)

Laplace's chapter is called "Des illusions dans l'estimation des probabilitiés" ("On illusions in the estimation of probabilities," in the 1995 translation by Andrew Dale from which we take all our quotes).[3] Dale's translation is of the fifth edition and includes many ideas that we associate with the heuristics-and-biases revolution in cognitive psychology and economics.

In his mathematical and statistical work in probability theory and its applications, Laplace was one of the architects of the structure of probability as a form of reasoning about uncertainty, and developed what is now referred to as Bayesian inference (Stigler, 2005). Thus, one could say that Laplace contained within himself the normative view of probability calculus of von Neumann, as well as the view identified with the behavioral revolution that humans systematically depart from the normative model.

We were stunned to see in Laplace's one chapter so many ideas, treated in such depth, that seemed so fresh when studied by Tversky, Kahneman, and their colleagues, nearly two hundred years later. In addition to identifying several cognitive illusions—and introducing the concept of cognitive illusion—Laplace also offered insightful explanations for these counterintuitive attitudes and behaviors.

To note Laplace's contributions is not to diminish the contributions of earlier writers who had considered the gambler's fallacy and other misconceptions of probability, nor should it reduce our admiration for the acute observations and trailblazing experiments of later cognitive and behavioral scientists and their transformative impact across the social sciences.

Rather, we believe that Laplace's insights can give a clearer view of the necessity of the heuristics and biases paradigm, or some version of it. Using a mixture of introspection, qualitative observation, and logic, Laplace was able to identify a large number of serious flaws in the naive view of humans as rational actors under uncertainty. And, in part, we believe he was able to do so because he took the normative model of probabilistic (Bayesian) decision making so seriously. The most effective critics and tinkerers with a model are those who use it.

Accordingly, beginning with the same introspections that evidently guided Laplace, modern researchers went on to offer clear experimental demonstrations of behavior that departed from the normative model. That many of Laplace's explanations coincide with modern accounts—arrived at independently—suggests that the contributions of the heuristics and biases approach to judgement and decision making will have an enduring legacy.

Laplace's work reminds us how fundamental are heuristics and biases to our cognitive processes, in that these insights were all there for the taking, nearly two hundred years before they were demonstrated experimentally—indeed, long before the field of experimental psychology even existed.[4] To draw a physics analogy, Laplace's combination of observation and logic reveals incoherence in the model of humans as rational actors, in the same way that applications of Maxwell's equations in the late 1800s revealed internal inconsistencies with the solar-system model of the atom and demonstrated the need for something like quantum theory.

We are not claiming in this article to have made any historical discoveries. Laplace's Philosophical Essay has always been recognized as a founding document of probability theory in all its complexity; for example, Ayton and Fischer (2004) write, "The idea that beliefs about probability show systematic biases is somewhat older than experimental psychology," and note that Laplace

[3]Laplace's first full presentation of the material for this chapter came in the fourth edition of the *Essai*, from 1819; see Stigler (2005, 2012).

[4]In the same chapter, Laplace also has a remarkable discussion of visual perception that anticipates later work in psychophysics, including a modern take on top-down visual processing. It is not clear how many of these insights are due to Laplace, and how many were common knowledge among the community of scientists with which he corresponded.

"was concerned with errors of judgment and even included a chapter concerning 'illusions in the estimation of probabilities.' It is here that we find the first published account of what is widely known as the gambler's fallacy..." But, as we discuss below, Laplace's observations and theorizing went far beyond the gambler's fallacy, and his anticipation of some of the more sophisticated later work in psychology and economics may not be so well known, as his name is not mentioned once in the classic judgment and decision making collection of Kahneman et al. (1982), nor in the follow-up volume of Gilovich et al. (2002), the well-known behavioral economics volume of Camerer et al. (2003), or the comprehensive judgment and decision making textbook of Baron (2008). Recognition of Laplace's contributions does not invalidate later work in psychology and behavioral economics; rather, it gives us a new perspective on these ideas as being more universal and less contingent on particular developments in the 1970s and later.[5]

2. What is remarkable about Laplace's chapter?

Anticipating the approach of the heuristics and biases literature, Laplace introduces the concept of a cognitive illusion by drawing an analogy to visual illusions:[6]

> *The mind, like the sense of sight, has its illusions; and just as touch corrects those of the latter, so thought and calculation correct the former.* (Laplace, 1825, p. 91)

Laplace's approach to identifying cognitive illusions follows the now familiar template: provide the rational benchmark as represented by the beliefs the ideal decision maker, which Laplace's probability theory served to model, then demonstrate how people's behavior deviates from this benchmark (Kahneman and Tversky, 1982). Anticipating the dual-process theory of James (1890) and later cognitive psychology literature (Kahneman, 2011), Laplace asserts that the use of intuition rather than well-reasoned judgment is the source of cognitive illusions:

> *One of the great advantages of the probability calculus is that it teaches us to distrust our first impressions. As we discover, when we are able to submit them to the calculus, that they are often deceptive, we ought to conclude that it is only with extreme circumspection that we can trust ourselves in other matters.* (Laplace, 1825, p. 94)

Laplace's chapter consists of a collection of anomalies that he had observed, many of which were already well known and exploited by purveyors of gambling games. For each anomaly, Laplace speculates on why it exists, using some combination of (i) elaborating on people's verbal justifications, (ii) introspection, and (iii) an application of the psychological theories in vogue at the time.

While we focus our discussion here on the chapter on illusions, Laplace's insights for psychology, and the other social sciences—which he referred to as the "moral sciences," or "political economy," depending on the edition—also appear elsewhere in his book. Most notably, in the seven preceding chapters Laplace employs Bernoulli's stylized balls-in-urns framework to illustrate how the insights from probability theory can improve the decision making of practitioners and the design of state

[5] A similar point has been made about the work of Adam Smith and its relation to behavioral economics (Ashraf et al., 2005).

[6] For example, Kahneman and Tversky (1982) write: "The emphasis on the study of errors is characteristic of research in human judgment, but is not unique to this domain: we use illusions to understand the principles of normal perception and we learn about memory by studying forgetting. Errors of reasoning, however, are unique among cognitive failures in two significant respects: they are somewhat embarrassing and they appear avoidable." See Kahneman (1991) and Gigerenzer (2005) for contrasting perspectives.

decision making bodies. Perhaps the most striking example from the perspective of the behavioral scientist comes from his chapter on the probability of testimony in which Laplace introduces what must be the first account of the base-rate fallacy later discussed by Meehl and Rosen (1955); Bar-Hillel (1980); Tversky and Kahneman (1974).[7] In particular, Laplace leverages its framework in order to develop novel (at the time) insights into how to assess the (posterior) credibility of witnesses to improve decision making in the courtroom, and in daily life.[8]

Laplace focuses on people's faulty reasoning in his chapter on illusions, but one should not conclude that Laplace viewed irrational behavior as the inevitable consequence. On the contrary, in the second chapter of the *Essai*, Laplace introduces inverse probability—reasoning backwards from (observed) events to their (hypothetical) causes—as his Sixth Principle of the probability calculus, and in so doing, anticipates the modern idea of *ecological rationality* (Todd and Gigerenzer, 2000). In particular, Laplace explains how the principle of inverse probability can rationalize why people have a tendency to perceive "regular" events in sequences, such as HHHHHH in a sequence of coin flips, as more surprising and indicative of an underlying cause than "irregular" events, such as HTTHTH. While Laplace acknowledges that people's verbal justifications for this tendency can be fallacious,[9] he argues that the implicit reasoning is not necessarily wrong by drawing an analogy to the presumably more typical (ecologically valid) case in which the underlying causal mechanism is uncertain. In Laplace's view it is reasonable to attribute a particular cause to the observation of a regular event because "this [regular] event must be the effect either of a regular cause or of chance, the first of these suppositions is more probable than the second" (Laplace, 1825, p. 9). Laplace uses this principle to great effect in his chapter, "Application du Calcul des Probabilités à la Philosophie naturelle," in which he illustrates the usefulness of null ("*hasard*") hypothesis significance testing in the natural sciences.[10]

[7] "Let us suppose that experience has shown that this witness lies once in ten times, so that the probability of the truth of his testimony is 9/10 ... Let us suppose now that the urn contains 999 black balls and one white one, and that after a ball has been drawn from the urn, a witness to the drawing declares that this ball is white" (Laplace, 1825, p. 65-67). Because the witness is 9 times more likely to declare white when the ball is white vs. when the ball is black, the posterior odds in favor of the ball being white are 9 times higher than the prior odds. Letting b be the number of black balls, the prior odds are 1:b, so the posterior odds become 9:b, i.e. the posterior probability that the witness is telling the truth is $9/(9+b)$, therefore when $b = 999$, the probability is 9/1008. Laplace notes that that as the number of black balls b increase, the posterior probability that the witness is lying $b/(9+b)$ approaches certainty. Laplace discusses that some authors (without common sense) entirely neglect base rates, and that even those with common sense may fail to take full cognisance of them: "... we find that the probability of an error or of a lie on the part of the witness increases with the increasing extraordinariness of the matter attested. Some authors have put forward the contrary view, basing their opinion on the assumption that, the appearance of an extraordinary matter being completely similar to that of an ordinary one, the same reasons ought to lead us to give the same credence to the witness, when he asserts one or other of these matters. Simple common sense rejects this very strange assertion; but the probability calculus, while supporting the conclusions of common sense, also takes cognisance of the unlikeliness of testimonies on extraordinary matters."

[8] After lamenting the proclivity of even great minds to believe in miracles Laplace writes: "The true principles of the probability of testimony having been thus misunderstood by philosophers to whom reason is chiefly indebted for its progress, I have thought it necessary to present at length the results of the calculus on this important matter." (Laplace, 1825, p. 71)

[9] "This principle explains why regular events are attributed to a particular cause. Some philosophers [e.g. d'Alembert, Laplace's sponsor] have thought that such events are less likely than others, and that in the game of heads or tails, for example, the combination in which heads turns up twenty times running is dispositionally inclined to occur less readily than those combinations in which heads and tails are intermingled in an irregular manner. But this opinion supposes that past events have an influence on the possibility of future events, which is not admissible. Regular combinations occur more rarely only because there are fewer of them." (Laplace, 1825, p. 9)

[10] "But in order not to lose oneself in vain speculations it is necessary, before looking for the causes, to be sure that they are indicated with a probability that does not allow them to be regarded as anomalies due to chance." (Laplace, 1825, p. 43)

In our discussion of Laplace's chapter on illusions in the following sections, we organize (loosely) the anomalies into the three categories of reasoning employed by Tversky and Kahneman (1974): representativeness, anchoring and adjustment, and availability.

2.1. Representativeness

Kahneman and Tversky (1972b) define representativeness as the degree to which an uncertain event, or sample is "(i) similar in essential properties to its parent population; and (ii) reflects the salient features of the process by which it is generated." Laplace provides several examples of systematic errors in people's subjective assessment of probability that he attributes to representativeness-type judgments.

Laplace's first example involves the first explicit account that we have seen of what Tversky and Kahneman (1971) call the belief in the "law of small numbers," which can lead to gambler's fallacy-like beliefs outside the confines of the casino. Laplace observes that people commonly believe that the ratio of boys to girls must be nearly balanced at the end of each month. Consequently, after learning that there has been a preponderance of boys in a given month, men hoping for a son become discouraged and mistakenly conclude that girls must be more probable in order to compensate for the current sex imbalance. Laplace identifies this thinking as a naive generalization of an effect of sampling without replacement from a finite urn, extrapolated to an infinite urn, which is a behavioral assumption that has also been explored in the psychology and behavioral economics literature (Morrison and Ordeshook, 1975; Rabin, 2002).[11]

Laplace's second example, which he observes to be the opposite illusion, is closely related to the hot hand fallacy (Gilovich et al., 1985; Miller and Sanjurjo, 2017, 2018).[12,13] Laplace recounts that in the French lottery, when certain numbers are drawn more often than what would be expected by chance, people come to believe that these numbers are lucky, despite the fact that the numbers are transparently generated by an independent and identically distributed process.[14] Laplace, continuing with this theme, observes how people overreact to streaks of consecutive outcomes in casino games, attributing them to fate when they could just as easily be attributed to the

[11] "I have seen men, ardently longing for a son, learning only with anxiety of the births of boys in the month in which they expected to become fathers. Thinking that the ratio of these births to those of girls ought to be the same at the end of each month, they fancied that the boys already born made it more probable that girls would be born next. In this way the drawing without replacement of a white ball from an urn that contains a limited number of white and black balls, increases the probability of drawing a black ball on the next draw. But this ceases to hold when the number of balls in the urn is unlimited, as should be supposed in order to compare this case to that of births." (Laplace, 1825, p. 93)

[12] Laplace makes no attempt to reconcile the seeming contradiction of holding both hot hand and gambler's fallacy beliefs. We are aware of three types of explanations: (i) rational inference conditional on (incorrect) gambler's fallacy beliefs (Rabin, 2002), (ii) people's default assumptions on the sign of serial dependence being contingent on perceived properties of the underlying data generating process (Ayton and Fischer, 2004; Oskarsson et al., 2009), and (iii) evolutionary adaptiveness of default assumptions when exploiting clumpy resources or exploring for them (Wilke and Barrett, 2009).

[13] Gilovich et al. (1985) conclude that the belief in the hot hand is an example of costly cognitive illusion outside of the laboratory. However, their two key analyses, one involving basketball shooting, and the other involving an incentivized prediction task, have been shown to be invalid. Moreover, a reanalysis of their data reveals hot hand shooting as well as an ability of hot hand believers to predict shot outcomes at levels meaningfully greater than chance (Miller and Sanjurjo, 2017, 2018). For our discussion here, though, all that is relevant is that in a game of chance with known probabilities, any belief in serial dependence is clearly an error.

[14] "Under an illusion contrary to the preceding ones, one may look in previous draws of the French lottery for the numbers that have most often been drawn, to form combinations on which one believes one's stake may advantageously be placed. But seeing how the numbers are shuffled in this lottery, we may conclude that the past ought to have no effect on the future." (Laplace, 1825, p. 93)

"capriciousness of chance."[15] Extending this example, Laplace then returns to the gambler's fallacy, providing perhaps the first account of one side of the disposition effect, the reluctance to realize one's losses (Shefrin and Statman, 1985; Odean, 1998; Imas, 2016). Laplace remarks that gamblers often bet more after losses, believing that their bad luck must eventually turn, as in Kahneman and Tversky (1979).[16]

Laplace's third illusion illustrates how the representativeness heuristic can lead to the neglect of prior probability, what was later called the base-rate fallacy (Bar-Hillel, 1980). Laplace employs a counter-intuitive conditional probability paradox that is analogous to the canonical base-rate neglect problem (Meehl and Rosen, 1955; Kahneman and Tversky, 1972a), though more complicated.[17] The paradox involves an urn of uncertain composition: it contains four balls, each black or white, but not all the same color (uniform prior). A ball is then drawn from the urn, and it happens to be white. Intuition correctly leads one to conclude that the urn is more likely to contain a majority of white balls, as that is more representative of a draw of a white ball. On the other hand, when asked about the probability of drawing (with replacement) four consecutive black balls, people wrongly conclude that this probability is lower than it would be if the urn were known to be precisely balanced (two white balls and two black balls). Intuitively, drawing four consecutive black balls is not representative of an urn that (likely) contains a majority of white balls, while it is relatively more representative of an urn that is balanced. What intuition neglects here is that uncertain urn has a $\frac{1}{3}$ prior chance of containing a majority of black balls, which is impossible with balanced urn. While the draw of the white ball from the uncertain urn reduces the probability that the uncertain urn contains mostly black balls (consistent with intuition), it does not eliminate it.[18]

Laplace's fourth example illustrates how the representativeness heuristic can lead to a mistake that may underly the well documented error of insensitivity to sample size (Kahneman and Tversky, 1972b), conservatism (Edwards, 1968; Tversky and Kahneman, 1974), and the failure to believe in the law of large numbers (Benjamin et al., 2016). Again using the balls-in-urns framework, Laplace observes that with an urn consisting one white ball and one black ball, offering a payout of 1:1 for drawing a white ball is a fair bet. On the other hand if the urn consists of one white ball and two black balls this payout is no longer fair. In order to provide a fair bet to people betting on white, intuition suggests that one should allow the bettor two chances to draw a white so that the base rate of two chances of black for every one chance of white is counter-balanced. The mistake here is that when moving from one draw to two draws, people fail to realize that there is more than one way to draw one black and one white. This mistake is a simple version of the general failure to appreciate how probability mass concentrates towards central values when summing together independent random variables, which may lead to an insensitivity to sample size, conservatism, and a failure to believe in the law of large numbers.

[15] "In a long series of events of the same kind, the very capriciousness of chance ought sometimes to produce those singular runs of good or bad luck that most players do not fail to attribute to a kind of fate" (Laplace, 1825, p. 93).

[16] "It often happens in games that depend simultaneously on chance and the skill of the players, that he who loses, unsettled by his loss, tries to make up for it by daring throws that he would eschew in another situation. He thus aggravates his own bad luck, and prolongs its duration." (Laplace, 1825, p. 93–94)

[17] As discussed in Section 2, Laplace was the first to construct a base-rate neglect problem.

[18] The number of black balls in the urn can be $b = 1, 2$, or 3, each with equal chance, so the prior odds in favor of the urn being mostly black is 1:2. The relative likelihood of drawing white from a mostly black vs. not mostly black urn is less than one, i.e. $\Pr(\text{draw white} \,|\, b = 3)/\Pr(\text{draw white} \,|\, b = 1 \text{ or } 2) = \frac{1}{4}/(\frac{1}{2} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2}) = \frac{1}{4}/\frac{5}{8} = \frac{2}{5}$, so posterior odds are $\frac{2}{5}$:2, or 1:5. Thus, $\Pr(b = 3 \,|\, \text{draw white}) = \frac{1}{6}$.

2.2. Anchoring and adjustment

Tversky and Kahneman (1974) define anchoring and adjustment as a heuristic procedure for estimating unknown quantities in which one begins with the information one knows, and adjusts until an acceptable value is reached, typically insufficiently (Epley and Gilovich, 2006).

While Laplace does not explicitly discuss anchoring effects, he does propose an explanation for the gambler's fallacy in games of chance that involves an endogenous form of anchoring. He begins with the first published account we have seen of the gambler's fallacy:

> *When one number has not been drawn for a long time in the French lottery, the mob is eager to bet on it. They fancy that, because the number has not been drawn for a long time, it, rather than the others, ought to be drawn on the next draw.* (Laplace, 1825, p. 92)

He then offers his explanation for the gambler's fallacy, which appears to be novel relative to the literature (Nickerson, 2002; Oskarsson et al., 2009; Bar-Hillel and Wagenaar, 1991):

> *So common an error seems to me to rest on an illusion, in which one involuntarily refers back to the source of events. It is, for example, very unlikely that in a game of heads or tails one will get heads ten times running. This unlikeliness, which surprises us even when the event has happened nine times, leads us to believe that tails will occur on the tenth toss.* (Laplace, 1825, p. 92)

Laplace's explanation here differs from his *law of small numbers* explanation of why people expect more girl births after an the excess of boy births (see Section 2.1). Instead, Laplace proposes a mental process in which, after observing nine heads in row, people assess the probability of heads on the next flip using the prior (past) probability as a reference point. In particular, instead of assessing the probability of heads on the next flip in isolation, people frame their problem as that of assessing the probability of observing ten heads in row. Because this event was exceedingly unlikely before they began flipping, they continue to anticipate being surprised should the event occur, even though they have already observed nine heads in a row.

This story rings a bell today. For example, the victory of Donald Trump in 2016, first in the Republican primary and then in the general election, provoked a general surprise that could be considered as a slowness or reluctance to update given the intermediate information of Trump's continuing viability in the polls even after pundits had so many times pronounced the death of his candidacy.

In the case of coin flips, this explanation amounts to a form of endogenous anchoring—the gambler refers back to her (correct) low ex ante belief of 2^{-10} for ten heads in a row, but does not sufficiently adjust (if at all) once nine heads have been flipped. Viewed another way, it is possible to interpret this explanation as form of confirmation bias in that the gambler's high prior belief of not observing ten heads in a row persists despite already having observed nine consecutive disconfirming events; see, for example, Nickerson (1998), for a review.[19]

Laplace's explanation was possibly influenced by the famous mistake of his sponsor, the eminent 18th century mathematician Jean-Baptiste le Rond D'Alembert, who earnestly argued that tails was more likely after three heads in a row and made a similar appeal to the low prior probability of

[19] Laplace also explicitly discussed confirmation bias (see Section 2.4).

four heads in a row.[20] Given d'Alembert's ability, and reputation, his blunder amounts to perhaps the most compelling evidence of the strength of the gambler's fallacy, in contrast to nearly costless exhibits of this error investigated, for example, by Croson and Sundali (2005), Sundali and Croson (2006), Clotfelter and Cook (1993), and Terrell (1994).[21]

2.3. Availability and related heuristics

Tversky and Kahneman (1974) define availability as a heuristic procedure for estimating the frequency or probability of an event by "the ease with which instances or occurrences can be brought to mind." Availability can be assessed by the ease of recall of previously experienced events, or the ease of imaginability of an event occurring, either due to its vividness, or other salient features that are associated with them. Tversky and Kahneman (1973)'s conception of the underlying mechanism behind availability is worth quoting at length, as it coincides remarkably with the associationist psychology that influenced Laplace:[22]

> *That associative bonds are strengthened by repetition is perhaps the oldest law of memory known to man. The availability heuristic exploits the inverse form of this law, that is, it uses strength of association as a basis for the judgment of frequency. In this theory, availability is a mediating variable, rather than a dependent variable as is typically the case in the study of memory. Availability is an ecologically valid clue for the judgment of frequency because, in general, frequent events are easier to recall or imagine than infrequent ones. However, availability is also affected by various factors which are unrelated to actual frequency.* (Tversky and Kahneman, 1973, p. 209)

Laplace places particular focus on the role of experienced frequencies and how they shape the memories, habits, and emotions that are available for the mind to draw upon when assessing probabilities.

Laplace's first example is a thought experiment, which can be viewed as a field analogue of the widely documented description-experience gap found in the laboratory (Hertwig et al., 2004; Erev et al., 2010). In particular, consistent with the later literature, he notes how the small chance of a negative outcome has less impact on decisions involving experienced frequencies, than on decisions involving explicit probabilities,

> *Probability based on daily experience . . . affects us more than a larger probability that is only a simple result of calculation. Thus, in return for small gains, we have no fear at all in exposing our lives to risks much less unlikely than the drawing of a quine in the French lottery; and yet one would not choose to get the same benefits, with the certainty of losing one's life if a Quine were to occur.* (Laplace, 1825, p. 91)

[20] In D'Alembert's response to the problem, "When a fair coin is tossed, given that heads have occurred three times in a row, what is the probability that the next toss is a tail?", he argues that the probability of a tail is greater than 1/2 because, from the ex ante perspective, it is unlikely that a probable event will never occur in a finite sequence of trials (D'Alembert, 1761, pp. 13–14); see Gorroochurn (2012, p. 124) for a discussion.

[21] The observation that anecdotal experience can sometimes be more compelling than controlled interventions is reminiscent of an observation of Thaler (1980, p. 48) that the "strongest support for the sunk cost hypothesis can be found in the classroom. Anyone who has ever tried to teach this concept knows that it is not intuitively obvious, even to some experienced business people." Also see Kamenica (2012, p. 17 footnote 39) for a similar comment, highlighting the ongoing contrast between the limited formal evidence supporting the sunk cost fallacy, and the strong evidence available from interaction in the classroom.

[22] Daston (1988, pp. 218–223) discusses how Laplace's availability-like explanations were influenced by (i) the associationist psychology of the time, and (ii) his recognition of the aberrations due these mental associations.

A "quine" here is getting all of the 5 drawn numbers from 1 through 90 correct—a 44-million-to-one event; see Stigler (2003).

Later in the chapter Laplace proposes an explanation for this (presumed) phenomenon, which is novel relative to the explanations that have been offered for the description-experience gap in the laboratory (Hertwig and Erev, 2009):

> *From what has been said, one sees how much our belief depends on our habits. Accustomed to judge and to conduct ourselves in accordance with certain probabilities, we give our assent to these probabilities, as if by instinct, and they in turn cause us to take resolutions with more force than very much greater probabilities that result from reflexion or calculation.* (Laplace, 1825, p. 110)[23]

Laplace's second set of examples illustrate how availability can be influenced by experience, salience, affect, and recency. Laplace observes the bias induced by overweighting of present events relative to those in the past:

> *Present evils and their cause affect us much more than the recollection of evils produced by the contrary cause: they prevent our correct appreciation of the disadvantages of both and of the probability of the appropriate means to guard against them. It is this that leads nations emerging from a state of peace (one to which they can never return except after long and painful disturbances), alternately to despotism and to anarchy.* (Laplace, 1825, p. 92)

Laplace observes how personal experience is overweighted relative to the experiences of others (an idea later studied by Simonsohn et al. (2008) and Miller and Maniadis (2013), among others):

> *This vivid impression, which we receive of the presence of events, and which scarcely allows us to notice the contrary events observed by others, is a prime cause of error, against which one cannot guard oneself too much.* (Laplace, 1825, p. 92)

Laplace further observes how the presence of a large jackpot and the publicity given to winners, but not losers, of those jackpots can lead people to be generally insensitive to probabilities and, in particular, overweight a small probabilities:

> *Most of those who take part in lotteries do not know how many chances are to their advantage and how many are against them. They see only the possibility of winning a large sum for a small stake, and the schemes that their imaginations produce exaggerate, in their eyes, the probability of getting that sum . . . All would, without doubt, be alarmed at the enormous number of wagers lost if they got to know of this: but, on the contrary, care is taken to give great publicity to the wins.* (Laplace, 1825, p. 92)

Later in the chapter Laplace emphasizes the role of affect in the distortion of subjective probabilities (a topic later studied by Loewenstein et al. (2001) and Fischhoff et al. (1978), among others):

> *Finally we shall establish, as a psychological principle, the exaggeration of probabilities by the passions. Something that is feared, or that is keenly desired, seems to us for that reason to be even more probable. Its image, strongly etched on the sensorium, weakens the impression of contrary probabilities, and sometimes obliterates them to the point of making one believe that the thing has happened.* (Laplace, 1825, p. 111)

[23] Laplace (1825, p. 109) also considers the roles of empathy, imitation, and social proof.

2.4. Miscellaneous biases, and a call for a new experimental science of psychology

Laplace's treatment of psychology in his chapter extends beyond the examples mentioned above. Laplace speculates at length on the human proclivity towards the belief in luck, superstition, and other forms of magical thinking.[24] He attributes the persistence of these beliefs to information processing biases such wishful thinking (desirability bias),[25] and the neglect of disconfirming events (confirmation bias).[26] Finally, Laplace discusses how these phenomena were well understood by the ancients, in particular the Greek philosopher Diagoras of Melos and the Roman orator Cicero (Laplace, 1825, p. 99).

While Laplace viewed the probability calculus that he developed as a "supplement to ignorance and to the weakness of the human mind," he acknowledged that the calculus alone was not sufficient to make good decisions (Laplace, 1825, p. 124). In particular, he advocated for improving people's decision making by modifying their choice architecture so as to include visual representations of probability, rather than numeric representations. In this, Laplace anticipated later systematic investigations of how alternative representations of uncertainty (decision aids) can improve decision making (e.g., Gigerenzer and Hoffrage, 1995):

> *In order to diminish this cause of illusion as much as possible, it is necessary to appeal to the imagination and the senses to aid reason. When the respective probabilities are represented by lines, their differences will be much more readily perceived. A line that would represent the probability of the evidence by which an extraordinary fact is supported, placed next to the line that would represent the improbability of this fact, would make the probability of an error in the evidence very obvious; as a diagram, in which the altitudes of mountains are compared, gives a striking idea of the relationships between these altitudes.* (Laplace, 1825, pp. 110–111)

Finally, Laplace devotes considerable space advocating for a new science of psychology, with an aim towards understanding its laws in the same way that physical laws are understood.[27,28] His thoughts appear to be influenced by inchoate psychological ideas in vogue at the time, including the notions of empathy and imitation explored by Adam Smith and Pierre Jean Georges Cabanis, the associationist psychology of the British philosophers John Locke, David Hume, and David Hartley, and the studies of perception carried out by his contemporaries; see Hatfield (1994) and Daston (1988). Laplace advocates for an experimental methodology similar to that later employed in perceptual psychophysics of the latter part of the 19th century.[29] Laplace's examples involve

[24] "Man's opinion that he has long been placed at the centre of the universe, considering himself as the special object of nature's solicitude, leads each individual to think of himself as the centre of a more or less extensive sphere, and to believe that chance specially favours him." (Laplace, 1825, p. 94)

[25] "Our passions, prejudices and prevailing opinions, by exaggerating the probabilities that are favourable to them, and by attenuating contrary probabilities, are rich sources of dangerous illusions." (Laplace, 1825, p. 92)

[26] "No consideration is taken of the large number of non-coincidences that make no impression on one, or of which one is ignorant. However, it is necessary to take cognisance of them, in order to estimate the probability of the causes to which the coincidences are attributed." (Laplace, 1825, p. 99)

[27] "At the limits of visible physiology there begins another physiology whose phenomena, much more varied than those of the first, are, like them, subject to laws that it is very important to understand. This physiology, which we shall denote by the name psychology, is without doubt a continuation of the visible physiology." (Laplace, 1825, p. 100)

[28] "I hope that the preceding thoughts, however imperfect they may be, may draw the attention of philosophical observers to the laws of the sensorium or of the intellectual world, for it is important that we examine these laws as thoroughly as those of the physical world." (Laplace, 1825, p. 112)

[29] "The external senses can learn nothing of the nature of these modifications, astonished by their infinite variety

concepts familiar to modern psychologists, including top-down visual processing[30] and selective attention.[31]

3. Discussion

What is the relevance today of a two-hundred-year-old book chapter on cognitive illusions? Is this a mere historical curiosity? We believe that there still is much to be learned from this story.

Laplace's approach to identifying behavior that departed from the enlightenment conception of rational decision making—an effort that occurred in parallel with his role as a major architect of this ideal, as it applied to inference and decision making under uncertainty—spurred him to search for the general principles of reasoning that underlay these departures. That many of his explanations happen to coincide with modern accounts, arrived at independently based on the same introspections that evidently guided Laplace, suggests that the heuristics and biases approach to judgement and decision making is a scientific contribution that will endure.[32]

More generally, Laplace's work as a proto-psychologist and applied statistician, which complemented his career as a mathematician and physicist, demonstrates the creative tension between normative and descriptive ideas of inference and decision making. From the modern vantage point, Laplace's perspective on the ability of humans to reason under uncertainty does not fit neatly into any single paradigm. While we have focused on his contributions towards understanding why human reasoning frequently departed from the normative benchmark of probability theory (Tversky and Kahneman, 1974), we have also seen that Laplace occasionally viewed the resulting behavior as reflecting a certain ecological rationality, anticpating work such as Todd and Gigerenzer (2000). Laplace showed an appreciation of the intuitive expertise found in Kahneman and Klein (2009), holding in great esteem the collective common sense of elite practitioners—*hommes éclairés*—while at the same time observing conditions under which their intuitions (and reasoning) could fail.[33]

and the distinction and the order that they maintain in the small space that includes them, modifications of which the so varied phenomena of light and electricity give us some idea. But on applying the method that has been used for observations of external senses to observations of the internal sense, which alone can understand them, one will be able to carry over to the theory of human understanding the same precision as in the other branches of natural philosophy." (Laplace, 1825, p. 110)

[30] "When an observer placed in utter darkness sees, at different distances, two luminous balls of one and the same diameter, they seem to him to be of unequal size. Their internal images will be proportional to the corresponding images depicted on the retina. But if, when the darkness lifts, he catches sight at the same time of the balls and all the space between them, this sight enlarges the internal image of the further ball, and makes it almost equal to that of the other ball. Thus it is that a man, seen at distances of two and four metres, seems to us of the same size: his internal image does not change, although one of the images depicted on the retina is twice the size of the other." (Laplace, 1825, p. 103)

[31] "The great influence of the attention on the impressions of the sensorium is a very remarkable psychological phenomenon; it is deeply embedded in the memory and there it increases the acuteness, at the same time as it weakens concomitant impressions. If we look fixedly at an object, in order to discern any peculiarities, the attention may render us insensible of the impressions of other objects formed at the same time on the retina." (Laplace, 1825, p. 108)

[32] The program that Laplace and his contemporaries engaged in to develop a probability theory that embodied a form of universal rationality was eventually abandoned, along with the concept of a universal rationality. "However, the upheaval of the Revolutionary and Napoleonic era appears to have shaken the confidence of probabilists in a way that d'Alembert's persistent criticisms had not. The conduct of reasonable men no longer seemed an obvious standard, nor a comprehensive basis for a theory of society. Distinguishing prudent from rash behavior in post-Revolutionary France was no easy matter, and just what constituted 'good sense' was no longer self-evident. With the demise of the reasonable man, the probabilists had lost both their subject matter and criterion of validity." (Daston, 1988, pp. 106–107)

[33] Daston (1988, p. xiv) observes that many probabilists from Laplace's era aimed for probability theory to be a

Laplace recognized a form of bounded rationality in unaided human judgment, anticipating ideas such as in Simon (1956) and Gigerenzer and Selten (2001), emphasizing that without the probability calculus as a prescriptive tool, even the common sense judgments of the *hommes éclairés* could be off by orders of magnitude.[34]

Modern behavioral science research has taken us far beyond Laplace. While Laplace was an early advocate for the scientific method to be applied to psychological questions, he was limited in his inquiry by his reliance upon observational data. Modern research, through the use of innovative and carefully designed experimental demonstrations, has provided insights and further directions of study into how and why human behavior departs from the normative model of probability theory (Kahneman et al., 1982). Looking at decision making from a different direction, as Laplace's faith in a clockwork universe that could be reduced to intelligible causes via the scientific method has been called into question with the discovery of quantum phenomena and emergent complexity, Laplace's assumption that probability theory could serve as a domain-independent prescriptive model for human judgement has been upended by research demonstrating the relative efficacy of simple domain-specific decision rules and predictive models that respect cognitive limitations, tacit knowledge, multidimensionality of goals, and the need to adapt to complex and changing environments (Meehl, 1954; Gigerenzer and Brighton, 2009; Todd and Gigerenzer, 2000).[35]

Nevertheless, Laplace's attempts to understand the underlying mechanisms for people's biases were highly original, insightful, in many ways were centuries ahead of their time, and in at least two instances produced novel conjectures that have not been tested to this day. We believe that modern-day social and behavioral scientists can benefit from revisiting Laplace's thinking on illusions in the estimation of probabilities, and beyond.

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descriptive model of how elite practitioners reason, and as a prescriptive model for improving the decision making of the non-elite. In particular, "The classical probabilists aimed at a model of rationality under uncertainty—of the intuitions of an elite of reasonable men—and when their results did not square with those intuitions, they tinkered with the model to bring about a better fit" (Daston, 1988, p. xii). Nevertheless, Laplace did not view the probability theory as a descriptive model of how human beings reason, elite included. Throughout the *Essai*, Laplace highlights examples of elite reasonable men such as d'Alembert, Pascal, and Locke, deviating from the normative benchmark of probability theory. On the other hand, Laplace seemed to believe that the common standards of good sense developed by elite reasonable men typically followed the normative benchmark provided by probability theory, though not precisely (Laplace, 1825, p. 68). "Laplace reiterated his view that the calculus of probabilities, when wielded with skill and perspicacity, would confirm and guide good sense." (Daston, 1988, p. 359)

[34] To illustrate the deficiencies of the French judicial system's requirement of a simple plurality for conviction, Laplace (1825, p. 80) writes: "But simple common sense is not sufficient to allow one to estimate the great difference between the probabilities of the errors in these two cases. In such a case it is necessary to resort to the calculus, and one finds that the probability of error in the first case is very near to $\frac{1}{5}$, while it is only $\frac{1}{8192}$ in the second case, a value which is not even a thousandth of that in the first case."

[35] Also see Pearson (1978) for a discussion of John Stuart Mill's critique of the practical utility of Laplace's probability calculus.

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