Rejoinder to discussion of Laplace's theories of cognitive illusions, heuristics, and biases[1]

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We appreciate the thoughtful comments from Glenn Shafer and from Daniel Kahneman and Maya Bar-Hillel and respond to each in turn.

Response to Shafer

Shafer highlights three issues deserving of additional exploration: (1) the originality of Laplace's insights, (2) the extent to which Laplace's approach speaks to norms of rationality rather than simply providing a method to calculate correct answer to objective problems, and (3) the parallel between Laplace's (misplaced) confidence in his probabilistic methods as the standard to correct human reasoning, and the (misplaced?) confidence that is currently placed in modern methods from mathematical statistics.

With regards to originality, we distinguish between Laplace's insights relating to errors in probabilistic judgement, and his psychological insights which appear later in the chapter.

We agree with Shafer that it would be valuable to explore the originality of Laplace's insights relating to errors in probabilistic judgement. Using the gambler's fallacy as an example, Shafer notes how Nicolas de Condorcet discussed the gambler's fallacy in his 1785 work, which predates even Laplace's original 1795 lecture at the Ecole Normale that informed the 1812 first edition of the Essai. The gambler's fallacy is an interesting case, as it is an error simultaneously so prevalent and so clearly egregious that one would not be surprised to find a written account from an ancient Roman source. Nevertheless, as we noted in the draft, Laplace was a student of d'Alembert, who famously fell victim to the gambler's fallacy in his 1761 work, which appears to be the first written account of that fallacy, though of course d'Alembert was not aware of his mistake. The first written account that we could locate that treats the gambler's fallacy as a mistake is in a short passage from Laplace in his 1773 contribution to l'Academie Royale des Sciences' Memoires de Mathematique et de Phyisique (published in 1776). Laplace discusses d'Alembert's error, emphasizing how the reasoning implicitly requires the absurd assumption that past coin flips will influence the outcome of future coin flips.[4] Laplace also provides a rationalization for why people could believe in autocorrelation or streakiness even with random

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[4] In particular, Laplace writes, ". . . mais il faudroit supposer pour cela, que les évènemens passés ont quelque influence sur ceux qui doivent arriver; ce qui n'est point admissible." He refers to "plusieurs Philosophes" rather than mentioning d'Alembert's name directly.

coin flips.[5] We would welcome work tracing the origins of Laplace's other insights relating to errors in probabilistic judgment.

A fact that we did not emphasize in our paper is that the final two-thirds of Laplace's chapter on illusions consists of a call for the establishment of psychology as a science and a discussion of the psychological mechanisms that may underly errors in probabilistic judgment. It is not clear how many of these insights are due to Laplace and how many were common knowledge among the community of scientists working at the time. The recent work of Bru and Bru (2018), cited by Shafer, discusses at length the originality and influences of Laplace's psychological speculations. To this topic we contributed a small part in the final paragraphs of Section 2 of our paper, including what we believe to be three original contributions from Laplace.[6] As Bru and Bru (2018) note, that part of Laplace's chapter was partially sourced from his unpublished manuscript *Sur les panorama*, which was recently rediscovered by Stephen Stigler (2012); this material first appeared in the fourth edition of the Essai (Laplace 1819).[7]

Shafer writes, "the late 20th century literature on heuristics and biases . . . including the work by Kahneman and Tversky in the 1970s, took as its starting point Laplace's picture, not the modern subjectivist dogma." We would modify this statement in two ways. First, Kahneman and Tversky investigated errors in probabilistic judgement relating to both objective (Tversky and Kahneman 1974) and subjective probability (Tversky and Kahneman 1983). Second, while Laplace did not posit a list of axioms, he did appear to treat Bayes' rule as a norm of rationality, chiding both Pascal and Locke for misunderstanding the principles of the probability of testimony (Laplace 1995, pp. 71-72).

Finally, we are intrigued by Shafer's claim that Laplace's legacy suffered a "comeuppance" that might similarly occur to mathematical statistics in the future. Shafer argues that, just as Laplace's method of deductive inference based on probability theory was made largely obsolete by the "unprecedented flood of data" produced by nineteenth century natural and social science, we are seeing something similar in the way that traditional statistics is being eclipsed by data science and machine learning. We agree with Shafer that statistical methods are products of their time and rise and fall in utility. But we don't see Laplace, or modern statisticians, as having "a haughty spirit" or conflating their methods with "extravagant claims about rationality." Rather, we see probabilistic inference, as promoted by Laplace and the statisticians and machine learning researchers of today, as a theoretical baseline for coherent

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[5]In particular, Laplace proposes a Bayesian explanation to rationalize why, after observing a streak, people tend to believe the streak is driven by an underlying causal mechanism, rather than chance, thus anticipating modern debates on the "hot hand" (Gilovich, Vallone, and Tversky, 1985, Miller and Sanjurjo, 2018).

[6] These contributions include Laplace's articulation of the methodology later used in psychophysics (footnote 31), and a modern cognitive-psychological explanation of top-down visual processing (footnote 32) and selective attention (footnote 33).

[7] In particular, Stigler notes that Laplace expanded the chapter on illusions from 17.4 pages to 48.7 pages between the third (1816) and fourth editions (1819), adding material on subjective impressions and psychology to the material on illusions related to objective probabilities of the previous. Stigler notes that the English-speaking world missed out on this material, as the widely available English-language translation published by Dover was based on an earlier edition.

reasoning. One paradoxical advantage of probabilistic learning is that, in making strong assumptions, it can fail spectacularly (as noted by Shafer), and these failures can give insights into the flaws of our models. Making strong assumptions need not be "prideful"; rather, it can be thought of as a way of exploiting mathematics to test these assumptions and see where they can go wrong and how they can be improved.

Response to Kahenman and Bar-Hillel

Kahenman and Bar-Hillel highlight the role of introspection as a driving force behind many of the insights that arose from the heuristics and biases program. Indeed, when Laplace advocated for a new science of psychology, he placed introspection (the "internal sense") center stage as a tool of inquiry and measurement. In particular, Laplace (1995, p. 100) wrote:

> At the limits of visible physiology there begins another physiology whose phenomena, much more varied than those of the first, are, like them, subject to laws that it is very important to understand. This physiology, which we shall denote by the name psychology, is without doubt a continuation of the visible physiology. The nerves, whose fibres disappear in the medullary substance of the brain, propagate there the impressions that they receive of {or from} external objects, and they leave there lasting impressions which, in an unknown manner, modify the sensorium or seat of feeling and thought. The external senses can learn nothing of the nature of these modifications, astonished by their infinite variety and the distinction and the order that they maintain in the small space that includes them, modifications of which the so varied phenomena of light and electricity give us some idea. But on applying the method that has been used for observations of external senses to observations of the internal sense, which alone can understand them, one will be able to carry over to the theory of human understanding the same precision as in the other branches of natural philosophy.

Laplace used introspection and also observation of others to gain insight and provide compelling evidence for the power of these illusions. While Laplace was a keen observer of the common errors in probabilistic judgment of the *hoi polloi*, he emphasized that these were not simply the errors of untrained minds. Even the eminent philosophers, *hommes éclairés* such as Locke and Pascal, violated the norms of rationality prescribed by probability theory. Perhaps there was no stronger example of this than Laplace correcting his mentor d'Alembert, who argued fervently, and sincerely, that tails was more likely to occur after a long streak of heads (Gorruchurn 2012, p. 124). If someone as capable as d'Alembert could allow his intuition to drive his reason into such folly, can anyone be sure that they are free of such biases?

We agree with Kahenman and Bar-Hillel that a key aspect of cognitive illusions, as with visual illusions, is that "discredited perceptual impression persists even when the observer knows better." But this brings us to a difference between these two sorts of illusion. With a visual illusion such as Muller-Lyer, anyone can bring out a ruler and confirm the correct answer with one's own senses. On the other hand, when faced with a cognitive illusion, a person always has the option of refusing to accept the verbal explanation, either in a direct sense (continuing to

believe in the gambler's fallacy, for example) or in the indirect sense, by arguing for an alternative interpretation of the problem. Thus, discussions of cognitive illusions involve a level of argumentation, controversy, and psychological theorizing going beyond the more technical level of discussions of visual illusions.

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