1. (25 points) A is an \( n \times p \) matrix (in typical multiple regression settings, \( n \) is the number of observations and \( p \) is the number of parameters, and \( n \geq p \)), prove that

(1) \( A'A \) and \( AA' \) are symmetric matrices. (\( A' \) denotes the transpose of \( A \))

(2) \( A'A \) and \( AA' \) are semi-positive-definite matrices. (An \( n \times n \) matrix \( M \) is semi-positive-definite if \( \forall x \in \mathbb{R}^n, x'Mx \geq 0. \))

(3) If \( A \) has full column rank (\( \text{rank}(A) = p \)), then prove \( A'A \) is a positive-definite matrix. (An \( n \times n \) matrix \( M \) is positive-definite if \( \forall \) nonzero \( x \in \mathbb{R}^n, x'Mx > 0. \))

2. (25 points) \( A \) is an \( n \times p \) matrix with full column rank. Let \( P \equiv A(A'A)^{-1}A' \)

(1) An \( n \times n \) matrix \( M \) is a projection matrix if it is symmetric and idempotent (i.e. \( A^2 = A \)). Prove that \( P \) is a projection matrix.

(2) Give the rank of \( P \) and \( I - P \). (\( I \) is the \( n \times n \) identity matrix)

(3) Prove that the projection \( P \) is orthogonal. (i.e. \( \forall x \in \mathbb{R}^n, (Px)'[(I - P)x] = 0 \))

3. (25 points) \( \vec{X} \) is a 3 dimensional Gaussian random vector, with distribution \( N_3(\mu, \Sigma) \), in which

\[
\mu = \begin{pmatrix} 3 \\ 4 \\ -3 \end{pmatrix}, \Sigma = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & -2 \\ 3 & -2 & 8 \end{pmatrix}
\]

Let \( Y_1 = X_1 + X_3 \) and \( Y_2 = 2X_2 \), determine the distribution of \( \vec{Y} = (Y_1, Y_2)' \) and the conditional distribution \( Y_1|Y_2 = 10. \)

4. (10 points) Prove \( \vec{Y} \sim N(0_{n\times 1}, I_{n\times n}) \) implies that all \( Y_i \) i.i.d. follow \( N(0, 1). \)

(This problem may seem ridiculously easy to you. Just write out the joint density and see what it tells us about the marginal distributions.)
5. **(15 points)** Assume that an \( n \) dimensional Gaussian random vector \( X \) is distributed as \( N(\mu, \Sigma) \)

(1) Find a transformation of \( X \), \( Y = f(X) \), such that \( Y \sim N(0_{n \times 1}, I_{n \times n}) \).

(2) Prove that

\[
(X - \mu)'\Sigma^{-1}(X - \mu) \sim \chi^2(n)
\]